Al-Hawārī’s *Essential Commentary*

Arabic Arithmetic in the Fourteenth Century

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Al-Hawārī’s *Essential Commentary*

Arabic Arithmetic in the Fourteenth Century

Mahdi Abdeljaouad and Jeffrey Oaks

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Cover: Folio 217r from *Support for students in the science of arithmetic* (*ʿUmdat al-ṭullāb fī ʿilm al-ḥisāb*) by Ibrāhīm ibn Muḥammad al-Qabāqibī (d. 851/1447-8), in which he solves the same problem that al-Hawārī works out at 207.6. From British Library MS Or 8416, copied 1023/1614. Available online: https://www.qdl.qa/en/archive/81055/vdc_100055934394.0x000001

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Preface

Al-Hawārī was not an important mathematician. His only known work, the *Essential Commentary* on a short book of Ibn al-Bannāʾ, contains no groundbreaking developments nor any original insights. His textbook is simply an introduction to practical arithmetic which enjoyed a modest circulation in the Western part of the Islamic world in late medieval times. And yet this unremarkable book has attracted our interest, and it should attract your interest, too.

The *Essential Commentary* is worth a modern edition, translation and commentary first of all for what it tells us about the state of Arabic arithmetic in the early fourteenth century. From an historical angle, it is remarkable how techniques and ideas from Greece, the Middle East, and India came to be selected and combined to form such an eclectic whole. Ibn al-Bannāʾ and al-Hawārī were not the first to present shortcuts for mental calculation in a book ostensibly devoted to rules for operating with Indian (i.e., "Arabic") numerals, and by their time the integration of portions of Greek number theory, from definitions to rules of calculation, was already an established part of arithmetical instruction. The very real tensions created by combining these disparate techniques and ideas could be disregarded by those interested in their utility, and could serve as a starting-points for philosophical discussion by people more interested in theory.

From the conceptual angle, we get to see how the numbers of Arabic arithmeticians differ radically from both Greek and modern numbers. Mathematicians today define real numbers as the elements of a set which, together with binary operations, satisfy the ordered field axioms. Historians of Greek mathematics, for their part, can point to Euclid's definition of number as a multitude of indivisible units. Arabic numbers were neither of these. For medieval arithmeticians numbers derive from the acts of counting, measuring, and weighing, and as a consequence they worked with any positive quantity that arises in calculation, including fractions and irrational roots. These numbers were validated through practice, so the people who worked with them had no need to ground them in definitions or axioms. Nevertheless, several Arabic authors, Ibn al-Bannāʾ included, attempted to give practical arithmetic some kind of foundation modeled on or based in Greek mathematics, usually in Euclid's *Elements*.

It is not just the concept of number that differentiates medieval from modern arithmetic. If we pay attention to how Ibn al-Bannāʾ, al-Hawārī, and other authors phrased their explanations and examples, we discover that they expressed them differently, too. Like the numbers in Greek logistic, numbers in Arabic arithmetic were regarded as being numbers *of something*. Rather than belonging all to the same abstract set, they come in different kinds: a three might be three units (3), three fifths ($\frac{3}{5}$), three roots of five (a trio of $\sqrt{5}$'s), three cubes ($3x^3$, again a trio), or, in problem-solving, three men or three dirhams. Further, their way of expressing binomials and apotomes, whether of fractions, roots, or algebraic terms, is without parallel in modern arithmetic and algebra. These and other seemingly innocent curiosities of language had a significant impact on how calculations were performed.

Becoming familiar with the ways numbers were conceived and expressed is not only interesting in itself, but it provides the necessary background for the study of the works of more brilliant Arabic writers on algebra and arithmetic such as al-Karajī, al-Khayyām, and al-Fārisī. These and other scholars took their inspiration and developed their concepts directly from the practical tradition. And because medieval Europeans learned much of their arithmetic (including algebra) from Arabic sources, it is only natural that the concepts and modes of expression exhibited in al-Hawārī's book should be found in their works, too. In fact, they are abundantly evident in Fibonacci, Italian abacus authors, Jean de Murs, Luca Pacioli, and others, and they persisted in sixteenth-century Europe even as significant conceptual developments were taking place. To pick just two examples: Michael Stifel (1550) conforms to the practice of Arabic algebraists by insisting on rational coefficients in algebra, like multiplying $\sqrt{6}$ by x to get $\sqrt{6x^2}$ instead of $\sqrt{6}x$; and François Viète (1590s), like his Arabic predecessors, always writes single roots, such as $\sqrt{1200}$ and $\sqrt{\frac{45}{16}}$, instead of $20\sqrt{3}$ and $\frac{3\sqrt{5}}{4}$.[1] These premodern ways of thinking about and expressing numbers and polynomials only truly began to give way with Descartes, whose 1637 *La Geometrie* gave a numerical reinterpretation to Viète's new geometric algebra.[2]

Finally, it is worthwhile to actually work through the numerical operations and to solve problems by the methods described in this book. For you adventurous types willing to perform a division of fractions or to work through a problem by double false position, you will leave paper and pencil behind and take up the dust board (a chalkboard will suffice). And by reading aloud the instructions provided by our authors, you can gain some insight into the prominent role played by recitation and memorization in student-teacher interactions. Supplementing notational board calculations with rhetorical presentation not only helps one remember the rules, but it also causes one to think about the meanings of the signs, which, on reflection, ultimately accounts for their curious (to us) ways of expressing arithmetical and algebraic quantities.

We have written this book with different readers in mind, mainly historians of mathematics and related fields, mathematicians, and mathematics educators. Please keep in mind that, for each group, we explain different aspects of al-Hawārī's book that people in other groups might find too elementary.

This book is divided into five parts. The introduction covers the basic historical, textual, social, and conceptual aspects of Arabic arithmetic that form the backdrop to al-Hawārī's book. This is followed by our translation of the *Essential Commentary*. Because books written in another language many centuries ago cannot speak entirely for themselves, we have written a passage-by-passage commentary to al-Hawārī's commentary which turned out to be longer than the translation itself. Appendices follow, which include a conspectus of problems, sample worked-out problems solved by various methods from other Arabic books, a brief chronological list of ancient and medieval mathematicians and other scholars, and a glossary of Arabic terms. The English portion finishes with the bibliography and an index of names. Last (or first, if you are an Arabic reader) is the corrected critical edition of the Arabic text together with its own introduction and conspectus, which we originally published in Tunisia in 2013.

Each passage of the translation and our commentary is linked to the Arabic edition by a reference to page and line number. For example, "201.14" in the margin of the translation, or in a box in our commentary, refers to page 201, line 14 of the Arabic edition. Cross

[1] (Stifel 1544, fol. 283a); (Viète 1646, 140, 309). In Stifel's notation, he multiplies √ℨ6 by 1ꝛ to get √ℨ6ℨ. See (Oaks 2017).

[2] (Oaks 2018b).

references are given throughout. For readers who will not consult the original Arabic, these references still serve to link our commentary with the translation. Hyperlinks connect the different parts of the introduction, translation, commentary, and appendices of the online version of this book. We were unable to make hyperlinks with the Arabic part because the English part was typeset in XeLaTeX while the Arabic part was typeset in MSWord.

Most of what we write in the commentary is directed specifically at the passage in question. But many times we offer remarks to clarify the mathematics from a more general point of view, whether mathematical, conceptual, linguistic, or historical. In our comments at 68.18, for example, we pause to give a breakdown of the vocabulary used in the writing of numbers; at 74.14 we devote a page to how zero was dealt with in calculations; and at 219.1 and 219.4, we devote more than five pages to an explanation of how Arabic ways of expressing operations and their results differ from ours, and so on.

Two bio-bibliographic works are indispensable for sorting out details of the works of medieval Arabic mathematicians. B. A. Rosenfeld's and E. Ihsanoğlu's 2003 *Mathematicians, Astronomers, and Other Scholars of Islamic Civilisation and Their Works (7th-19th c.)* lists the names and works of 1,711 scholars known to have written books in the mathematical sciences, and Driss Lamrabet's 2014 *Introduction à l'Histoire des Mathématiques Maghrébines* does much the same for 983 scholars who worked in the western part of the Islamic world. References after names are to these two works. For example, for al-Ḥaṣṣār we write (#532, M55), which means that he is scholar #532 in Rosenfeld and Ihsanoğlu and scholar M55 in Lamrabet. The "M" stands for "Maghreb". For others, like Ibn al-Samḥ, the "A" stands for "al-Andalus". Numbers are given for each scholar listed in Appendix C. For authors not listed in Appendix C, numbers are given on mentioning their names.

We follow the International Journal of Middle East Studies system for transliteration of Arabic words.[3]

It was Mahdi's idea to write this book. With some input from me, he collected the manuscripts and produced the critical edition. We each then translated the book independently, and I compared the two translations to produce the version you see here. I then wrote the commentary, the introduction, and the appendices, with valuable input from Mahdi.

Finally, we thank Len Berggren, Margaret Gaida, and Robert Morrison for their comments and suggestions on an earlier version of this book.

Jeff Oaks,
Indianapolis, May 4, 2019.

[3] https://www.cambridge.org/core/services/aop-file-manager/file/57d83390f6ea5a022234b400/TransChart.pdf

Chapter 1
Introduction

1.1 Ibn al-Bannāʾ and al-Hawārī

1.1.1 Ibn al-Bannāʾ's textbook and al-Hawārī's commentary

Shortly before the year 1301 CE, in the western part of North Africa, the celebrated mathematician and astronomer Ibn al-Bannāʾ (1256-1321) wrote what was to become a very popular arithmetic textbook. His *Condensed [Book] on the Operations of Arithmetic* (*Talkhīṣ aʿmāl al-ḥisāb*) describes rules for operating with Arabic numerals and methods of arithmetical problem-solving. One factor that made the book appealing was that it is brief. Condensed expositions were popular because students then were often expected to memorize textbooks, and this one covered just the right amount of material.

But brevity is also a drawback. Ibn al-Bannāʾ's little book contained no numerical examples to illustrate how the calculations were to be performed, and his concise exposition left no room for proofs or supplementary remarks. So, over the course of the next three centuries, the *Condensed Book* inspired a number of people to write commentaries elaborating on its contents in various ways. Ibn al-Bannāʾ himself became the first to write such a commentary in 1301 CE (701H). In *Lifting the Veil from the Face of the Operations of Arithmetic* (*Rafʿ al-ḥijāb ʿan wujūh aʿmāl al-ḥisāb*) he expounds on his short work by providing philosophical remarks, proofs, numerical examples, and expanded explanations.

Al-Hawārī was among Ibn al-Bannāʾ's students in Marrakesh early in 1305 (704H). Ibn al-Bannāʾ was teaching his *Condensed Book*, and already during the course of the lectures al-Hawārī was writing his own commentary. As he tells us in his introduction, al-Hawārī's specific purpose was to provide numerical examples for Ibn al-Bannāʾ's rules. The present book contains our edition, translation, and commentary of his *Essential Commentary on the Condensed [Book] on the Operations of Arithmetic* (*al-Lubāb fī sharḥ Talkhīṣ aʿmāl al-ḥisāb*), which he completed June 12, 1305. In it, al-Hawārī reproduces passage by passage the entire text of the *Condensed Book*. After each of Ibn al-Bannāʾ's rules he gives his own numerical examples, and, where he finds it necessary, he supplements Ibn al-Bannāʾ's explanations with further remarks often extracted from his teacher's *Lifting the Veil*.

1.1.2 The authors and their works

Ibn al-Bannāʾ's full name is Abū al-ʿAbbās Aḥmad ibn Muḥammad ibn ʿUthmān al-Azdī al-Marrākushī. For those not familiar with the forms of Arabic names, we will dissect this one. His given name is Aḥmad, and he gained the sobriquet Abū al-ʿAbbās when his wife gave birth to a son they named al-ʿAbbās (*Abū* means "father"). The word *ibn* means "son", so he was the son of Muḥammad, who in turn was the son of ʿUthmān al-Azdī. The designation al-Marrākushī indicates that he was from Marrakesh, a city now in modern

Morocco. He is best known today by the nickname Ibn al-Bannāʾ, which means "son of the builder".

Ibn al-Bannāʾ was a renowned scholar known to have written nearly 120 works on topics ranging from the rational sciences (mathematics, astronomy, logic) to religion (including law), language (rhetoric and prosody), the occult sciences (including astrology), agronomy, philosophy, and medicine.[1] He was born in Marrakesh and remained in Morocco his whole life.

Four of his books are relevant to us, and the Arabic text of each has been published:

- *Condensed [Book] on the Operations of Arithmetic* (*Talkhīṣ aʿmāl al-ḥisāb*, [M1]).[2]
  This is Ibn al-Bannāʾ's brief introduction to calculation with Indian (i.e. Arabic) numerals, including methods for finding unknown numbers. Because it does not include examples, the book does not show the numerals. It was completed before 1301. Souissi's 1964 edition also contains a French translation.

- *Lifting the Veil from the Face of the Operations of Arithmetic* (*Rafʿ al-ḥijāb ʿan wujūh aʿmāl al-ḥisāb*, [M8]).[3]
  Ibn al-Bannāʾ's own commentary to his *Condensed Book* was completed in 1301. Although it contains many numerical examples that would be of help to students, the theoretical and philosophical nature of many of his remarks suggest it was written for a more sophisticated audience. Al-Hawārī copied many passages from this commentary into his own book, mainly those giving further explanations of the rules.

- *Essays on Arithmetic* (*Maqālāt fī l-ḥisāb*, [M2]).[4]
  This is another arithmetic book that covers much of the same material as the *Condensed Book*. In it, Ibn al-Bannāʾ sparingly shows the Indian numerals. He does not cover algebra or double false position, but he does include a collection of problems solved by single false position and proportion. Problems [1] and [2] in Appendix B are translated from this book.

- *Book on the Fundamentals and Preliminaries in Algebra* (*Kitāb al-uṣūl wa l-muqaddimāt fī l-jabr wa l-muqābala*, [M6]).[5]
  Ibn al-Bannāʾ's book on algebra, dating from the late thirteenth century, builds on Abū Kāmil's late ninth century book on the same topic. This book explains problem-solving by algebra at a level appropriate for students. Problem [6] in Appendix B is translated from this book.

Al-Hawārī's full name is ʿAbd al-ʿAzīz ibn ʿAlī ibn Dāwud al-Hawārī al-Miṣrātī. The designation "al-Miṣrātī" indicates that he was descended from the Libyan tribe of Miṣrāta. The "al-Hawārī" tells us that he hailed from the Berber tribe named Hawārī, which we know emigrated from Libya to Morocco in the ninth century.[6]

The only secure dates we have on al-Hawārī are found in his book. The earliest extant manuscript, that of Medina (described below), reports that he completed it on Saturday, 18 Dhū al-Qaʿda, 704H, which is the Julian date 12 June 1305. Also, at line 204.3 he relates that Ibn al-Bannāʾ was dictating to him on Wednesday, the 28th of the month of

[1] (Lamrabet 2014, 164ff); (Samsó 2007).
[2] Published in (Ibn al-Bannāʾ 1969).
[3] Published in (Ibn al-Bannāʾ 1994).
[4] Published in (Ibn al-Bannāʾ 1984).
[5] Published in (Saidan 1986, vol. 2).
[6] (Abdeljaouad and Oaks 2013, 11).

Rajab. This can only be in 704H, which corresponds to 24 February 1305. Because we know that Ibn al-Bannāʾ was in Marrakesh at that time, it places al-Hawārī there as well. We know nothing else about his dates, his locations, or his career. His commentary on the *Condensed Book* is his only known work: *Essential Commentary on the Condensed [Book] on the Operations of Arithmetic* (*al-Lubāb fī sharḥ Talkhīṣ aʿmāl al-ḥisāb*, [M1]).[7]

Al-Hawārī delivered just what he said he would in his introduction: a book supplementing each of Ibn al-Bannāʾ's rules with numerical examples. There is nothing innovative or theoretical about it. Like the *Condensed Book*, al-Hawārī's commentary makes no advances in either ideas or techniques. He gives no proofs, and he generally avoids philosophical discussions. And yet, the very lack of innovation in this book makes it a good source for learning about the nature of arithmetic in the epoch of its author, especially when it is compared with other Arabic arithmetic books. By reading how these authors worked with and expressed numbers and equations, we gain insights into the concepts underlying their arithmetic. Furthermore, the apparent lack of interest of both Ibn al-Bannāʾ and al-Hawārī in prior mathematical traditions, when coupled again with an examination of other books, allows us to identify influences from Greece, the Middle East, and India on the practical arithmetical tradition in Arabic.

1.1.3 Manuscripts of al-Hawārī's *Essential Commentary*

The fourteen known surviving manuscripts of al-Hawārī's book were copied between the fourteenth and nineteenth centuries CE. We published our Arabic edition based on the following five manuscripts:

- Medina, MS Ḥikmat 21 ḥisāb. 63 ff, 16 lines per page, 16 × 21 cm. The copyist completed it on 18 Rabīʿ I, 746H (Julian 18 July 1345). This manuscript does not distinguish between Ibn al-Bannāʾ's and al-Hawārī's words.
- Oxford, MS Marsh 378/3, fols. 109a-162a. Copied in 1444 according to Woepcke.[8] In this manuscript a word or two is written in red ink to denote the beginning of a new idea, or what we might consider as a new paragraph. This often corresponds to a shift in author (Ibn al-Bannāʾ to al-Hawārī, or vice-versa), but many times it does not.
- Istanbul, Süleymaniye Library, MS Şehit Ali Paşa 1977/2 (Türkiye Yazma Eserler Kurumu Başkanlığı Süleymaniye Yazma Eser Kütüphanesi, Şehit Ali Paşa Collection 01977), fols. 54a-103b. Copied 20 Ramaḍān 880H/Julian 16 January 1476 in Constantinople. The copyist distinguished Ibn al-Bannāʾ's text from al-Hawārī's comments by placing in front of Ibn al-Bannāʾ's extracts the letter "ص" (*ṣād*), which stands for *muṣannif* ("Author"), while passages from al-Hawārī are preceded by the letter "ش" (*shīn*), which is the first letter of the word *sharḥ* ("commentary").
- Tehran, Library of Parliament MS 2672/2, fols. 10a-56b, copied before 972H/1564. Red ink is employed here like in the Oxford manuscript.
- Tunis, National Library of Tunis MS 9940. 32 ff., 22 × 26 cm, 29 lines per page. Copied 4 Jumādā II, 1082H/ Julian 28 September 1671 in Damascus. In this manuscript Ibn al-Bannāʾ's passages are preceded by the letter "م" (*mīm*), which either stands for *matn* or *muʾallif*, both of which mean "[original] text", and al-Hawārī's comments start with a "ش" (*shīn*) for *sharḥ* ("commentary").

[7] Published in (al-Hawārī 2013) and in the present volume.
[8] (Ibn al-Bannāʾ 1969, 8).

For those not familiar with the notation, the Tehran manuscript is number 2672 in the collection, and the “/2” indicates that al-Hawārī’s treatise is the second treatise contained in that manuscript. In this manuscript, al-Hawārī’s treatise begins on the front (a) of folio (sheet) 10 and ends on the back (b) of folio 56. The Tunis manuscript 9940 contains only al-Hawārī’s treatise, and it is 32 folios long.

As is common with manuscripts, there are variations from one copy to another. In the case of al-Hawārī’s commentary, these differences are all minor. Some are clearly errors, while for others it is not easy to tell which variation the author originally wrote. When in doubt we have sided with the oldest manuscript, that of Medina, copied only four decades after the book was finished. There are, however, greater differences between the figures drawn in the manuscripts. We discuss them below.

On the following pages are samples from each of the five manuscripts. We have chosen the page that contains the double false position diagram just before 208.8. (Double false position is described below in chapter 1.5.)

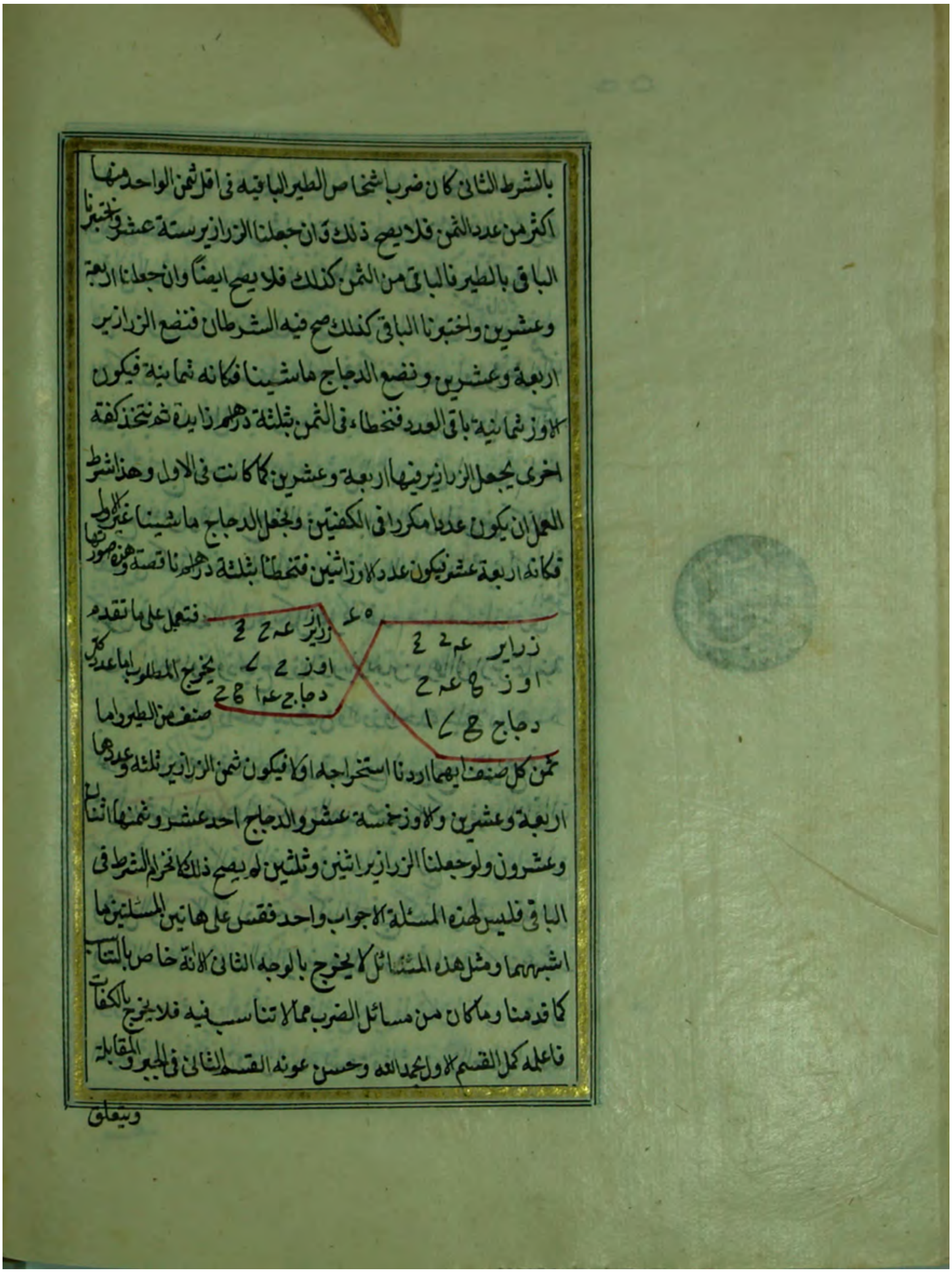

Figure 1: Medina Manuscript fol. 55b, covering the text from 207.16 to 209.2.

Figure 2: Oxford manuscript fol. 154b, covering the text from 208.6 to 211.15. The scan is in black and white. Lighter shaded portions, which can be seen in the text in the middle of the page, in the lines of the diagram, and as bars over some words, are in red ink.

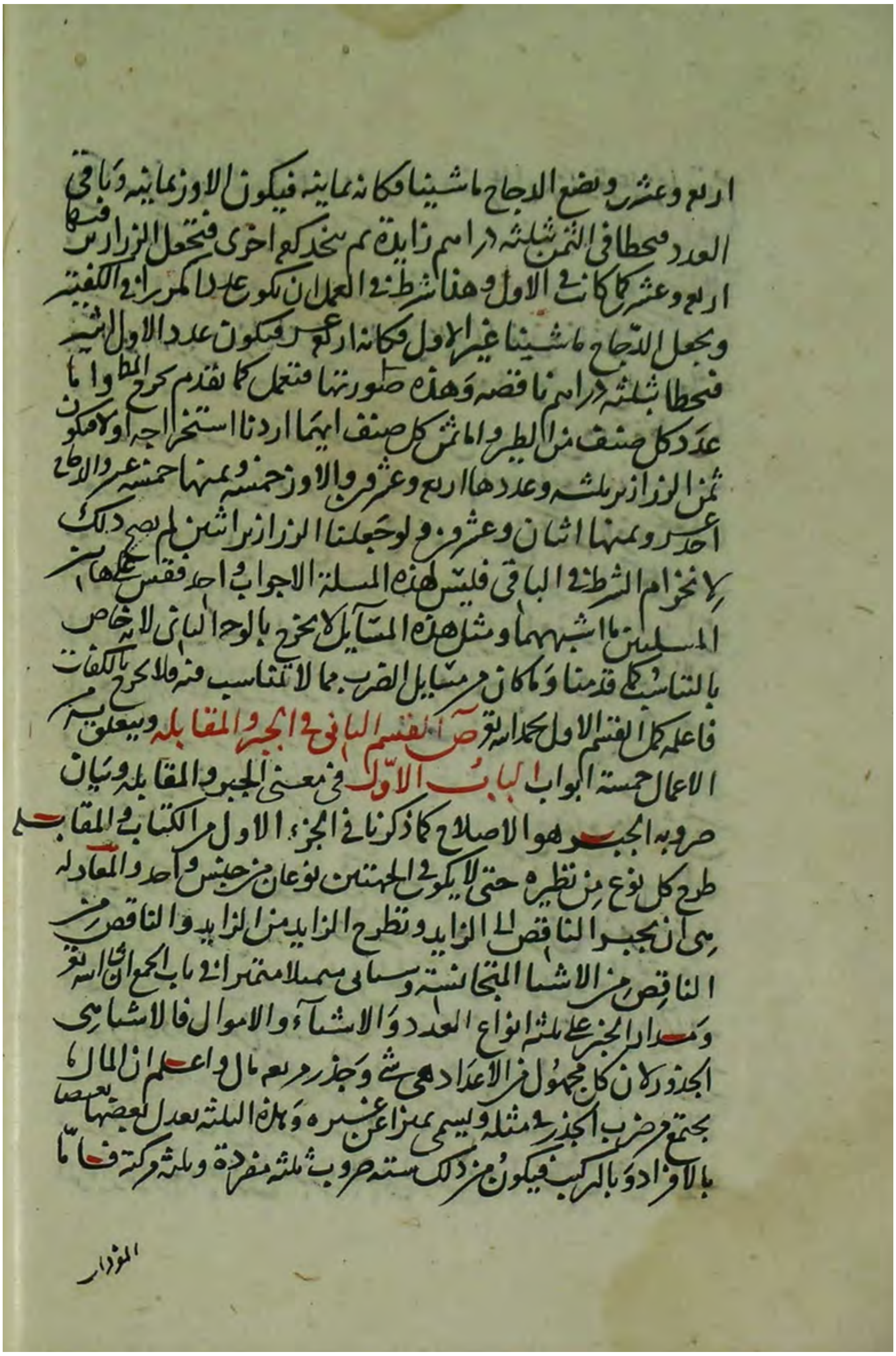

Figure 3: Istanbul manuscript fol. 97b, covering the text from 208.2 to 211.15. This manuscript does not show the diagram.

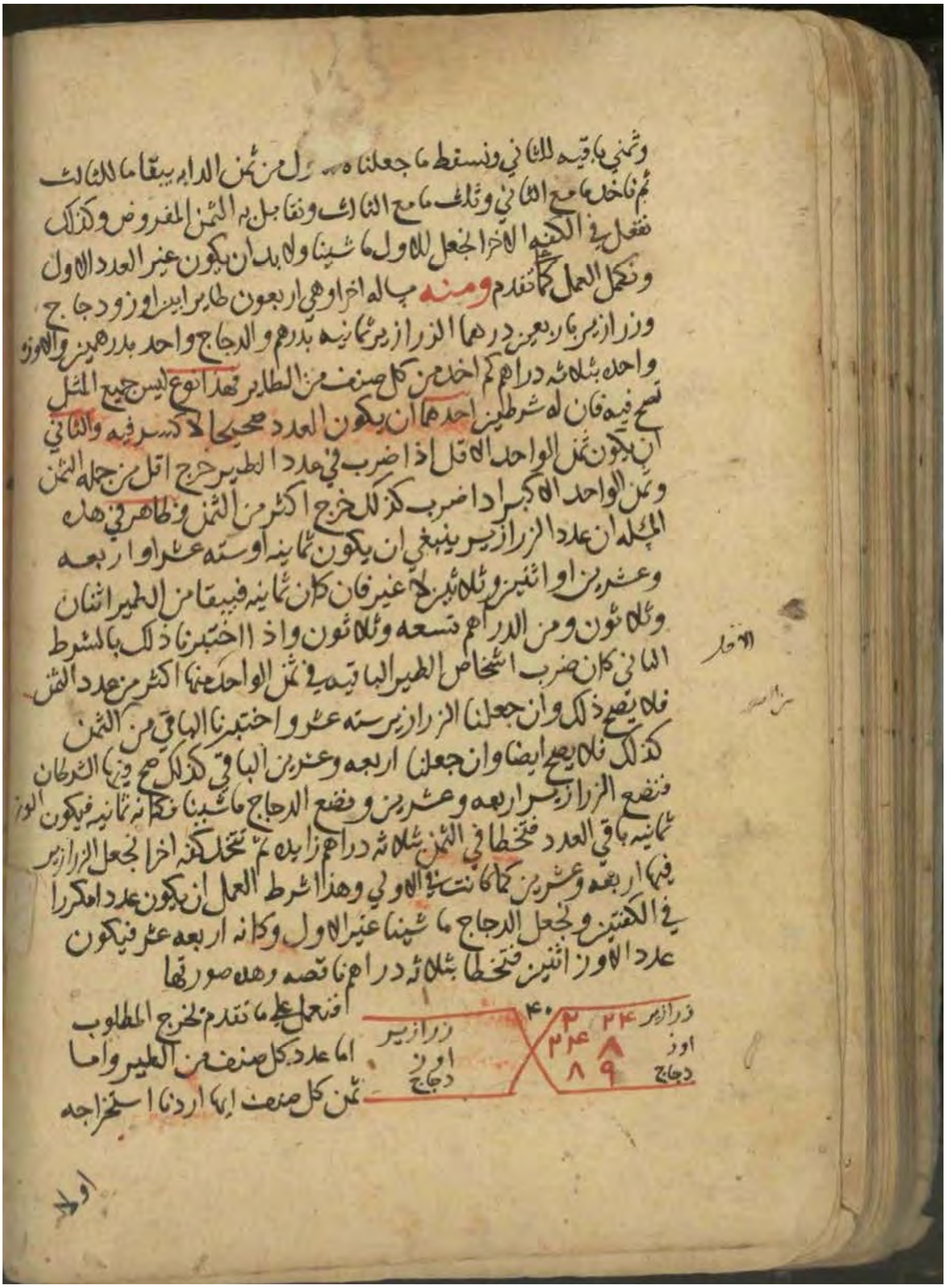

Figure 4: Tehran manuscript fol. 50b, covering the text from 207.1 to 208.9.

Figure 5: Tunis manuscript fol 28a, covering the text from 208.2 to 213.5. This scan is in black and white, so it is difficult to tell if there is anything written in red ink.

1.2 Historical setting

Even a quick look over the translation reveals that al-Hawārī's arithmetic textbook differs from its modern counterparts in format, contents, and methods. These differences run deeper than what might be seen as accidental variations. Medieval Islam was an intellectual melting-pot of ideas from the various cultures it was in contact with, and al-Hawārī's book exhibits these different influences, each with qualities of its own, and which were not always amenable to merging. Further, medieval people conceived of numbers differently than we do today, and their system of education and their very attitude toward books affected the way they presented their calculations. These historical, mathematical, and conceptual issues warrant a few words to put the *Essential Commentary* in context, and we begin in this section with history.

When the prophet Muḥammad died in 632 CE, his empire already covered most of the Arabian peninsula. The military push by his successors resulted in rapid conquests both east and west, so that by the time the Umayyad dynasty fell to the ʿAbbāsids in 750 CE the empire stretched from the Iberian peninsula in Western Europe all the way to the Indus River. The Arab conquerors found themselves ruling over territory once controlled by Greece to the west, Persia in the Middle East, and bordering India to the east.

One remarkable cultural phenomenon that arose from this expansion is known as the Arabic translation movement. From roughly the middle of the eighth century until the end of the tenth, the ruling class and other wealthy patrons subsidized the large-scale collection, translation, and study of scientific and other knowledge. Books were sought out from any available source, whether in Greek, Sanskrit, Persian, or Syriac. Topics included the mathematical sciences (arithmetic, geometry, optics, mathematical astronomy, etc.) as well as philosophy, astrology, geography, mechanics, medicine, agriculture, alchemy, and assorted other topics. Concurrent with this translation activity, several authors wrote down in books the arithmetic (including algebra), mensuration, and folk astronomy that had previously been transmitted orally among people working in the trades in the Middle East.

The causes of the Arabic translation movement are complex and are still being debated. We will only note that it appears to have been spurred by the demands of managing such a large empire combined with the demands of political expediency involving Persian influence.[9] The causes of its demise around the latter tenth century are necessarily equally complex, but one factor seems to have been that they had just about run out of relevant books to translate, and most of those that were left had been surpassed by original works in Arabic.

In conjunction with the translation movement, scholars composed original works in various branches of mathematics. The circulation in this singular environment of books originating from different cultures naturally gave rise to the mixing of ideas and techniques. To pick just a few examples: in astronomy, Indian trigonometry was adopted in books based on Greek planetary theory; in geometry, definitions and constructions from Greek texts were introduced to books on practical mensuration; and we find applications of algebra to Greek geometry and of Greek geometry to algebra. This combining of elements from different traditions was a lasting characteristic of Arabic mathematics and science, despite their occasional incompatibility.

The prevalence of the word "Greek" in this last paragraph is not accidental. The majority of works translated and studied by Arabic scholars were originally written in that

[9](Gutas 1998); (Saliba 2007).

language. And, just as important, Arabic authors consciously structured their mathematical works, at least the more theoretical ones, on Greek models. Arabic scholars were drawn in particular to the logical-deductive structures that guided Euclid's *Elements* in geometry and Ptolemy's *Almagest* in astronomy. The philosophical underpinnings of Greek-inspired mathematics in Arabic were also Greek in origin, and were drawn mainly from the works of Aristotle and some late antique neoplatonists. While mathematicians in other cultures had also engaged in stating definitions, outlining principles, and writing proofs, it was the unified construction of a science based on these elements, together with underlying philosophical principles, that for Arabic scholars set Greek mathematics apart.[10]

After the translation movement had run its course, the production of new books continued unabated. Innovative work continued in algebra, geometry, optics, mechanics, and various aspects of astronomy that included planetary models, trigonometry, and numerical interpolation. In some branches of learning this momentum persisted longer than in others. After the thirteenth century, for example, we see little advancement in algebra from a theoretical perspective (including the algebra in al-Hawārī's book), while new and innovative books on planetary theory continued to appear at least to the middle of the sixteenth century.[11]

Even if Arabic scholars distinguished between practical (*ʿamalī*) and theoretical (*naẓarī*) mathematics,[12] it was common for the two to appear together in the same book. A good example in which the two approaches mingle is the *Completion of Arithmetic* of al-Baghdādī (died 1037).[13] This book gives rules for practical calculation while at the same time incorporating and building on various aspects of theoretical Greek number theory. Practical topics could also serve theoretical needs: in the latter eleventh century, al-Khayyām adapted the algebra of merchants and surveyors to aid in theoretical geometric problem-solving by providing solutions to irreducible cubic equations via conic sections.[14] Often, practical techniques attracted the attention of mathematicians who crafted Greek-style proofs for them. One example from the late ninth century is Qusṭā ibn Lūqā's short work proving the validity of the method of double false position in the manner of Euclid's *Data*.[15]

Ibn al-Bannāʾ states his practical intent when he writes that his *Condensed Book* "is useful for inheritance and business transactions and other [purposes]".[16] The commentaries it spawned, however, frequently take it in philosophical and theoretical directions. For example, Ibn al-Bannāʾ ensured that the nature of the unit would become a source for philosophical debate by the elaboration he gave the issue in his *Lifting the Veil*.[17] And many commentators, Ibn al-Bannāʾ included, provided proofs to many of the rules in the *Condensed Book*. Al-Hawārī's commentary, on the other hand, retains the practical orientation by focusing almost exclusively on numerical examples.

Like many Arabic mathematics books, Ibn al-Bannāʾ's *Condensed Book* is a kind of hybrid, exhibiting techniques originating from Indian, Greek, and Middle Eastern sources.

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[10] The classic work on Greek influence is (Rosenthal 1975). For mathematics and philosophy, see (Endress 2003).

[11] See (Saliba 2007); (Ragep 2007); (Brentjes 2007).

[12] As, for example, al-Fārābī does in his *Enumeration of the Sciences* (first half of 10th c.) (al-Fārābī 1953, 54.7).

[13] *Al-Takmila fī l-ḥisāb*. Saidan published the Arabic text in (al-Baghdādī 1985). See also (Saidan 1987).

[14] (Oaks 2011a). Al-Khayyām himself used algebra this way to solve a particular problem of cutting a quadrant of a circle to produce a particular ratio.

[15] A German translation is published in (Suter 1908–1909).

[16] (Ibn al-Bannāʾ 1994, 202.5).

[17] See our commentary at 65.2.

Part I of the book is devoted to calculation with Indian numerals (we call them Arabic numerals), which originated in India. Certain chapters contain material extracted ultimately from Greek sources, like Euclid's treatment of quadratic irrationals recast in numerical terms (beginning at 173.4), and portions of theoretical number theory borrowed from Nicomachus of Gerasa (starting at 65.10 and 127.10). Last, the methods of mental multiplication (beginning at 108.9) and the techniques of problem solving by proportion, double false position, and algebra comprising Part II derive ultimately from local oral tradition. These different elements will now be described.

1.3 Arabic arithmetic

The two common notations for writing numbers in medieval Arabic were *jummal* and Indian. *Jummal* notation is sometimes called *abjad*, after the first four letters in the traditional ordering of the Arabic alphabet: *alif*, *bā*, *jīm*, and *dāl*. In this system, the 28 letters of the alphabet are assigned values 1, 2, 3,…, 9, 10, 20, 30,…, 90, 100, 200, 300,…, 900, and 1000.[18] It is an additive system, so to express the number 542, for example, one writes the letters whose values are 500 (ث), 40 (م), and 2 (ب): ثمب. *Jummal* notation makes its appearance in al-Hawārī's book only in the rule for casting out sevens, in the figures just before 89.10 and 90.1 (see our commentary at 88.10). *Jummal* numerals were associated with two forms of calculation, finger reckoning and sexagesimal arithmetic, which will be described below.

We acquired our "Arabic" numerals from Arabic sources, but Arab-speaking people called them "Indian" (*al-hindī*) numerals because they learned them from Indian sources. Indian numerals consist of the signs for 1, 2, 3, …, 9, together with a 0 for the empty place. This is the system taught in Ibn al-Bannāʾ's textbook and illustrated by al-Hawārī's examples. The shapes of the numerals have varied greatly over time and place, even among the manuscripts of al-Hawārī's book. See our commentary at 69.2 for some of the forms.

There was another way of writing numbers that was practiced mainly in the far western part of the Islamic world. The earliest extant mention of *rūmī* ("Roman") signs is a brief description in the *Chapters on Indian Arithmetic* by al-Uqlīdisī, completed in Damascus in 952-53 CE (341H).[19] Other texts mentioning these numerals date from the latter twelfth century or later, and were written in al-Andalus and Morocco.[20] Ibn al-Bannāʾ wrote a book on calculation with these numerals, which were popular with public administrators. They were also called "Fez signs", after the Moroccan city. Like *abjad* numerals, this is a decimal and additive system, with 27 individual signs for 1, 2, 3,…, 9, 10, 20, 30,…, 90, 100, 200, 300,…, 900. The peculiar shapes of the numerals cannot be securely said to derive from any other known system. No examples are known in al-Andalus after the thirteenth century, but the system remained popular with accountants in Morocco until about a century ago. Ibn al-Bannāʾ and al-Hawārī mention *rūmī* signs in passing at 85.16 and 104.1.

The three main methods of calculation in medieval Islam were finger reckoning, sexagesimal arithmetic, and Indian arithmetic. Sexagesimal arithmetic and finger reckoning had already been in use in the Middle East prior to the rise of Islam. And although Indian

[18]There were minor differences between east and west in the Islamic world in the schemes used.

[19](al-Uqlīdisī 1978, 310); (al-Uqlīdisī 1984, 386). The signs are likely related in some way with special signs appearing on Roman coins minted in Siscia in the period 348-350 CE (Kent 1981, 343, 359, 364-67).

[20]Al-Ḥaṣṣār (d. before 1194), writing in the Maghreb, mentions them in his *Complete [Book] on the Art of Number* (Guergour 2000, 68). The signs, which varied between east and west, are also mentioned in the writings of Mozarabs (Christians writing in Arabic) in Toledo around the same time (Colin 1933, 204).

numerals are attested in Syria by the mid-seventh century CE, it seems that rules for calculating with them became known to Muslims about a century later through direct contact with Indian scholars, at the start of the translation movement. The three systems remained popular throughout the medieval period, and were often described together in the same book.

Finger reckoning was a method popular among merchants, government secretaries, and surveyors. Calculations were preformed mentally in base ten, with intermediate results "stored" by positioning the fingers in particular ways. It was called *ḥisāb al-yadd* (hand arithmetic), *al-ḥisāb al-hawāʾī* (aerial calculation), *al-ḥisāb al-maftūḥ* (open calculation), or sometimes *ḥisāb al-rūm wa l-ʿarab* (calculation of the Romans (i.e., Byzantines) and Arabs). In one of many examples, Abū l-Wafāʾ (tenth century) explains how to multiply 46 by 28:

> We begin by multiplying forty by twenty, to get eight hundred, and by eight to get three hundred twenty. Hold them. Then we multiply the six by twenty to get one hundred twenty, and by eight to get forty-eight . Then we add them all to get one thousand, two hundred eighty-eight.[21]

The command "hold them" means "store the intermediate sum" 1,120 by positioning the fingers. Then, after the next sum 168 is calculated, the two can be added to get the answer. When the result of such a calculation was recorded on paper, *jummal* notation was used. Finger reckoning and *jummal* numerals played roles similar to the abacus (counters manipulated on tabletops) and Roman numerals in the West: the first was for calculating and the second for recording the numbers.[22]

Most Arabic arithmetic books covering mental calculation do not address the actual positioning of the fingers. One that does is ʿAlī ibn al-Maghribī's fifteenth-century *Poem on Reckoning with Finger-Joints*.[23] Units were stored by positioning the last three fingers on the right hand, tens with the right-hand thumb and index finger, hundreds with the left hand thumb and index finger, and thousands with the last three fingers on the left hand. The earliest illustration we know for the positioning of the fingers is found in an Italian textbook, the 1494 *Summa de Arithmetica* of Luca Pacioli, shown below.[24] It differs from ʿAlī ibn al-Maghribī's system mainly by switching the hundreds and thousands, and by switching the left and right hands.

We know from literary references that finger reckoning had been practiced by the ancient Greeks and Romans, and it was probably known in other parts of the ancient world as well.[25] The passage from Abū l-Wafāʾ translated above is from his *Book of What is Necessary for Scribes, Businessmen, and Others in the Science of Arithmetic*.[26] This book, composed in the period 961-976 CE, is the earliest extant Arabic book explaining the techniques of mental calculation. This and later books on the topic cover mainly multiplication, division, and ratios, including some computational shortcuts. They also typically

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[21] (Saidan 1971, 142.17).

[22] For the Roman and later European abacus, see (Pullan 1968).

[23] Saidan describes the poem and translates the relevant parts in (Saidan 1968). Pellat gives an edition and French translation of a similar poem from the previous century, by Abū ʿAbd Allāh Muḥammad ibn Aḥmad al-Mawṣilī al-Ḥanbalī (#768), in (Pellat 1977, 52-59).

[24] The illustration is in (Pacioli 1494, fol. 36b). We reproduce a scan from the 1523 printing from the Columbia University Library copy. This illustration is identical to that in the 1494 printing.

[25] (Smith 1958, vol. II, 196ff); (Pellat 1977).

[26] Edited in (Saidan 1971). See (Saidan 1974) for a description of the contents.

Figure 6: Finger positions in Luca Pacioli's 1494 *Summa de Arithmetica.*

have chapters on mercantile and other practical problems in which the rules are applied. The more common methods for solving these problems are the rule of three, single false position, double false position, and algebra. Abū l-Wafāʾ's book, like many others on finger reckoning, also gives rules for base sixty calculations. Ibn al-Bannāʾ covers several rules for mental multiplication, most copied from Ibn al-Yāsamīn's late twelfth century *Grafting of Opinions*, beginning at 108.9; and his rules on division from 119.18 through 124.19 are also from finger reckoning.

Sexagesimal arithmetic is calculation in base sixty. This place-value system is like our base ten system, with sixty taking the place of ten. Where, for example, our 9062 is $2+6\cdot10+0\cdot10^2+9\cdot10^3$, the sexagesimal number 12,0,45,1 is $1+45\cdot60+0\cdot60^2+12\cdot60^3$ or, in base ten, 2,594,701. Sexagesimal arithmetic dates back to late third millennium BC Mesopotamia, where the numbers were recorded in cuneiform on clay tablets. It became the standard system for Babylonian astronomers, and later it was adopted by astronomers writing in Greek, Arabic, Sanskrit, and other languages. It is because of sexagesimal arithmetic that today we still have sixty seconds in a minute and sixty minutes in an hour. In Arabic the numbers 1 through 59 were written in *jummal* form so, reversing our 12,0,45,1, the number would have looked like يبōمه ا. The "ō" is the sexagesimal zero, or empty place. Sexagesimal arithmetic is not covered in al-Hawārī's book.

Indian arithmetic is calculation in our base-ten place-value system using the nine figures 1, 2, …, 9 and the zero (0). This system originated in India probably around the first century CE, though the earliest Indian text showing the numerals dates from the late sixth century. The numerals had reached Syria by 662 CE when the bishop Severus Sebokht mentioned them, and they were known to the Muslims by 760 CE.

The earliest known Arabic treatise on Indian numerals is the *Book on Indian Calculation*,[27] written by Muḥammad ibn Mūsā al-Khwārazmī in Baghdad in the first half of the ninth century. Both the Arabic original of this book and a Latin translation made in the twelfth century are lost, but we do have a reworking of the Latin translation. It appears that al-Khwārazmī's original treatise described how numbers are formed by the nine figures, and then covered in order the operations of addition, subtraction, doubling, halving, multiplication, division, all for whole numbers; then operations on fractions were discussed, and finally square root extraction. This book is already a hybrid, since the fractions are taken from finger reckoning and sexagesimal arithmetic.

Al-Khwārazmī's rules are intended to be worked out on a dust board or some other ephemeral surface that allows for the easy erasing and shifting of digits. A dust board is a flat board covered in dust or fine sand on which one wrote with a finger or a stylus. It was a medium for working through calculations much like today's chalkboard or whiteboard. It is because of this association with the dust board that "Indian arithmetic" (*al-ḥisāb al-hindī*) was also called "board arithmetic" (*ḥisāb al-takht*) or "dust arithmetic" (*ḥisāb al-ghubār*). Another erasable writing surface was the wax tablet, with wax instead of sand covering the board.[28]

The earliest extant Arabic book on Indian reckoning is al-Uqlīdisī's mid-tenth century *Chapters on Indian Arithmetic*, mentioned above.[29] This book covers much the same material as al-Khwārazmī's, but with some innovations. It is the earliest we know that explains decimal fractions, like 4.75 instead of $4\frac{3}{4}$. Al-Uqlīdisī also gave rules for working out the computations with pen and paper, without any erasing. He famously explains that one advantage of switching to ink and paper is because the dust board is associated with "the misbehaved who earn their living by astrology on the streets".[30]

Al-Uqlīdisī's innovations did not catch on quickly. The dust board remained popular for centuries, only disappearing in some parts of the Muslim world at the beginning of the 1900s. Most of Ibn al-Bannāʾ's rules, in fact, are intended for the dust board or wax tablet, since they call for the erasing of digits. He does give three rules for multiplication that require no erasing, at 101.16, 103.14, and 106.11. These rules will work for pen and paper, but they may have been intended for the *lawḥa* (board; tablet). This was a board covered in soft clay on which one wrote with a cane stick dipped in ink. Because the *lawḥa* is reusable it would have been more economical than paper. Decimal fractions did not catch on quickly, either. Ibn al-Bannāʾ does not describe them at all, but instead works with the common forms of fractions from finger-reckoning.

[27] *Kitāb al-ḥisāb al-hindī*. Following (Saidan 1987, 44), we attribute this and some other books listed in Ibn al-Nadīm's *Fihrist* under Sanad ibn ʿAlī (#48) to al-Khwārazmī. This is our source for the Arabic title of the book.

[28] For more on the dust-board, and on the transmission of Arabic numerals generally, see (Kunitzsch 2003).

[29] The Arabic text is published in (al-Uqlīdisī 1984), and Saidan's English translation is published in (al-Uqlīdisī 1978).

[30] Saidan's translation, from (al-Uqlīdisī 1978, 247).

1.4 Greek mathematics and Arabic arithmetic

1.4.1 Greek and Arabic concepts of number

Some Arabic arithmeticians gave their subject the semblance of a rigorous foundation by adding definitions from Greek number theory. This might have been a good way to merge different approaches to arithmetic were it not for the incongruity of their number concepts. The numbers of the Arabic arithmeticians (*ḥussāb*) include any positive quantity that arises in calculation, including fractions and irrational roots. But numbers in Euclid's *Elements* Books VII-IX and Nicomachus's *Arithmetical Introduction*, the two main Greek sources, are restricted to positive integers.[31]

Aristotle's works provided the main philosophical backdrop to this particular Greek concept of number, at least as it circulated in medieval Islam. In his *Categories*, Aristotle divided the genus of "quantity" into discrete and continuous: "Of quantities some are discrete, others continuous…Discrete are number and language; continuous are lines, surfaces, bodies, and also, besides these, time and place".[32] Geometric magnitudes are continuous because they are infinitely divisible. Any line, surface, or body can be divided into as many parts as one wishes. By contrast, numbers are discrete because the unit (1) is necessarily indivisible. How can it be a unified whole if it can be split into parts? Euclid was repeating the accepted definition when he wrote "[a] number is a multitude composed of units".[33] There are two important consequences of these characterizations of the unit and number. First, "one" cannot be regarded as a number because it is not a multitude. One is instead the origin or cause of number, and the sequence of numbers begins with two. Second, there can be no fractions due to the indivisibility of the unit, so the only true numbers are integers.[34]

Although the relevant Greek works had been translated into Arabic by the end of the first quarter of the ninth century, the Greek notions of unit and number took some time to make their way into practical Arabic arithmetic books. The earliest books we consulted give no definition of number, and instead begin with instructions on how to perform operations.[35] Philosophical treatments of arithmetic could not ignore the issue, however. Two early books whose chapters on arithmetic depend heavily on Nicomachus are the *Treatises of the Brethren of Purity* (10th c.) and the *Book of Instruction of the Elements of the Art of Astrology* (11th c.) by al-Bīrūnī. These two books get around the problem of the incompatibility of Greek and Arabic numbers by asserting that although the unit is properly speaking indivisible, in practice we work with a divisible quantity that we call a "unit".[36] Some later authors cited both views without trying to reconcile them, like the Persian al-Fārisī (died 1319) in his *Foundation of Rules on Elements of Benefits*. Al-Fārisī first

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[31] (Djebbar 2004). There was no single Greek concept of number. Diophantus's *Arithmetica* (ca. fourth century CE) shows many fractions, and the works of Hero of Alexandria (first century CE) give many rational approximations to irrational numbers. Also, because number concepts, whether in Greek or Arabic, were based in counting or measuring, it would not have occurred to ancient or medieval mathematicians that negative or complex numbers could exist.

[32] *Categories* 4b20-24, translated in (Aristotle 1963, 12). Language is discrete because it consists of indivisible syllables.

[33] (Euclid 1956, vol. 2, 277).

[34] For these ideas in Aristotle, see (Cleary 1995, 345ff).

[35] We checked the arithmetic books of al-Khwārazmī (Latin redaction), al-Uqlīdisī, Abū l-Wafāʾ, Kūshyār ibn Labbān, al-Karajī, al-Baghdādī, Ibn al-Samḥ, and Ibn al-Haytham.

[36] The Brethren of Purity write "The 'one' is said in two ways: in its proper usage, and by way of metaphor. In its proper usage it is a thing that cannot be partitioned or divided…As for 'one' in metaphor, it is every aggregate that is considered a unity. So, for example, ten is called a 'unit', and a hundred is called a 'unit',

reviews Aristotle's classification of continuous and discrete quantity before he presents the Greek definitions of the unit and of number. He then cites the definition of the practical arithmeticians (*ḥussāb*) of his time: "Number is a quantity you obtain from one by repetition or partition or both, and it is clear by this meaning that the type is divided into whole numbers and their fractions".[37]

Evidence of Greek influence on practical arithmetic in Western Islam appears in the earliest extant texts from the late 1100s. Al-Ḥaṣṣār, who probably worked in Salé in Morocco, added to his descriptions of the digits that one "is the origin of number" and that two "is the first number", as did his contemporary Ibn al-Yāsamīn.[38] About a century later, Ibn al-Bannāʾ became the first arithmetician we know of to give Euclid's definition of number in a book on practical calculation. He begins his *Essays on Arithmetic* with

> Number is a multitude composed of units…Its origin emerges from the arithmetical one. And the arithmetical one is by its essence not a number, since it is the cause and number is the effect.[39]

Ibn al-Bannāʾ took a different approach in the *Condensed Book* by condensing the definitions of Euclid *and* the arithmeticians in one short statement: "A number is a collection of units, and it is divided according to how it is produced into two kinds: whole and fraction". (See our commentary at 65.2.) The friction between the Greek and Arabic number concepts also comes into play in Ibn al-Bannāʾ's definitions of multiplication (95.2), division (117.2), and fractions (133.1).

The incompatibility problem was solved by the Persian mathematician and poet ʿUmar al-Khayyām (Omar Khayyam), though his work does not seem to have been noticed by arithmeticians. Where geometric magnitudes in Euclid and other early Greek geometers do not possess any numerical measure, al-Khayyām worked with arithmetized lines, planar regions, and bodies, and he identified the numbers of the arithmeticians with the dimensionless measures of such continuous magnitudes.[40] This way a number like $\sqrt{10}$ can be the length of a line, the area of a planar figure, or the volume of a body. Consequently there are two different units, the indivisible arithmetical unit and the divisible geometric unit.

1.4.2 Greek number theory and geometry in Arabic arithmetic

Books VII to IX of Euclid's *Elements* and Nicomachus's *Arithmetical Introduction* describe theoretical arithmetic for positive integers, or what we would call elementary number theory. Topics include the classification of numbers into even, odd, and their subspecies; prime numbers and divisibility; amicable and perfect numbers; ratio and proportion; figurate numbers; and series summation.

and a thousand is called a 'unit'" (translated in (El-Bizri 2012, 67). See also (Goldstein 1964, 136)). Al-Bīrūnī writes "Although 'one' is in reality indivisible, nevertheless the unit, one as a technical expression, employed in dealing with sense-objects, whether by weighing, measuring by bulk, or length or number, or merely in thought, is obviously capable of sub-division" (translated in (al-Bīrūnī 1934, 24)). Al-Baghdādī tacks onto his *Completion of Arithmetic* a chapter on number theory based in Nicomachus (Chapter 6), but he does not address the nature of the unit.

[37](al-Fārisī 1994, 71.8). Sibṭ al-Māridīnī (died 1506), working in Egypt, similarly contrasted Euclid's definition with that of the "Arithmeticians" (Sibṭ al-Māridīnī 2004, 73.5). See also (Djebbar 2004, 315-318).

[38](al-Ḥaṣṣār manuscript, fol. 3b); (Zemouli n.d., 5).

[39](Ibn al-Bannāʾ 1984, 121.4).

[40](Oaks 2011a, 59ff).

Ibn al-Bannāʾ's *Condensed Book* covers the classification of even numbers (65.10) and the sieve of Eratosthenes (127.10) for finding prime numbers. Both are taken directly or indirectly from the Arabic translation of Nicomachus, as is the classification of different kinds of proportion that al-Hawārī copied from *Lifting the Veil* (195.2). Several definitions from Book VII of the *Elements* found their way into Ibn al-Bannāʾ's and al-Hawārī's books, certainly through some intermediary Arabic source. Besides definition VII.2 for "number" at 65.2, these include definition VII.16 for "side" and "surface" at 66.17 and 67.3, VII.3-4 for "part" and "parts" at 133.1, and also possibly VII.10 for "oddly odd" at 66.7 and VII.15 for "multiplication" at 95.2. Like the words we translate as "square" and "cube", "side" (*ḍilʿ*) and "surface" (*saṭḥ*) are arithmetical terms borrowed from geometry.

Rules for summing finite series were known in both India and Greece before the advent of Islam, and probably in other parts of the old world as well.[41] The rules we read in Arabic books ultimately derive from Greek sources, with some innovations introduced by the Arabic authors.[42] Ibn al-Bannāʾ likely took his rules (starting at 79.13) from another Arabic source.

It is largely because numbers could take only integer values that geometric magnitudes in Euclid, Apollonius, and Archimedes are without numerical measure. For the Pythagorean Theorem, for example, Euclid drew literal squares on the three sides of a right triangle and showed that the square on the hypotenuse is the same size as the other two together.[43] While working in the Greek tradition, Arabic mathematicians properly kept numbers out of geometry. One example is Thābit ibn Qurra's treatise proving that the volume of a section of a paraboloid is half the volume of the enclosing cylinder.[44]

By contrast, Arabic practitioners held a concept of number deriving as much from measurement as from counting, allowing them to work freely with fractions and irrationals. By routinely assigning numerical values to geometric magnitudes in surveying work and in architecture, they were able to foster the intimate connection between arithmetical and geometrical calculation. So when Euclid's *Elements* became available in Arabic translation in the late eighth century, mathematicians writing in a practical vein did not hesitate to reinterpret propositions from the geometric books in terms of arithmetic.[45] Thābit ibn Qurra, for example, gave a numerical reading to *Elements* propositions II.5 and II.6 in his proofs for the rules to solve three-term algebraic equations.[46]

Euclid's treatment of quadratic irrational lines in Book X was particularly amenable to numerical interpretation. One treatise from the ninth century, probably written by al-Māhānī, gives numerical calculations of the square roots of binomials and apotomes.[47] For example, a line in Euclid that is divided into two incommensurable parts satisfying a certain condition could now be identified with the number $5+\sqrt{45}$. Ibn al-Bannāʾ briefly presents the arithmetical version of the theory of quadratic irrationals in his *Condensed Book*, and al-Hawārī gives it a thorough review with the help of some passages from *Lifting the Veil* (starting at 173.4). Al-Hawārī also copied from *Lifting the Veil* the manipulations

[41] Two Indian books containing such rules are the *Āryabhaṭīya* of Āryabhaṭa (ca. 500 CE) and the *Pāṭī-gaṇita* of Śrīdhara (8th or 9th c.) (Plofker 2009, 131-132); (Sridhara Acarya 1959).
[42] (Saidan 1996, 341); (Djebbar 2004, 310).
[43] Proposition I.47.
[44] (Rashed 1996, 319ff).
[45] Even Greek geometers in late antiquity had begun to assign numbers to magnitudes.
[46] (al-Khwārizmī 2009, 34-42). We deliberately cited the same author in this paragraph and the last. For Thābit ibn Qurra, a proposition in the tradition of Greek geometry should be free of any arithmetizations, but for algebra this rigor is relaxed. It is the setting, not the mathematician, that determined which approach is appropriate.
[47] (Ben Miled 2005).

of geometric proportion that Ibn al-Bannāʾ took, once again directly or indirectly, from definitions 12-15 in Book V of an Arabic translation of Euclid's *Elements* (196.16).

1.5 Arithmetical problem solving

Part II of the *Condensed Book* covers arithmetical problem-solving techniques that were originally associated with finger reckoning. Here, instead of solving problems that ask for the result of a calculation on given numbers, Ibn al-Bannāʾ explains methods for finding unknown numbers. For instance, instead of working out a calculation like "divide thirty-five by fifteen" (at 120.16), al-Hawārī poses the problem "a quantity: taking away its third and its fourth leaves ten. How much is the quantity?" (at 199.1).

Today, a problem asking for an unknown number is typically viewed as an algebra problem. We would solve the problem just mentioned by naming the quantity x, and then setting up and solving the equation $x - \frac{1}{3}x - \frac{1}{4}x = 10$. But problems like this were regarded by medieval mathematicians as belonging to arithmetic, since algebra was just one of several methods available to solve them. In fact, al-Hawārī works out this particular problem not by algebra, but by double false position (described below). The independence of problems from the methods of solution is most evident in other books that show the same problem solved by two or three different methods. One example is a problem of al-Ḥaṣṣār translated in Appendix B, problem [7].

Here is a brief list of the problem solving methods covered by Ibn al-Bannāʾ. These are the most common among the methods described in Arabic arithmetic books:

- The method Arabic authors called "the four proportional numbers" is known to us also as "the rule of three". Given a proportion $a : b :: c : d$, this method shows how to find one of the values given the other three. It is explained starting at 195.9. Al-Hawārī does not solve any problems by this method, so we translate one from Ibn al-Bannāʾ's *Essays on Arithmetic* in Appendix B, problem [1].

  Single false position is a method that directly applies the rule of three. One posits a convenient, but probably incorrect, value for the solution to a problem, then the answer is calculated from the value and the error via proportion. This method is not taught in Ibn al-Bannāʾ's *Condensed Book*, but it was very common in Arabic arithmetic. Problems [2] and [7] translated in Appendix B are solved by single false position. The first of these is from Ibn al-Bannāʾ's *Essays on Arithmetic*.

- Double false position. Two (false) values are posited for the solution to a problem, and the correct answer is found from the values and their respective errors. This method is also based on proportion, and is explained beginning at 198.2 with examples at 199.1, 200.1, 201.1, 202.3, 203.1, 205.5, 207.6, and Appendix B, problem [7].

- Algebra. In Arabic algebra the powers of the unknown are assigned particular names corresponding to our x, x^2, x^3, etc. To solve a problem an unknown is assigned one of these names, or sometimes a combination of them. The conditions of the enunciation are worked out to set up an equation, which is then simplified and solved. The basic rules of algebra are explained starting at 209.1, though no sample problems are provided. Examples worked out by algebra from other books are given in problems [3] through [7] in Appendix B.

The rule of three was practiced across the Old World in antiquity, from Europe to China. As mentioned, it is the basis for the method of single false position, which dates back at least to a problem solved in the Rhind Papyrus in Egypt, ca. 1650 BC. A version of single false position is also attested to in cuneiform tablets from the end of the Old Babylonian period (that is, around 1700-1600 BC), and it appears in Sanskrit sources before the advent of Islam.[48] This method was probably as widespread in antiquity as the rule of three. It was practiced and taught among merchants and others over the course of many centuries and over a vast territory before we encounter it in medieval Arabic books.

Double false position also circulated among people who solve problems on the job. The only evidence we have of its use before the ninth century comes from a few problems in a single ancient Chinese text, the *Nine Chapters on Mathematical Procedures*, written some two thousand years ago.[49] Contrary to some reports, double false position is absent from Hero of Alexandria's approximation of $\sqrt[3]{100}$ (ca. 100 CE), nor has it been found in Babylonian or Sanskrit sources.[50] This does not mean that it was *not* practiced in those places. We just have no positive evidence that it was.

Because of confusion in many modern accounts, algebra will require a bit more explanation. Algebra in medieval Arabic was called *al-jabr wa l-muqābala*, literally "restoration and confrontation".[51] The phrase was often shortened to just *al-jabr*, and transliterations into Latin and Italian eventually led to our word "algebra". But algebra then was quite different from algebra now. Many of the various meanings we impart to the word "algebra" today are due to modern developments and to the disappearance of other problem-solving methods. Today, people often apply the words "algebra" and "algebraic" to any kind of formal, abstract reasoning, or to any technique of finding unknown numbers. Two current definitions of "algebra" in the Oxford English dictionary support the former view:

> 2a. As a mass noun: (originally) the branch of mathematics in which letters are used to represent numbers in formulae and equations; (in later use more widely) that in which symbols are used to represent quantities, relations, operations, and other concepts, and operations may be applied only a finite number of times.
>
> 3. In extended use and *fig*. Something, esp. a system or process, that resembles algebra in substituting one thing for another, or in using symbols, signs, etc., to represent ideas and concepts.[52]

Neither of these characterizations apply to Arabic algebra. Letters and other symbols are only incidentally employed in Maghrebi texts (see below), and in any case they represent *kinds* of numbers and not numbers themselves.[53] Further, there is no abstract reasoning that would take the art beyond arithmetic. These modern views of algebra have muddled the attempts of some historians to properly identify just what *al-jabr wa l-muqābala* is, and to distinguish it from other techniques and modes of reasoning in Arabic and other premodern mathematics.

By "Arabic algebra" we mean the art of *al-jabr wa l-muqābala* as it was understood by those who practiced it. This was a specific technique of numerical problem-solving,

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[48] For Egypt: (Imhausen 2003, 37, 51); (Gillings 1972, Chapter 14). For Babylonia: (Høyrup 2002, 59-60, 102, 311-313). For Sanskrit: (Hayashi 1995, 396-399).
[49] (Chemla 1997).
[50] (Høyrup 2002, 103); (Plofker 2002, 182-183); (Plofker 2009, 259).
[51] The uses of these two terms in algebra are described briefly at 211.1, and through examples in the section beginning at 223.1.
[52] http://www.oed.com/, accessed August 26, 2018.
[53] See our commentary at 229.8; for a more thorough explanation, see (Oaks 2017).

with it own vocabulary and rules. As Ibn al-Bannāʾ and al-Hawārī explain, the powers of the unknown are given names. The first power of the unknown is called a "thing" or a "root", akin to our x. The second power is called a "*māl*", an Arabic word which ordinarily means "sum of money" or "wealth", and which corresponds to our x^2.[54] The cube of the thing, our x^3, is called a "cube", and higher powers are expressed by combinations of *māl* and cube, such as *māl māl* for the fourth power and *māl* cube for the fifth.

In most problems solved by *al-jabr wa l-muqābala* an unknown number is named "a thing", though sometimes it is named as one of the other powers or some combination of them. The conditions of the problem are then applied to set up a polynomial equation expressed in terms of the names of the powers, and the equation is then simplified and solved. One important distinction between *al-jabr* and other methods like single and double false position is that, in the latter, operations are performed only on known numbers, while in algebra calculations can be performed on the names of the powers. In other words, in algebra one operates on the unknown.

The earliest known Arabic books on algebra are the *Book of Algebra* by al-Khwārazmī and the book of the same title by Ibn Turk.[55] Both were written in Baghdad during the reign of the caliph al-Maʾmūn, 813-833 CE. Some historians have thought that because no Arabic book on this topic is known before al-Khwārazmī's that he must have invented algebra. This view does not take into account the oral traditions of calculation before the start of the translation movement. In fact, evidence of oral transmission is manifest even in al-Khwārazmī's book.[56] Analogously, the earliest known work on double false position is the *Book on Proof of the Method of Double False Position* by Qusṭā ibn Lūqā (ca. 820-ca. 912/3).[57] No one would claim that Qusṭā *invented* double false position. He was merely among the first we know to write down a book describing the method.

With a clear idea that *al-jabr* must predate the time of al-Khwārazmī, historians have looked for the technique in the mathematics of previous civilizations. The most prominent candidates are (a) the *Brāhma-spuṭa-siddhānta* of Brahmagupta and other works from India (seventh century CE and later), (b) the *Arithmetica* of Diophantus of Alexandria (ca. early fourth century CE), and (c) ancient Babylonian cuneiform tablets (mainly ca. 2000-1600 BC). The method practiced in India is certainly a kind of algebra – names are given to unknowns and equations are formed and solved. But because the vocabulary is of an entirely different nature and the technique is more advanced in some respects than what we find in Arabic algebra, we can exclude direct influence of this sophisticated Indian algebra on the latter.[58] What is called Babylonian algebra is likewise distant from Arabic algebra. There the known and unknown quantities are represented by lines and rectangles and the manipulations take place in the context of a diagram. Although operations can be performed on the unknown, there is no simplification to a canonical set of equations, so

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[54] There is no good English translation of *māl*, so we leave it transliterated.

[55] Rashed's edition of al-Khwārazmī's book, with an English translation, is published in (al-Khwārizmī 2009). Only a portion of Ibn Turk's book survives. It is edited with an English translation in (Sayili 1962). In his tenth-century *Kitāb al-Fihrist*, Ibn al-Nadīm reports a book on algebra by Sanad ibn ʿAlī (#48), a contemporary of al-Khwārazmī and Ibn Turk. This attribution is probably an error, and the book belongs instead to al-Khwārazmī. See (Saidan 1987, 439-440).

[56] For evidence of oral transmission, see (Oaks 2012b); (Saidan 1974, 369); (King 1988).

[57] *Kitāb al-burhān ʿalā ʿamal ḥisāb al-khaṭaʾān*. Suter published a German translation in (Suter 1908–1909). We have also consulted the British Library, Oxford, and Cairo manuscripts. Abū Kāmil (late ninth century) is also reported to have written a book on double false position about the same time, but it is not extant.

[58] Léon Rodet made a detailed analysis of these differences in (Rodet 1878).

each problem requires its own trick to solve.[59] Babylonian algebra, then, seems to be a precursor to Arabic algebra.

Diophantus's way of solving problems, on the other hand, matches Arabic algebra in its vocabulary, overall structure, and concepts. More specifically, they agree in their systems of naming the powers of the unknown, in their basic procedures of setting up, simplifying, and solving equations, and in their conceptions of monomials, polynomials and equations.[60] Diophantus wrote in Alexandria in late antiquity. His book was translated into Arabic in the latter ninth century, after the time of al-Khwārazmī.[61] Arabic authors called the *Arithmetica* a book on *al-jabr wa l-muqābala* or *al-jabr*,[62] and many of Diophantus's problems found their way into Arabic algebra books.

We also know from two passages in Ibn al-Nadīm's *Fihrist* that the Greek astronomer Hipparchus of Bithynia wrote a book on algebra in the second century BC that was also translated into Arabic.[63] This book is unfortunately lost in both languages. Neither Hipparchus nor Diophantus can be said to have invented algebra, however. Like al-Khwārazmī some centuries later, both most likely composed books inspired by the algebra of practitioners that was already circulating orally in their times. The identity and even the nationality of the person or people who first practiced this premodern algebra are lost to history. There is certainly some historical link between Greek and Arabic algebra, either through texts, through oral practice among those who performed the calculations in their work, or both.

1.6 Education, books, and notation

One peculiar feature of medieval mathematics books, at least to modern readers, is their general lack of notation. Ibn al-Bannāʾ wrote all numbers in his *Condensed Book* in words, even if he was teaching the use of Indian numerals! And al-Hawārī, whose purpose was to give numerical examples, still writes the numbers and operations in words before showing the notation. This is perhaps the most glaring of the many differences between medieval and modern arithmetic books. The causes of these differences have to do largely with how the material was taught, and more generally with how people in al-Hawārī's time related to books.

1.6.1 Education in medieval Islam

Where today education is structured around institutions, curricula, and degrees, in medieval Islamicate societies it was structured around individual teachers, books, and licenses (*ijāzāt*).[64] Schools did exist and provided support for both teachers and students, and from what we can gather they often had curricula; but throughout the medieval period education remained centered on the student-teacher relationship. To learn a topic, a student chose a teacher (*shaykh*) with a good reputation to study a particular book. Once the stu-

[59] (Høyrup 2002).
[60] (Christianidis and Oaks 2013).
[61] The four surviving books of the Arabic translation have been edited and translated twice, in (Sesiano 1982) and (Diophantus 1984).
[62] (Sesiano 1982, 8-13).
[63] (Ibn al-Nadīm 1871–1872, 269, 283); (Ibn al-Nadīm 1970, 642, 668). Both references mention Abū l-Wafāʾ's commentary on Hipparchus's "book on the art of algebra (*al-jabr*)". In addition, Abū l-Wafāʾ himself mentions this commentary in his *Book of What is Necessary for Scribes, Businessmen, and Others in the Science of Arithmetic* (Saidan 1971, 126.7).
[64] For Islamic education, see (Makdisi 1981), (Berkey 1992), and (Chamberlain 1994).

dent had learned the book to the satisfaction of the *shaykh*, he (or rarely, she) received an *ijāza*, or license, granting him authority over the text. The *ijāza* not only certified mastery of the contents of the book, it gave its possessor authorization to teach it himself. The *ijāza* recorded the names in the chain of transmission of that particular text, listing the student, his teacher, his teacher's teacher, ideally all the way back to the author of the book.

Instruction was centered around dictation and memorization. The *shaykh* read the book to the student, who memorized it and dictated it back to the *shaykh*. This emphasis on speaking and listening reflects the notion that books properly reside in the minds of those who had memorized them, and not in manuscript copies. Students recited aloud even when studying alone after the lecture, "for what the ear hears becomes firmly established in the heart".[65] Manuscripts played an essential role in this oral environment. Copying the words of a teacher on paper aided memorization, and written books gave the student access to the text when studying alone. But true learning was thought to take place in the presence of the teacher, who possessed an understanding of the book that manuscripts lacked.

Instruction was not restricted to dictation of the book under study. The *shaykh* would often supplement it with examples, illustrative remarks, and material from other sources. Al-Hawārī cites additional procedures for double false position that Ibn al-Bannāʾ dictated to him at 204.3, and he signals other added material at 163.11 and 232.9. Many other passages are attributed in the manuscripts to Ibn al-Bannāʾ that are not in the *Condensed Book*, and most likely they were dictated to al-Hawārī as well.[66] Whatever a textbook lacked in examples and explanations would have been covered somehow by the *shaykh* teaching it.

Many authors wrote their books concisely to make memorization easier, which is certainly the case for Ibn al-Bannāʾ's *Condensed Book*. Some authors even put down the mathematical ideas in verse, since poetry is easier to memorize than prose. Ibn al-Yāsamīn's famous *Poem on Algebra* covers the basic rules of algebraic problem-solving in just 54 lines,[67] and Ibn Ghāzī reduced the contents of Ibn al-Bannāʾ's *Condensed Book* to 461 verses in his *Desire of Reckoners*.[68] In the other direction, many authors filled their books with detailed explanations and many worked-out exercises, making them too long to have been memorized in their entirety. For example, Ibn al-Hāʾim's 1387 *Commentary on the Poem of al-Yāsamīn* amounts to an encyclopedia of algebraic knowledge that takes up 257 pages in the modern printed edition,[69] and Ibn Ghāzī himself expounded on his arithmetical poem to make a commentary that occupies 323 pages of the modern edition.[70] Al-Hawārī's commentary falls into this category, too.

1.6.2 The role of notation

Because of the oral nature of learning in medieval Islam, books read like transcriptions of lectures.[71] Notation serves no purpose to the student listening to a lecture, so there is no advantage in including it in the running text of a book. This is why Ibn al-Bannāʾ wrote his

[65] Abū l-Hilāl al-Ḥasan al-ʿAskarī, quoted in (Berkey 1992, 27).
[66] These passages are shown in SMALL CAPS in the translation.
[67] Published in (Abdeljaouad 2005a). Ibn al-Yāsamīn also wrote a 55-line poem on root extraction and an 8-line poem on double false position.
[68] Note also the short poems quoted by al-Hawārī describing the shapes of the Indian numerals at 69.2, the *abjad* numerals for casting out sevens at 88.14, and ʿAlī ibn al-Maghribī's *Poem on Reckoning with Finger-Joints* mentioned above.
[69] (Ibn al-Hāʾim 2003).
[70] The commentary is titled *Aim of the Students in Commentary on Desire of Reckoners* (Ibn Ghāzī 1983).
[71] (Chamberlain 1994); (Berkey 1992).

book entirely in words, and why the notation for the calculations in al-Hawārī's book only appear as figures or illustrations. These figures play a role much like geometric diagrams or other illustrations, and were intended to show the student what is to be put down on the dust-board or other surface. For example, he begins this calculation in the chapter on addition, at 75.9:

> Suppose we want to add nine hundred seventy-eight to four hundred fifty-six. We put them down on two lines, as mentioned, as in this figure: $\begin{matrix}978\\456\end{matrix}$.

The visual notation (here $\begin{matrix}978\\456\end{matrix}$) is always set apart from the audible text in al-Hawārī's book with some phrase like "and its figure is" or "so we write it down like this".

Some Arabic authors included the Indian notation in the spoken parts of their arithmetic books, but there the numbers were still meant to be recited aloud like the words that surround them. Naṣīr al-Dīn al-Ṭūsī, for example, begins this problem in his *Gathering of Arithmetic by Means of Board and Dust* (thirteenth century):[72]

> For example, we want to multiply 123 by 456. We write them in two lines like this: $\begin{matrix}4 & 1\\5 & 2\\6 & 3\end{matrix}$.

Here the 123 and 456 are still intended to be spoken, while the same numbers shown in columns form a figure that was to be apprehended visually, like any illustration.

There were thus two ways of writing arithmetic. To solve a problem, one wrote in notation on a dust-board or other temporary surface. Then, if one wished to communicate the result to others, a rhetorical version was composed. For this reason, the numerals tend to appear in arithmetic books only to show students what is to be put down on the board.

Because board calculations were for personal, immediate use, the notation can tolerate ambiguities. For example, the notation for excluded fractions of both connected and disconnected types were often represented the same way in notation. But this does not matter, because the person working out the calculation knows which is which while doing so (see our commentary at 140.14). Ambiguity would only be a problem if one wanted to consult the book later or have others read it. Dust does not permit the former, and for the latter a rhetorical version in unambiguous prose was composed.

1.6.3 Developments in Indian notation: fractions, roots, and algebra

Arabic practitioners devised notations to extend Indian numerals to show fractions and roots in arithmetic and polynomials and equations in algebra.[73] There are differences in these notations that can usually be ascribed to geography, mainly between east and west. The Persian mathematician al-Fārisī, a contemporary of Ibn al-Bannāʾ, shows fractions with the denominator over the numerator, and without the bar. His "three sevenths" is shown as $\begin{smallmatrix}7\\3\end{smallmatrix}$, for example.[74] By contrast, al-Kāshī, working in Samarkand in eastern Persia

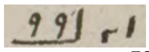
[72](Saidan 1967, 125). Some other books that show notation in the spoken parts of the text include al-Uqlīdisī's *Chapters on Indian Arithmetic* (952/3 CE), Ibn Munʿim's *Understanding Calculation* (12th-13th c.), and al-Fārisī's *Foundation of Rules on Elements of Benefits* (early 14th c.).

[73]For algebraic notation, see (Abdeljaouad 2005a) and (Oaks 2012a).

[74](al-Fārisī 1994, 199).

in 1427, put the numerator over the denominator. His “eight and four sevenths” is shown in notation as $\begin{smallmatrix}8\\4\\7\end{smallmatrix}$.[75] The earliest known book from the Maghreb showing fractions in notation is al-Ḥaṣṣār’s late twelfth century *Book of Demonstration and Recollection in the Art of Dust Board Reckoning*. There, fractions are shown just the way we write them today, with the numerator over the denominator separated by a horizontal line. Here, for example, is his $\frac{9}{11}$ from a manuscript copied in 1194 CE: .[76] Other books from the west, including al-Hawārī’s commentary, also show fractions this way.

Ibn al-Yāsamīn’s late twelfth century *Grafting of Opinions* is the earliest known book that indicates the square root by placing an elongated letter *jīm* (ج) above the affected numbers, much like the way we use our elastic sign “$\sqrt{\quad}$”. This notation became common in the Islamic West, though al-Hawārī does not show it. We have not (yet) seen a notation for roots in Arabic books written east of Egypt.

Initially, there seems to have been no notation specific to Arabic algebra. The little evidence we have indicates that polynomials were written on a board as a list of coefficients. In the early eleventh century, al-Karajī, working in Baghdad, wrote out in words the polynomial that we would express as $x^{12} + 4x^{11} + 10x^{10} + 20x^9 + 35x^8 + 56x^7 + 84x^6 + 104x^5 + 115x^4 + 116x^3 + 106x^2 + 84x + 49$, and then wrote “you put it down in this figure: 1 4 10 20 35 56 84 104 115 116 106 84 49”.[77] A century and a half later al-Samawʾal, working in Persia, shows polynomials in his *Dazzling [Book] on the Science of Calculation* the same way, but with the names of the powers written in words above the Indian numerals for each coefficient to keep straight which number goes with which power.[78] The large majority of problems solved by algebra do not require anything higher than the second power, so when working out a problem on a board there was little need to indicate the name of the power when working through the calculations. We have found one Western manuscript that shows an equation in the margin in Eastern notation: . We can transcribe this as “1 2 = 99”, and in modern notation we write it as $x^2+2x = 99$.[79]

Later, in the western part of the Islamic world, a notation came into common use in which the first letter of the name of the power was placed above the coefficient. Our earliest glimpses are two brief figures in Ibn al-Yāsamīn’s late twelfth century *Grafting of Opinions*.[80] Most authors working in the Maghreb or al-Andalus in the fourteenth and fifteenth centuries also show it, while some in the region, Ibn al-Bannāʾ and al-Hawārī included, do not. Like the Indian notation, it is only presented to instruct students on how to write it to perform calculations on a board. In almost every instance, we are treated to a single expression or equation in notation here and there during the course of a rhetorical explanation or solution to a problem. We are fortunate, then, that in one place Ibn Ghāzī shows an entire problem worked out using this algebraic notation in nearly a dozen lines in his 1483 *Aim of the Students*.[81] Incidentally, it is on that page in the Library of Congress manuscript that the Eastern equation shown above appears. The Western version is written like this: . Algebraic notation is neither mentioned nor shown in more

[75](al-Kāshī 1969, 89). Al-Baghdādī writes fractions the same way (al-Baghdādī 1985, 102ff).
[76](al-Ḥaṣṣār manuscript, fol. 43b).
[77](al-Karajī 1964, 52). The same polynomial is written similarly by al-Samawʾal (al-Samawʾal 1972, 67).
[78](al-Samawʾal 1972, 45ff). Here al-Samawʾal is performing operations on the coefficients in tables.
[79](Ibn Ghāzī manuscript, fol. 108b). It appears that the Eastern marginal annotator felt the need to clarify the final calculations shown in the manuscript, which were written sloppily in Western notation.
[80](Zemouli 1993, 137, 231).
[81](Ibn Ghāzī 1983, 302); (Ibn Ghāzī manuscript, fol. 108b); (Oaks 2012a, 62).

advanced books on algebra, like Ibn al-Hāʾim's *Commentary* mentioned above. Because it is better suited to represent medieval algebraic calculations than modern notation, we describe the Arabic notation in our commentary beginning at 211.13 and further at 219.1, and we use it to explain al-Hawārī's calculations.

1.6.4 Nesselmann's stages revisited

Addressing Nesselmann will help some readers apprehend better the algebra in al-Hawārī's book, and indeed Arabic algebra in general. To make sense of the historical development of algebra, the German orientalist G.H.F. Nesselmann devised a three-stage scheme that he described in his 1842 book *Versuch Kritischen Geschichte der Algebra* (*Critical Essay on the History of Algebra*).[82] He dubs the most primitive stage "rhetorical algebra", in which all calculations are written verbally. Next is "syncopated algebra", which was still essentially rhetorical but with some recurring abbreviations. The third stage is "symbolic algebra", where calculations are represented in a language independent of oral presentation. Nesselmann identified Arabic algebra, quoting al-Khwārazmī in particular, as belonging to "rhetorical algebra", and Diophantus's *Arithmetica* as belonging to "syncopated algebra".

Although historians of mathematics in the past few decades have moved beyond Nesselmann's classification,[83] it is still frequently cited in popular accounts of the history of mathematics and in studies in mathematics education. Unfortunately, it is misleading as a scheme for describing historical developments in algebra or for identifying conceptual shifts. Nesselmann was unaware of board calculations when he classified Arabic algebra as "rhetorical". Although the algebraic calculations we find in the books of al-Khwārazmī, al-Khayyām, and others fall into his "rhetorical" category, the same work on a dust-board or wax tablet in the Arabic notation would be safely classified as "symbolic" according to his definition.[84] The mode of presentation is thus not an indication of a stage in algebraic development. Diophantus's instructions for abbreviating algebraic vocabulary read much like the instructions for writing algebra in notation in the later Arabic books. His abbreviations were probably used when working through the calculations on a wax tablet, so in this respect it is no different from the later Arabic practice. In manuscripts of the *Arithmetica* the abbreviations were evidently intended to be expanded and pronounced, like the numbers in Naṣīr al-Dīn al-Ṭūsī's arithmetic book quoted above. And finally, Nesselmann did not notice the difference between premodern and modern algebraic notations, where the letters or signs in the former designate types, and in the latter designate values.[85]

1.6.5 The figures in al-Hawārī's book

One consequence of the distinction between visual figure and audible text is that in European languages we read numbers expressed with Indian numerals backwards. Arabic is written right-to-left, so when reading a number like "214" in that language one starts from the 4 in the units place. When Europeans translated Arabic texts on arithmetic into Latin in the Middle Ages they preserved the orientation of figures. This is why "214" did not

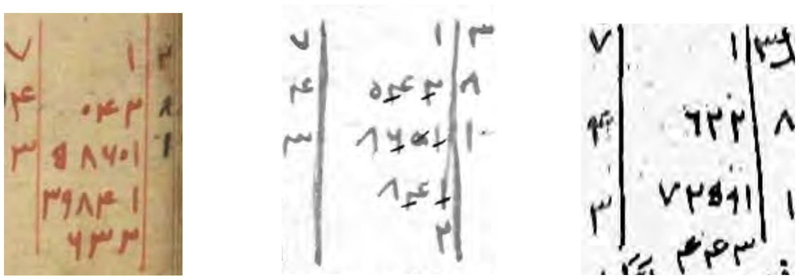
[82] (Nesselmann 1842, 301ff).
[83] Albrecht Heeffer, in particular, has recently criticized Nesselmann (Heeffer 2009).
[84] Ibn Ghāzī's page showing an entire problem worked out in notation is the one example we have of what must have been commonly written on a board. See (Oaks 2012a, 62).
[85] See (Oaks 2017) for an explanation of this difference. We also discuss it briefly in our commentary at 229.8.

become “412”. Because European languages are written left-to-right, we read the number 214 starting from 2 in the hundreds place.

Like the Latin translators, we have preserved the orientation of the Arabic figures in our translation. We could not reverse figures showing numbers horizontally, like $\begin{matrix}4043\\2685\end{matrix}$ (at 74.17), or else the “four thousand forty-three” would look like 3404. It would not make sense to reverse some figures and not others, so we kept the orientation of all figures as they appear in the manuscripts. Keep in mind when reading the translation that figures would have been viewed right to left.

Although copyists of al-Hawārī’s book transcribed his words more or less faithfully, the same cannot be said for the figures. For example, the figure we typeset for the passage after 103.3 is shown below, followed by the figures from the Tehran, Oxford, and Tunis manuscripts.

7					1	3
4			0	4	2	8
3	5	8	6	0	1	1
	3	9	8	4	1	
			6	3	2	



Translation Tehran fol. 23b Oxford fol. 126a Tunis fol. 10a

Figure 7: Comparing figures in the manuscripts

We deem our translation to be correct because it agrees with the verbal description of the operations in the text. Of the five manuscripts we consulted, only the Tehran manuscript shows this figure correctly. The Oxford and Medina manuscripts leave off the last round of calculations, and although not incorrect, the Oxford manuscript is the only one that crosses off numbers. Most of the numbers in the Tunis manuscript are wrong, and in the Istanbul manuscript (not shown here) the numbers between the lines are shifted up one row. All five manuscripts, including the Tehran manuscript, have errors in many figures, and it is rare to find a mistake that is common to all manuscripts. In our translation we show figures in the style found in the manuscripts which agree with the calculations described in the text.

1.7 The legacies of the *Condensed Book* and al-Hawārī’s commentary

The popularity of Ibn al-Bannāʾ’s *Condensed Book* is evident in the number of commentaries, poems, and abridgments it inspired. After his own *Lifting the Veil* and al-Hawārī’s *Essential Commentary*, we know of ten people who wrote commentaries on the *Condensed Book* between the fourteenth and sixteenth centuries.[86] In the other direction, Ibn al-Hāʾim

[86] Al-Ghurbī (M158, 2nd half 14th c.), al-Mawāḥidī (M176, ca. 1382), Ibn Zakariyyā (A336, #793, died 1403-4), Ibn Qunfudh (died 1407-8), al-ʿUqbānī (died 1408), Ibn Haydūr (M196, died 1413), Ibn Majdī (#815, died 1447), al-Ḥabbāk (M219, #831, died 1463), al-Qalaṣādī (died 1486), and Muḥammad al-Ghazzī (#998, 16th c.).

(ca. 1400) wrote an even more condensed version of Ibn al-Bannāʾ's little book, and four authors, including Ibn Ghāzī (15th c.), set the contents of the *Condensed Book* to verse.[87]

Although by no means as well known as his professor's *Condensed Book*, the number of surviving manuscripts of al-Hawārī's *Essential Commentary* is testimony to its popularity across several centuries. We have found four later works that reference or copy from al-Hawārī. ʿIzz al-Dīn al-Ḥanbalī (d. 1409) copied many numerical examples from al-Hawārī's book into his own commentary on Ibn al-Bannāʾ's *Condensed Book*, but without citing his source. Later, Ibn Ghāzī likewise copied many numerical examples into his *Aim of the Students* (1483). He mentions al-Hawārī by name twice, though not in connection with his copying. Sibṭ al-Māridīnī's (1423-1506) *Student's Guide to the Way of Arithmetic* paraphrases al-Hawārī's remarks on the ontology of ratios (at 133.3), and cites the source.[88] And while still a student in the 1740s in Istanbul, future mathematics teacher Şeker Zāde put together a notebook titled *Examples from Ibn al-Bannāʾ's Condensed [Book] and Ibn al-Hāʾim's Contents [of Calculation]*.[89] Şeker Zāde copied over a hundred numerical examples from al-Hawārī's commentary, always citing his source. This notebook is remarkable because the problems are all worked out in the Arabic notation, showing what would have been put down on the dust-board. And finally, a two-page extract from al-Hawārī's book, consisting of the subsection "on finding deaf parts" (127.9-128.16), appears in the middle of MS Berlin Landberg 199, on folios 31b-32a. This manuscript contains several works on arithmetic and algebra by other authors, including Ibn Fallūs (d. 1239) and al-Khwārazmī. Neither al-Hawārī's name nor the title of his book are given on those two pages.

We last hear of al-Hawārī in 1875, in connection with the proposed mathematics curriculum for the Zaytūna mosque school in Tunis. Until then, mathematics instruction at the mosque had been limited to five stylistic commentaries of a poem on inheritance and arithmetic. In an attempt to modernize instruction, Prime Minister Hayreddin Pasha oversaw the creation of an entirely new curriculum made public on December 25, 1875. Although a great improvement over the dismal curriculum in place at the time, the ten recommended texts on mathematics were themselves already hopelessly out of date. They included al-Hawārī's *Essential Commentary* in the upper division, even though it was already over 500 years old! The proposed curriculum received harsh resistance from the *ulama*, the religious authorities, and it was likely never implemented.

[87] The others are Ibn Marzūq (M205, died 1438), al-Wansharīsī (M257, died 1548-9), and Ibn al-Qāḍī (M315, died 1616).

[88] (Sibṭ al-Māridīnī 2004, 145).

[89] (Abdeljaouad 2011); (Abdeljaouad and Oaks 2013). Ibn al-Hāʾim's *Contents of Calculation* is his abridgment of the *Condensed Book* mentioned above. Incidentally, Şeker Zāde's figure for the passage after 103.3 is shown correctly.

Source

Contents of al-Hawārī's book

Pages for the translation and our commentary are given below, followed by the page and line number of the Arabic edition. The Arabic portion of this book receives a separate pagination. It is placed after the English part, or at the beginning if you are an Arabic reader. The chapter and section titles in this index are sometimes paraphrased from the Arabic.

Translation

Note: The Istanbul and Tunis manuscripts differentiate between passages attributed to Ibn al-Bannāʾ and those attributed to al-Hawārī, while the other three manuscripts we consulted do not make this distinction. In our translation, passages attributed to Ibn al-Bannāʾ that are taken from his *Condensed Book* are in **bold font**, while passages attributed to al-Hawārī are not in bold font. Passages borrowed from Ibn al-Bannāʾ's *Lifting the Veil*, often with minor changes in wording, are in italics: ***bold italics*** for those attributed to Ibn al-Bannāʾ, and *regular italics* for those attributed to al-Hawārī. Passages attributed to Ibn al-Bannāʾ that are neither in the *Condensed Book* nor in *Lifting the Veil* are in SMALL CAPS. Words and phrases added by us to b make the meaning clearer are placed in [square brackets]. Arabic words behind our translation are placed in parentheses, like "cube (*mukaʿʿab*)".

59.1 In the name of God, the Merciful and Compassionate, the humble and submissive servant to his Lord, hoping for His reward and His pardon for his sin, ʿAbd al-ʿAzīz ibn ʿAlī ibn Dawūd al-Hawārī al-Miṣrātī – may God forgive him – said: Praise be to God, the provider of graces and creator of life, who brings things into existence from nothingness, with praises the counting of which is uninterrupted. And full prayers on Muḥammad, His prophet and servant, and for His acceptance of the ancestors, who followed the revealed sunnas while not overstepping their boundary. And a prayer of supplication to our lord, the Commander of the Muslims, son of the Commander of the Muslims, Abū Yaʿqūb, deliverer of cherished victories from God. Then, may God prolong the term [in office] of our sovereign minister, the exalted, the radiant, the noble, the blessed, the honored, the jurisprudent, the learned, the all-knowing, the venerated of lofty virtue and superior intention, the superior, the proud, the issuer of laws guided by the light of the auspicious Marinid state, Abū Muḥammad ʿAbdallāh, son of our magistrate, the Shaykh Jurist, the virtuous, the pious, the ascetic, of excellent conduct and behavior, the late, sanctified, purest Ibn Madyan, who was made one of God's intercessors at the Gathering Place. May God perpetuate their memories among the pious, and protect the era, their return, and their place of honor from disgrace.

59.15 I begged almighty God to help me in my task of commenting on the book named *Condensed Operations in Arithmetic*, written by the incomparable scholar, our most eminent and open-minded master, Sheikh Abū l-ʿAbbās Aḥmad ibn Muḥammad ibn ʿUthmān al-Azdī, may God continue to bestow his divine blessing, care, and grace on him. I enhance my work with your name and embellish it with the splendor of your merits, so that I may deserve your blessing and your protection. To this end I asked our above-mentioned master, the scholar Abū l-ʿAbbās, for permission to undertake this work, to which request he kindly assented. Since the book I undertake to comment on has already been enriched by its own author, God bless him, with *Lifting the Veil*, containing all that is needed but providing only a few examples, I shall, almighty God willing, illustrate them with examples in the right place and when necessary. I call it the *Essential Commentary on the Condensed [Book] on the Operations of Arithmetic*. With God's help and assistance in all circumstances, it is time for me to begin the work, imploring the Almighty to guide and bless my endeavor.

61.1 **Our Master, the jurisprudent, the teacher, the leader, the learned, the radiant, the all-knowing, the guardian [of religion], Abū Aḥmad ibn Muḥammad ibn ʿUthmān al-Azdī, may God forgive him, said:**

61.3 **The goal of this book is to condense the operations of arithmetic, and to bring within reach its sections and its concepts, and to master its rules and its structures. It is comprised of two parts: the first is on operating with known numbers, and the second is on the basic rules through which the access to knowledge of the required unknown from the posited known is established, whenever a connection between them is provided. And I ask of God, praise be to Him, for assistance and success, and guidance toward the correct path.**

61.9 **Part One, on known numbers, which is divided into three chapters. The first is on working with whole numbers, the second on working with fractions, and the third on working with roots.**

63.1 **Chapter One, on whole numbers. For our purpose, this chapter is divided into six sections.**

65.1 **Section One, on the divisions of numbers and their ranks.**

65.2 **A number is a collection of units, and it is divided according to how it is produced into two kinds: whole and fractional.** Examples of whole numbers are fifteen, eighteen, and the like. Examples of fractions are a half, three eighths and half an eighth, a ninth and a fourth, six sevenths of seven eighths, and five sixths less a ninth.

65.6 **Whole numbers come in two varieties: even and odd.** An even number begins with a two or a four or a six or an eight, or does not begin with units, such as ten or fifty or the like. An odd number begins with a one or a three or a five or a seven or a nine.

65.10 **Even numbers come in three species: evenly-even, evenly-odd, and evenly-evenly-odd.** As for the evenly-even number, it is any number that can be halved, and each of its halves can be halved, until the halving reaches one. For example: thirty-two can be halved, and its half is sixteen. And each of the sixteens can be halved, and its half is eight. Half of the eight is four, half of the four is two, and half of the two is one, and the same for similar examples.

65.17 As for the evenly-odd number, it is any number that can first be halved and gives an odd number other than one. For example, fourteen can be halved, and its half is seven, which is an odd number other than one. The same goes for similar numbers. For this reason the [number] two is of the first species. So know it.

66.1 An evenly-evenly-odd number is any number that can be halved in such a way that each of its halves can be halved repeatedly until one reaches an odd number other than one. For example, twenty-eight can be halved, and its half is fourteen. Each of these fourteens can be halved, and its half is seven, which is an odd number, as we saw before. Since it can first be divided into evens, it may look like an evenly-even number, and since it reaches an odd number other than one, it may look like an evenly-odd number. So think it over.

66.7 **Odd numbers come in two species: prime and oddly-odd.** As for the odd prime number, it is any number that cannot be counted except by one, like eleven and twenty-nine and the like. They are also called deaf parts or simple, which will become clear in the work on the sieve.[1] As for the oddly-odd number, it is any number that can be counted by odd numbers, like fifteen, which is composed from the product of three by five. The same for similar examples.

66.13 I say we need to provide here an introduction in which we mention the names of the composite numbers according to their differences, and we will give examples of them. We say that the number, in relation to its composition, is either even or odd. Even numbers come in two varieties, either non-composite simple prime, which is uniquely two, or composite, which are of three kinds.

66.17 The composition of two equal numbers is called a square, or a number that has a root, and each of the two numbers is called a side or a root. For example, thirty-six is composed from six by six. The whole thirty-six is called a square or a number that has a root. Each of the sixes is called a side or a root.

67.3 The composition of two or more different numbers is called a surface, and each of these numbers is called a side. An example of the composition of two different numbers is eighteen, which is composed from three by six, or two by nine. The whole eighteen is called a surface. Each of the two and the nine, or the three and the six, is called a side. An example of the composition of three numbers is twenty-four, which is composed from three by four by two. So the whole twenty-four is called a surface, and each of the three and the four and the two is called a side. Likewise for more [numbers].

67.12 And the composition of three equal numbers is called a cube, and each of these numbers is a side and a cube root. For example, sixty-four is composed from the product of four by four by four. The whole sixty-four is called a cube, and each of these fours is called a side or a cube root. Likewise for similar situations. Some call the cube (*mukaʿʿab*) a cube root (*kaʿb*), which is a name for its side.

67.17 Odd numbers also come in two varieties: either simple, as we saw before, or composite, which are of three kinds, like the even numbers:

67.19 The composition of two equal numbers is called a square, or a number that has a root, and each of these numbers is a side or a root. For example, twenty-five is composed from five by five.

68.1 The composition of two or more different numbers is called a surface, and each of these numbers is a side. An example of the composition of two different numbers is thirty-five, composed from five by seven. An example of the composition of three different numbers is one hundred five, composed from three by five by seven. Likewise for more [numbers].

[1] The work on the sieve begins at 127.10.

68.8 The composition of three equal numbers is called a cube, and each of these numbers is called a side, or cube root. For example, twenty-seven is composed from three by three by three.

68.11 Subsection. The side and the cube root have come to have the same meaning, just as the root and the side also have the same meaning. They differ only as general and particular [substances] differ formally, but they are equal in usage. Taking a side of the cube[2] is a lengthy work of little benefit, which is why he[3] did not mention it, may God be satisfied with him. However, it can be found by means of decomposition, which is easy. You decompose the cube into the numbers of which it is composed. With them, you piece together three equal numbers by composition such that one of them is the required side. Study this with attention. You shall attain your aim, by God's will.

68.18 **As numbers increase indefinitely, they are placed in three ranks.** *They are called ranks because one of them follows another, and the units in each rank are greater than the units that came before them and smaller than the units that follow.*[4] **They are also called places,** BECAUSE THE NUMBER RESIDES IN THEM. THE PLACES OF EACH NUMBER REPEAT PERIODICALLY.

69.2 IN EACH RANK ARE NINE NUMBERS. THE FIRST RANK CONSISTS OF ONE TO NINE, AND IS CALLED THE UNITS RANK. Their figures are 1, 2, 3, 4, 5, 6, 7, 8, 9. Someone has written a poem about them:

> *Alif* and *ḥā* then *ḥajja* followed by *ʿuw*
> and after the *ʿuw* by an *ʿayn*. Draw
> *Hā* followed by a distinct figure
> looking like an anchor, and also you position
> Two zeros for eight with an *alif* [between them]
> and *wāw* is the ninth [digit]; so understand it.

69.9 The poet is right. He is also being clever when he states that eight is in the form of two zeros, since this helps us know the [figure for] zero.

69.10 **The second [rank] consists of ten to ninety, and is called the tens rank.** Their figures are 10, 20, 30, 40, 50, 60, 70, 80, 90. **And the third consists of one hundred to nine hundred, and is called the hundreds rank.** Their figures are 100, 200, 300, 400, 500, 600, 700, 800, 900.

69.15 **The names of numbers are formed from twelve simple names.**[5] **The first nine are those of the units, the tenth is for the tens, the eleventh is for the hundreds, and the twelfth is for the thousands, which is the units place**, in that it is the first of the three ranks, just as the units were first before its tens and its hundreds. And from there the cycle begins again.

70.2 And of the fourth rank, which is the thousands, we say: units, tens, hundreds, which all are thousands, and together they differ with the first three ranks only by the word "thousands".

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[2] I.e., calculating a cube root.
[3] Ibn al-Bannāʾ.
[4] Copied from (Ibn al-Bannāʾ 1994, 212.9-10).
[5] Literally, "Number has twelve simple names from which all of its names are formed".

Likewise for the three third ranks, which are thousands of thousands: they are also units and tens and hundreds, and differ with the preceding only by the word "thousand" twice. And likewise for the three fourth ranks, which [differ] with the preceding according to the previous description. And likewise for succeeding numbers. So know it.

70.8 For example,[6] five and twenty and two hundred and four and eighty thousands and a hundred thousands and seven and sixty thousands thousands and three hundred thousands thousands and nine thousands thousands thousands, and its figure is 9367184225. The four and eighty thousands and the hundred thousands are the units, tens, and hundreds like the first three ranks, differing only by the particular word "thousands". Likewise, the seven and sixty thousands thousands and three hundred thousands thousands are units, tens, and hundreds, and they vary from the preceding in being thousands of thousands. Likewise for the nine thousands thousands thousands. They are units, tens, and hundreds,[7] which differ from the preceding only by the repetition of "thousands" three times. So know it.

70.15 **It should be known that each number can be indicated by its index and its name. The index is a term for the rank of the number. The index of the units is one,** since they are in the first rank. **The index of the tens is two,** since they are in the second rank. **The index of the hundreds is three,** since they are in the third rank, **and so on.**

70.23 Take, for example, five and twenty and seven hundred and four and eighty thousands. Its figure is 84725. You find the five, which are units as mentioned before, in the first rank, and that is its index and the index of the other units. Likewise, the twenty belongs to the tens, which is in the second rank, and that is its index and the index of the other tens. So if it were, for example, thirty or sixty or eighty, it could have replaced the twenty, since tens have index two and are in the second rank, as mentioned. Likewise, the seven hundred is in the third rank, and that is its index, and the index of other [hundreds]. Likewise the four thousands belongs to the fourth rank, which is its index, and the index of other [thousands]. Likewise the eighty thousands is in the fifth rank, which is its index, and the index of other [tens of thousands]. So understand this. The same holds for greater numbers and for comparable numbers.

71.6 **The name is the term for the number that occupies some rank. The name of the one is units, of two is tens, and of three is hundreds.** Take for example three and forty and a hundred. Its figure is 143. You find that it has three places. The index of the first is one, and the name of this one is units, since it is the rank of the units rank. The index of the second is two, and the name of this two is tens, since it is the rank of the tens rank. The index of the third is three, and the name of this three is hundreds, since it is the rank of the hundreds rank. So understand it.

71.12 **Subsection on knowing the index of the repeated number.**[8]

[6] From here to the end of the section we write the numbers in the order they are written in Arabic, and with the stated plurals of "thousand". Unlike thousands, hundreds are not made plural when there is more than one because the numbers "two hundred", "three hundred", up to "nine hundred" are single compound words in Arabic. After that, we switch back to the English order, and without the plural.

[7] They are only units, since there is only a 9.

[8] I.e., a number expressed with a repetition of the word "thousands".

71.13 **You multiply the number of repetitions by three, and you add to the result the index of the species of that number to get the required number.** And the repetition is the number of times you say "thousand".

71.16 For example, suppose someone said to us, "What is the index of ten thousands thousands?" You find the number of repetitions to be two. You multiply it by three, giving six. Likewise if the repetitions are more or less than two: you must always multiply it by three. Then you add to this six the index of the species of ten thousands thousands, which is two, since it was stated before that the units of the thousands thousands are in the units place and the tens are in the tens place, and the hundreds are in the hundreds place. So the sum is eight, which is the index of the given number. So know it. Its representation is made of seven zeros and a one, as in this figure: 10000000.

72.1 **Conversely, if you have a great many places and you want its name, then divide it by three. The division leaves you with three or less. The quotient is the number of repetitions of the number obtained in the remainder.**

72.4 For example, suppose someone said to you, "What is the name of a number falling in the tenth place?" You divide the ten, which is the number of places, by three. Our quotient is three, and the remainder is one. Likewise if the number of places is greater or smaller than ten: you must always divide it by three. The resulting three is the number of repetitions for the name of the remainder one. And the name of the remainder one is units. The tens rankis the index of units of thousands thousands of thousands, and its figure is 1000000000.

72.10 If the remainder from the division were two, it would be tens of thousands of thousands of a thousand, and if it were three, it would be hundreds of thousands of thousands of a thousand.[9] So know it.

73.1 **Section two, on addition.**

73.2 **Addition is the joining of numbers one to the other in order to express them with one expression. This is divided into five types. One of them is addition with no known relation, and the second is addition with a known disparity**[10], which are divided into two kinds.

73.7 *[The first is] a disparity in quality, which occurs when the numbers are in geometric progression. The disparity of the numbers consists of different numbers which are equal in quality in the sense that the ratio of one number to the next is equal to the ratio of a half or a third or something else.*[11] This kind comes in two types, since the ratio can either be the ratio of a half, which is what he means when he says "addition with a disparity like that of the squares of the chessboard and similar [problems]",[12] where the one in the first square is half of the two in the second square, and the two is half of the four in the third square, and so on to the end; or the ratio can be a third or a fourth or a seventh or any other ratio, which is what he means when he says "if the numbers have another disparity".[13] Examples of these will be presented later, almighty God willing.

[9] Literally, "tens thousands thousands thousand" and "hundreds thousands thousands thousand".
[10] Copied from Ibn al-Yāsamīn, (Zemouli 1993, 131.2-5).
[11] Copied from (Ibn al-Bannāʾ 1994, 214.3-5).
[12] From 76.7 below.
[13] From 78.1 below.

73.17 *The second kind is a disparity in quantity, in which the numbers are in arithmetic progression, such as consecutive numbers with a disparity of one, or consecutive odd numbers with a disparity of two, and the like. So the numbers differ by equal numbers, and it is a difference in quality when considered as the ratio of one number to the next.*[14] This is what he means when he says "If the disparity of the numbers is a known number other than doubling".[15] He, may God be satisfied with him, said "known disparity" and not "known relation" because *whenever the word "relation" is uttered, it suggests the well-known geometric [progression].*[16]

74.1 **The third [type] is the addition of consecutive numbers, their squares, and their cubes, the fourth [type] is the addition of consecutive odd numbers, their squares, and their cubes, and the fifth [type] is the addition of consecutive even numbers, their squares, and their cubes.**[17]

74.4 Note: *Adding consecutive numbers, consecutive odd numbers, and consecutive even numbers are truly [conducted in] arithmetic progression. The specific aim of these last three types is to find sums of [consecutive] squares and cubes. He included in these [types] the sums of [consecutive] numbers, even though they were of the preceding type, because they are a foundation for adding the squares and the cubes, and because working with a particular case makes the approach to the general one easier.*[18]

74.9 **For addition with no known relation, the aim is for you to add a number made up of several digits to a similar number. You should put one of the addends on a line, and below it you put the other addend, with each place below its counterpart. Then you add each digit of one of the addends to its counterpart in the other. If there is no counterpart, then the answer is the addend, as if it had a counterpart. So the sum is the answer.**[19] **One can start adding from the first digits or the last, but choosing the first is most orderly.**[20]

74.17 Here is an example of adding from the first digit. Suppose we want to add four thousand forty-three to two thousand six hundred eighty-five. We write the addend on a line and the augend below it on another line parallel to it, as mentioned, as in this figure: $\begin{matrix}4043\\2685\end{matrix}$.

74.20 We add the five, which is in the first [place] of the lower line, to its counterpart in the upper line, which is the three, giving eight. We put it above them, since they are units of their species. Then we add the eight that is in the tens rank of the lower line to its counterpart in the upper line, which is the four, giving twelve. We put the two above them, since they are also the same species, and we add the ten, in the form of a one, to the six in the hundreds rank of the lower line, giving seven, since for every rank, the units which are after it are the tens before it.[21] Nothing corresponds to the seven in the upper line, so it is considered to be the sum of that rank and that of its counterpart as if it had something. We put the

[14] Copied from (Ibn al-Bannāʾ 1994, 214.13-15).
[15] From 79.1 below.
[16] Copied from (Ibn al-Bannāʾ 1994, 214.18-215.1).
[17] Copied from Ibn al-Yāsamīn, (Zemouli 1993, 131.5-8).
[18] Copied from (Ibn al-Bannāʾ 1994, 226.5-7).
[19] Copied from Ibn al-Yāsamīn, (Zemouli 1993, 131.9-14).
[20] Copied from Ibn al-Yāsamīn, (Zemouli 1993, 131.15-16).
[21] I.e., the units in a particular rank are the tens for the rank before it.

seven above the zero. Then we add the two in the thousands rank of the lower line to its counterpart in the upper line, which is the four, giving six. We put it above them, and this completes the work. The sum is six thousand seven hundred twenty-eight, and this is the figure for that: 6728.

75.9 Here is another example, adding from the last rank. Suppose we want to add nine hundred seventy-eight to four hundred fifty-six. We put them down on two lines, as mentioned, as in this figure: $\begin{matrix}978\\456\end{matrix}$.

75.11 We add the four in the hundreds rank of the lower line to its counterpart in the upper line, which is the nine, giving thirteen. We put the three above them, and the ten, in the form of a one, after the three. Then we add the five in the tens rank of the lower line to its counterpart in the upper line, and that is seven, giving twelve. We put the two above them and we add the ten, in the form of a one, to the three that is above the addends, giving four. We replace it with it. Then we add the six in the units rank of the lower line to its counterpart in the upper line, which is the eight, giving fourteen. We put the four above them, and we also add the ten, in the form of a one, to the two that is above the addends, giving three. We replace it with it, and this completes the work. The sum is one thousand four hundred thirty-four, and the figure for that is 1434.

75.20 **The most one can gain by addition is one place.**[22] For example, suppose we want to add nine to nine. These are the maximum numbers that can occur in those two places on the two lines. So we say nine [added] to nine gives eighteen, and its figure is 18. The sum gains one place.

76.4 **To check the addition, you subtract one of the two lines from the answer. This leaves the other line.** For example, if we want to check this problem, we subtract the nine, which is one of the addends, from eighteen, which is the answer. This leaves the other nine. So understand.

76.7 **For addition with a disparity like that of the chessboard squares and similar [problems], a one [is placed] in the first square, then one proceeds by doubling from the first [square] to another assigned [square]. You add one to the one that is in the first square to get what is in the second square. Then you multiply that by itself, so the outcome is what is in the second square and what is before it, with an added one. Then you also multiply that by itself, so the outcome is what is in the fourth square and what is before it, with an added one.[23] Continuing, you also multiply the result by itself, and you double the squares,[24] until you reach the assigned [square], and you drop the one from the sum. The remainder is the required number.**

76.15 For example, suppose we want to add what is in sixteen squares arranged as described. We add one to the one in the first square, giving two. We multiply it by itself, giving four. This is what is in the second square and what is before it, with an added one. It is also what is in the third square alone. Because the places of the squares are like the places of the number, no doubt that if we multiply a place by another place, the index of the result is always the

[22] Copied from Ibn al-Yāsamīn, (Zemouli 1993, 131.17).
[23] Ibn al-Bannāʾ copied 76.7-12 from Ibn al-Yāsamīn, (Zemouli 1993, 132.1-6).
[24] I.e., double the number of squares, like from the fourth square to the eighth square.

index of the two multiplicands[25] less one, according to what we will show in [the section on] multiplication.[26] So when we multiply what is in the second square by itself, it most certainly results in what is in the third. So know it.

76.22 Then we likewise multiply the four by itself, giving sixteen, which is what is in the fourth square and what is before it, with an added one. It is also what is in the fifth square alone. Then we multiply the sixteen by itself, giving two hundred fifty-six, whose figure is 256. This is what is in the eighth square and what is before it, with an added one, and it is also what is in the ninth square alone. Then we likewise multiply the two hundred fifty-six by itself, giving sixty-five thousand five hundred thirty-six, whose figure is 65536. This is what is in the sixteenth square and all of what is before it, with an added one, and it is also all of what is in the seventeenth square alone. So it is clear that each number in a square exceeds all of what is before it by one. So you drop the added one, leaving sixty-five thousand five hundred thirty-five, which is the required number. Do it the same way if the required [number of squares] is greater or smaller. So know it. This is the figure for the table:

128	64	32	16	8	4	2	1
32768	16384	8192	4096	2048	1024	512	256

77.9 **If the situation is different, then multiply the remainder by the first square to get the required number.**[27] A DIFFERENT SITUATION IS WHEN THE FIRST SQUARE IS SOMETHING OTHER THAN ONE.

77.11 For example, suppose we want to add what is in eight squares, and a four is in the first square. We suppose that a one is in the first square, and we find the sum as before, and we drop the one. This is what he meant when he said "the remainder". It yields two hundred fifty-five, and its figure is 255. We multiply it by the four that is in the first square, which gives one thousand twenty. Its figure is 1020, and it is the required number.

78.1 **If the numbers have another disparity, then multiply the smallest by how much the greatest exceeds it, and divide [the result] by the difference between the smallest and the number that follows it. Then add the result to the greatest. This gives the required number.**[28]

78.4 For example, suppose we want to add five numbers with a ratio of, say, two thirds, like sixteen, twenty-four, thirty-six, fifty-four, and eighty-one. We put them on a line and we put dots between them, as in this figure: 81 ∴ 54 ∴ 36 ∴ 24 ∴ 16. We multiply the smallest, which is the sixteen, by how much the eighty-one exceeds it, since it is the greatest of the numbers, and that is sixty-five. The result is one thousand forty, whose figure is 1040. Then we divide it by the difference between the sixteen and the number after it, which is the twenty-four, and that is eight. The result of the division is one hundred thirty, whose figure is 130. We add it to the greatest to get two hundred eleven, whose figure is 211, and it is the required number.

[25] I.e., the sum of their indexes.

[26] At 104.1 below.

[27] This sentence is copied from Ibn al-Yāsamīn, (Zemouli 1993, 136.6).

[28] Copied from Ibn al-Yāsamīn, (Zemouli 1993, 136.7-9).

78.13 *This procedure is common to all numbers that are in a geometric progression, by generalization with the ratio of a half or anything else. The preceding [solution] is particular. And the particular is a basis for the technique as much as the general is a basis, since working it out in general instead of the particular becomes unbeneficial due to its prolixity. And from this general procedure it is clear to you that any number in the squares of the chessboard exceeds the sum of what is before it by one.*[29] So know it.

79.1 **If the disparity of the numbers is through a known number other than doubling, then multiply the disparity by the number of numbers less one. Adding the first number to the result gives the last of the numbers. Add it to the first, and multiply it by half of the number of numbers. It yields the required answer.**[30]

79.4 For example, suppose we want to add six numbers, the first being ten, which is the smaller extreme, with a disparity of three down to the last. There are two unknowns in this problem: the greater extreme, which is the last, and the sum. We multiply the three, which is the disparity, by the five, the number of numbers less one, giving fifteen. We add to it the ten, which is the first number, giving twenty-five, which is the last of the numbers. We also add it to the first, yielding thirty-five. We then multiply it by three, which is half of the number of numbers, giving one hundred five. Its figure is 105, and it is the total of these numbers, I mean their sum. In this way the sum is clear to you, and it is necessarily the way to do it. So know it.

79.13 **For the addition of consecutive numbers, you multiply half of the upper extreme by the upper extreme and one.**[31] For example, suppose we want to add from one to ten consecutively. We add one to the ten, the upper extreme, and that is eleven. We multiply it by half of the ten, giving fifty-five, which is the sum, and its figure is 55.

79.18 **And squaring them is given by multiplying two-thirds of the upper extreme increased by a third of one, by the sum.** For example, suppose we want to add from a square of one to a square of ten consecutively. We take two thirds of the ten, the upper extreme, giving six and two-thirds. We always add a third of one to it, giving seven. We multiply it by the sum, and that is fifty-five. The result is three hundred eighty-five, which is the required number, and its figure is 385.

80.1 **And cubing them is given by squaring the sum.** For example, suppose we want to add from a cube of one to a cube of ten consecutively. We multiply the sum, which is fifty-five, by itself. The result is three thousand twenty-five, which is the required number, and its figure is 3025.

80.5 **For the addition of consecutive odd numbers you square half of the established upper extreme with the one.** For example, suppose we want to add the odd numbers from one to nine consecutively. We add one to the nine, the upper extreme, giving ten. We then multiply half of the ten by itself, giving twenty-five, which is the required number, and its figure is 25.

[29] Copied from (Ibn al-Bannāʾ 1994, 220.8-12).
[30] This passage belongs in the *Condensed Book*, but it is not in Souissi's edition. It is copied from Ibn al-Yāsamīn, (Zemouli 1993, 136.9-12).
[31] Copied from Ibn al-Yāsamīn, (Zemouli 1993, 136.13-14).

80.10 **And squaring them is given by multiplying a sixth of the upper extreme by the surface of the two numbers that come after it.** For example, suppose we want to add the odd numbers from a square of one to a square of nine consecutively. We multiply a sixth of the nine, the upper extreme, which is one and a half, by the surface of the ten by the eleven, and that is one hundred ten, since they are the two numbers which are after the nine. The result is one hundred sixty-five, which is the required number, and its figure is 165.

80.15 **And cubing them is given by multiplying the sum by its double less one.** For example, suppose we want to add the odd numbers from a cube of one to a cube of nine consecutively. We multiply the sum, which is twenty-five, by its double less one, which is forty-nine. The result is one thousand two hundred twenty-five, which is the required number, and its figure is 1225.

80.20 **For the addition of consecutive even numbers, you always add two to the upper extreme, and you multiply half of the sum by half of the upper extreme.** For example, suppose we want to add the even numbers from two to ten consecutively. We add two to the ten, the upper extreme, giving twelve. We multiply its half by half of the ten, giving thirty, which is the sum, and its figure is 30.

81.4 **And squaring them is given by multiplying two-thirds of the upper extreme and two thirds of one by the sum.** For example, suppose we want to add the even numbers from a square of two to a square of ten consecutively. We take two-thirds of the ten, the upper extreme, giving six and two-thirds, and we always add to it two thirds of one, giving seven and a third. We multiply it by the sum, which is thirty, so the result is two hundred twenty, which is the required number, and its figure is 220.

81.9 **And if you wish, multiply a sixth of the upper extreme by the surface of the two numbers that come after it.** For example, suppose we want to add from a square of two to a square of twelve. We take a sixth of the twelve, the upper extreme, giving two. We multiply it by the surface of thirteen by fourteen, which is one hundred eighty-two, since they are the next two numbers after the twelve. The result is three hundred sixty-four, which is the required number, and its figure is 364.

81.15 **And cubing them is given by multiplying the sum by its double.** For example, suppose we want to add the even numbers from a cube of two to a cube of ten consecutively. We multiply the sum, which is thirty, by its double, which is sixty. The result is one thousand eight hundred, which is the required number, and its figure is 1800.

82.1 For the preceding three types, ***whenever it begins with something other than one, you add from one to the upper extreme, then from one to the number before the beginning number. Then you drop the smaller from the greater. For the even numbers the two takes the place of the one.***[32] By understanding all of this and managing it you shall succeed, almighty God willing.

83.1 **Section Three, on subtraction.**

83.2 **Subtraction is the search for the remainder after the dropping of one of two numbers from the other. It comes in two types. [One] type is subtracting the smaller from**

[32] Copied from (Ibn al-Bannāʾ 1994, 228.13-15).

the greater one time. And [another] type is subtracting the smaller from the greater more than one time, until the greater vanishes or it leaves a remainder less than the smaller. This type is called testing by casting out.

83.6 **For the first type you should write the minuend on a line, and below it the subtrahend, arranged as in addition. You subtract each digit from its corresponding digit if you find it has a counterpart. If you do not find a counterpart, or if it is smaller than the subtrahend, then subtract the minuend from the subtrahend, and subtract the remainder from the next digit, and you then put the remainder in the place which agrees with its given rank.**

83.11 **If you wish, you can always add ten to the counterpart and subtract [the digit in the subtrahend] from the sum, and add one to the next digit of the subtrahend. Then continue in the same way until you have finished all of the subtrahend and minuend. You can start a subtraction from the first of the ranks or from the last. It is preferred to start from the last, contrary to what is preferred in addition.**

83.16 Here is an example of subtraction beginning from the last rank. Suppose we want to subtract four thousand nine hundred sixty-eight from five thousand thirty-five. We put them down on two parallel lines, like we mentioned for addition, as in this figure: $\begin{array}{r}5035\\4968\end{array}$.

83.19 We subtract the four which is in the thousands rank from its counterpart in the minuend, which is five. The remainder is one, so we put it above it. Then we likewise subtract the nine that is in the hundreds rank from its counterpart. There is nothing but a zero there, and the zero is nothing. So we subtract this nothing of the minuend from the nine of the subtrahend, leaving nine. We subtract it from the one that is above the five, since it is ten with respect to the zero, as mentioned. This leaves one. We put it above the zero, and nothing is above the five. Then we likewise subtract the six that is in the tens rank from its counterpart, which is three. [This is] smaller than the six of the subtrahend, so we subtract the three of the minuend from the six of the subtrahend, leaving three. We subtract it from the ten that is above the zero, leaving seven. We put it above the three, and nothing above the zero. Then we likewise subtract the eight that is in the units rank from its counterpart, which is five. [This is] smaller than the eight, so we subtract the five of the minuend from the eight of the subtrahend, leaving three. We subtract it from the seventy that is above the three. The remainder is sixty-seven. We put the seven above the five, and the sixty, in the form of a six, above the three. It is the remainder, and the figure for that is: $\begin{array}{r}0067\\ \hline 5035\\4968\end{array}$.

84.13 Here is another example of subtraction from the first [rank]. Suppose we want to subtract three thousand four hundred sixty-nine from six thousand five hundred forty-three. We put them down in two parallel lines, as mentioned, as in this figure: $\begin{array}{r}6543\\3469\end{array}$. We subtract the nine that is in the units rank from its counterpart in the minuend, which is three. [This is] smaller than the nine, so we add ten to the three of the minuend, giving thirteen. We subtract the nine from it. The remainder is four. We put it above the three, then we add one to the six that is in the tens rank of the subtrahend, giving seven. We likewise subtract it from its counterpart, which is four. [This is] smaller than the seven, so we add ten to the four, giving fourteen. We subtract the seven from it, leaving seven. We put it above the four. Then we add one to the four that is in the hundreds rank, giving five. We likewise subtract it from its counterpart, which is five, and it vanishes. So we put a zero above the five. Then we likewise subtract the three that is in the thousands rank – with nothing added to it, since nothing remains from the previous position – from its counterpart, which is the six. Three remains, and we put it above the six. This completes the work. The remainder is three thousand seventy-four, and the figure for that is 3074.[33]

85.8 **The most one can reduce by subtraction is one place.** For example, suppose someone said to you, "Subtract one from ten". The remainder is nine. The minuend is reduced by a place. So understand it.

85.11 **To check the subtraction, you add the remainder to the subtrahend. The result is the minuend. Or you subtract the remainder from the minuend, leaving the subtrahend.** For example, in the above-mentioned problem the remainder is nine. We add it to the one that is the subtrahend, giving ten. This is the minuend. And if we subtract the remainder, which is the nine, from the minuend, which is the ten, there remains one, which is equal to the subtrahend.

85.16 IN THE FIRST WAY ***of checking***[34] ***a subtraction by addition you should look for a number which, if you add it to the subtrahend, then [the sum] is equal to the minuend. One begins this from the first of the ranks***, JUST LIKE THOSE WHO WORK WITH *RŪMĪ* SIGNS DO.

86.1 ***Whenever you are faced with subtracting a number from a number, and then the remainder from another number, and so on, then you work it out in the steps as indicated in this section.***

86.3 ***If we wish, we can piece the problem together with addition and subtraction. We collect the even terms from among the subtrahends, namely the second, the fourth, the sixth, and so on, and we add them together with the minuend. We likewise add the odd terms from among the subtrahends, namely the first, the third, the fifth, and so on, and we drop their sum from the first sum. The numbering with the subtrahends truly begins after the minuend [and continues] to the last.***[35]

86.9 *For example, subtract two from five, and the remainder from seven, and the remainder from eight, and the remainder from ten. Sometimes this is expressed with exclusions, so one says: ten less eight less seven less five less two.* We put it down in a line like this: 2 ﻻ 5 ﻻ 7 ﻻ 8 ﻻ 10. *We subtract the two from the five, and the remainder from the seven, and the remainder from the eight, and the remainder from the ten, leaving six, which is the required number.*

86.15 *If we wish, we can add the even subtrahends, which are the seven and the two, with the minuend ten, to get nineteen. Then we add the odd subtrahends, which are the eight and the five, giving thirteen. We subtract it from the nineteen, leaving six.*

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[33] The Medina and Oxford manuscripts show $\begin{array}{c}\underline{3074}\\ \overline{6543}.\\ 3469\end{array}$

[34] Literally, "working out".

[35] Copied from (Ibn al-Bannāʾ 1994, 245.8-15).

86.18 *And if we wish, we can consider three successive terms. We subtract the middle from the sum of the extremes, leaving the remainder as one number.* We insert it in [their] place in the subtrahend. *Then we likewise consider it and the two remaining numbers. Take for example the ten and the eight and the seven. We drop the eight from seventeen, the sum of the extremes, leaving nine which, with the two and the five, are three successive numbers. We drop the five from the sum of the extremes, leaving six.*

87.3 *Or we first consider the seven and the five and the two. We drop the five from the nine, the sum of the extremes, leaving four which, with the eight and the ten, are three successive numbers. We drop the middle number, eight, from the sum of the extremes, leaving six.*

87.6 *Or we consider the eight and the seven and the five first. We drop the middle seven from the sum of the extremes, leaving six. This is now the middle term between the ten and the two. We drop it from their sum, leaving six.*

87.9 *And if we wish, we can subtract the eight from the ten, and add the remainder to the seven, and subtract the five from it, and add the remainder to the two, yielding six.*

87.11 *The cause of this is that when subtracting the deleted from the appended, it is deleted, but when subtracting the deleted from another deleted, it is appended. Thus the deleted second and fourth and sixth of the even terms are always appended, since each of them is deleted from a deleted. And the odd terms are always deleted, since they are deleted from an appended.*[36] So know it.

87.15 **The second type [of subtraction] consists of three [kinds of] subtractions. These are frequently used in checking work. One is casting out nines, the second is casting out eights, and the third is casting out sevens.**

87.17 **Casting out nines leaves one from each power of ten. You collect the number from its digits as if they were units, and then you cast out nines from them.**

87.19 For example, suppose we want to cast out [nines from] six thousand four hundred thirty-five. We put the number on a line like this: 6435. We add the five to the three, giving eight. We add it to the four, giving twelve. We subtract nine from it, leaving three. We add it with the six, giving nine. This is cast out entirely, and it is the answer.

88.1 **Casting out eights leaves two from each ten, four from each hundred, and a pair of hundreds and what is above it are cast out entirely. So four is the remainder from odd hundreds, and you multiply the tens by two**, SINCE THE REMAINDER FROM EACH TEN IS TWO**. You add that with the four and with the units, and you cast out eights from it.**

88.5 For example, suppose we want to cast out [eights from] five thousand three hundred ninety-three. We put down the number on a line like this: 5393. Five thousand is cast out entirely, and the remainder from the three hundred is four. Keep it in mind. Then we multiply the ninety[37] by two, giving eighteen. We add it to the remembered four with the three units, to get twenty-five. Casting out eights leaves one, which is the answer.

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[36] Copied from (Ibn al-Bannāʾ 1994, 245.16-247.2).

[37] I.e., the nine, called “ninety” here because it is positioned one place forward.

88.10 **In casting out sevens, three remains from each ten, two from each hundred, six from each thousand, four from each ten thousand, five from each hundred thousand, one from each million, and from there the cycle begins again. You can come to know it with these letters: A J B W D H, repeated below the digits.**

88.14 The A is one, the J is three, the B is two, the W is six, the D is four, and the H is five. Someone arranged these elements into verse. He said, "Three and two and six and four ∗ and five and one; that is casting out sevens".

88.17 If you wish, you can write down the letters as mentioned, or if you wish you can make them dust numerals. **You multiply each digit by what is below it in number – sound out the letters – and you cast out sevens. You leave its residue above it. Then you add all the digits of the remainders as if they were units and you cast out sevens.**

89.4 For example, suppose we want to cast out [sevens from] twenty-three million seven hundred eighty-six thousand four hundred thirty-five. We write it down on a line and we draw a line above it. We write the letters below it, each letter below a number consecutively. If we run out of letters and we have not run out of digits, we repeat the letters for the remaining [digits]. This is what he meant when he said "and from there the cycle begins again". That is, after the H, and for the rest of the number, the letters are repeated along with the quantity with it. The figure always looks like this:

2	3	7	8	6	4	3	5
J	A	H	D	W	B	J	A

89.10 We multiply the five by the number of the A below it, giving five. We put it over the line above the five. Then we likewise multiply the three by the number of the J below it, giving nine. We cast out sevens, leaving two, and we put it over the line above the three. Then we multiply the four by the number of the B, giving eight, leaving one. We put it above the four. Then we multiply the six by the number of the W, giving thirty-six, leaving one. We put it above the six. Then we multiply the eight by the number of the D, giving thirty-two, leaving four. We put it above the eight. Then we multiply the seven by the H, giving thirty-five, which vanishes. So we put a zero above the seven. Then we likewise multiply the three by the repeated A, giving three. We do not cast out, so we put it above the three. Then we multiply the two by the J, giving six. We do not cast out here, either. We put it above the two. Its figure is:

6	3	0	4	1	1	2	5
2	3	7	8	6	4	3	5
J	A	H	D	W	B	J	A

90.1 Now we work with the remainders five, two, one, one, four, three, and six. We add them as if they were units, and we cast out sevens. The remainder is the answer, and that is one.

90.3 **If you wish, multiply what is in the last place by three. You cast out sevens, and you add the remainder to what is before it. If there is no number in the place before it, then you multiply the accumulated residue by three and you cast out sevens. Keep doing this until you reach the units [place].**

90.7 For example, suppose we want to cast out [sevens from] fifty-eight thousand sixty-four. We put it down on a line like this: 58064. Then we multiply what is in the last place, which is five, by three, giving fifteen. We cast out sevens, leaving one. We add it to the eight that is before it, giving nine. We multiply it by three, giving twenty-seven, whose remainder is six. There is nothing in the preceding place, so this six is considered to be the sum of the residue and the rank as if it had something. We multiply it by three, giving eighteen, and its remainder is four. We likewise add it to the six that is before it, giving ten. We multiply it by three, giving thirty, and its remainder is two. We add it to the four in the units place, giving six. This completes the work, and the six is the answer. So know it.

90.15 **If you wish, make the last digit tens and add what is before it as if it were units, and then cast out sevens. Then you make the remainder tens and you add it to what is before it as if it were units, and you cast out again.**

90.18 For example, in our preceding example we make the last five tens, and we add to it the eight that is before it as if it were units, giving fifty-eight. We cast out sevens, and the remainder is two. Then we make it tens and we add to it the zero before it. Since there is no number there, it gives twenty, whose remainder is six. We make it tens, and we add to it the six before it, giving sixty-six. Its remainder is three. We make it tens and we add to it the four that is before it, giving thirty-four. Its remainder is six, which is the answer.

91.1 **Subsection on the way to test [calculations] by casting-out.**

91.2 **For addition, you cast out each line, add their remainders, and cast it out. The remainder is the answer. Then you cast out the sum in the problem, which will agree with the answer.**

91.4 For example, we added forty-three, whose figure is 43, to sixty-four, whose figure is 64. The sum is one hundred seven, whose figure is 107. We can check this problem and other problems by casting out nines or eights or sevens. However, many people cast out sevens. We want to check this problem likewise, as well as the other examples. We add the one, the remainder of the addend, to the one, the remainder of the augend, giving two, which is the answer. If the sum of the two remainders can be cast out, we cast it out also, and its remainder is the answer. Then we cast out the sum, leaving two, which agrees with the answer. So know it.

91.12 **For subtraction, you cast out the minuend and you keep the remainder in mind. Then you cast out the subtrahend and you drop its residue from the remembered number. If it is smaller, add the modulus to it and drop it from the sum. The remainder is the answer. Then cast out the remainder in the problem. This should agree with the answer. Or, you add the residue of the subtrahend to the residue of the remainder. This should agree with the residue of the minuend.**

91.16 For example, we subtracted seventy-four, whose figure is 74, from ninety-six, whose figure is 96. The remainder is twenty-two, whose figure is 22. If we want to check it, we keep in mind the residual five of the minuend. Then we cast out the subtrahend, whose remainder is four. We drop it from the remembered five, leaving one, which is the answer.

92.3 And if this residual four of the subtrahend were greater than the residual five of the minuend, then we add [the modulus] to the five, the number of the minuend: nine if it is nine, eight if it is eight, or seven if it is seven. Then we cast out the remainder [in the problem], leaving one, which agrees with the answer. And if we add the four, the residue of the subtrahend, to the one, the residue of the remainder, it is the same as the five, the residue of the minuend. So know it.

92.8 **For multiplication, you cast out the two multiplicands and you multiply the remainder of one of them by the remainder of the other, and you cast it out. The remainder is the answer. Then you cast out the result of the multiplication, which should agree with the answer.**

92.10 For example, we multiply twelve, whose figure is 12, by sixteen, whose figure is 16. The result of the multiplication is one hundred ninety-two, whose figure is 192. We multiply the residual five of the multiplicand by the residual two of the multiplier, giving ten. Its remainder is three, which is the answer. Then we cast out the result of the multiplication, leaving three, which is equal to the answer.

92.15 **This generalizes to whole numbers and fractions after numerating them, *that is, when each of them in the problem becomes of a type [that is] one fraction.***

92.17 *For example, suppose someone said, "Multiply a third by fourteen and a fourth". The result of the multiplication is four and three fourths*, according to what will come in the work on fractions,[38] almighty God willing. *If we want to check it [by casting out sevens], we multiply the third, which is the remainder of one of the two multiplicands, by the fourth, the remainder of the other multiplicand, resulting in a third of a fourth. Its remainder after its numeration is one, which is a third of a fourth, which is the answer. Then we numerate the result of the multiplication, which gives nineteen fourths. Its remainder is five fourths. We numerate it by a third by multiplying it by three, giving fifteen. Its remainder is one, which is a third of a fourth, equal to the answer in quantity and quality.*[39]

93.6 **For division and denomination, you cast out the result and the divisor or denominating number. You multiply the remainder of one of them by the remainder of the other, and keep it in mind. The remainder is the answer. Then you cast out the dividend or denominated number, which should agree with the answer. This method also generalizes to whole numbers and to fractions, after numerating them.**

93.10 Here is an example of division. We divide one thousand four hundred eighty-eight, whose figure is 1488, by twelve. The result of the division is one hundred twenty-four, whose figure is 124. If we want to check it, we multiply the residual five of the result by the residual five of the divisor, giving twenty-five. Its remainder is four, which is the answer. Then we cast out the dividend, whose remainder is four, which agrees with the answer.

93.15 Here is an example with fractions. *Suppose someone said to you, "Divide five sixths and three fourths by a half". The result of the division is three and a sixth, and its remainder,*

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[38] I.e., the section on multiplying fractions, at 149.1.
[39] Copied from (Ibn al-Bannāʾ 1994, 248.11-17).

after numerating it, is five sixths. We multiply it by one, the numerator of the half, which is the divisor, giving five halves of a sixth. We then multiply it by two so it becomes fourths of sixths, agreeing in the numeration with the numerated dividend. It gives ten. Its remainder is three fourths of a sixth, which is the answer. Then we cast out the numerated dividend, which is thirty-eight fourths of a sixth. Its remainder is three fourths of a sixth, which is equal to the answer.[40]

94.1 Here is an example of denomination. Suppose someone said, "Denominate eleven with fifteen". The result of the denomination is three fifths and two thirds of a fifth. The remainder after numerating it is four. We multiply it by the one, the remainder of the denominating number, giving four, which is the answer. Then we cast out the denominated number. Its remainder is four, equal to the answer.

94.5 Here is an example with fractions. *Suppose someone said, "Denominate two sixths and two thirds of a sixth with five eighths and a third of an eighth". It results in two-thirds, and the remainder of its numerator is two. We multiply it by the two, the remainder from numerating the denominating number, giving four thirds of a third of an eighth. Then [we multiply it] by the six, giving twenty-four thirds of a third of a sixth of an eighth. Its remainder is three, which is the answer. Then we cast out the numeration of the numerator. Its remainder is one. We multiply it by the three, then by the eight. It gives twenty-four thirds of a third of a sixth of an eighth. Its remainder is three, which equals the answer.*

94.11 *It is necessary that everything in the problem be reduced to a finer fraction, which is the part named with all of the denominators. The meaning of numeration shall be clarified later, with the help of almighty God, may he be exalted.*[41]

95.1 **Section Four, on multiplication and understanding its subtleties.**

95.2 **Multiplication consists of the duplication of one of two numbers by however many units are in the other.**

95.3 *This section covers two types. [In one] type, in putting down the multiplier, each one of them is equal to the one of the multiplicand. Here the duplication clearly occurs in both the term and the meaning.*

95.6 *In the second type, all of what is in the multiplier in units is equal to the one of the multiplicand. So the [number of] units in the multiplier is the number of what is in one of the multiplicand in parts. This type is called conversion, and the duplication occurs in the term but not in the meaning.*

95.10 *Suppose someone said, "Three men: each of them has five dirhams". You multiply five by three, which gives fifteen dirhams. This is duplication in the term and in the meaning.*

95.12 *Suppose someone said, "Five dirhams: how many thirds does it contain?" You multiply five by three, which gives fifteen thirds. Here the duplication is only in the term, and as for the meaning, fifteen thirds are exactly five.*[42]

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[40] Copied from (Ibn al-Bannāʾ 1994, 248.18-249.4).

[41] Copied from (Ibn al-Bannāʾ 1994, 249.5-11). Ibn al-Bannā gives a section on numeration in *Lifting the Veil* at p. 272.14. Al-Hawārī covers numeration beginning at 135.8.

[42] Copied from (Ibn al-Bannāʾ 1994, 255.3-13).

95.15 **[Multiplication] is divided into three kinds;** THAT IS, WITH REGARD TO PROCEDURE. **The first kind is by shifting, the second by half-shifting, and the third without any shifting.**

95.17 **The first kind, which is multiplication by shifting, calls for erasing, and is called sleeper [multiplication]. You write the multiplicand and the multiplier on two lines so that the first digit of the multiplier is below the last digit of the multiplicand. Then you multiply it by all the digits of the multiplier. You begin by writing the result in its place, traversing the line, continuing along the line of the multiplicand. Then you shift the number of the multiplier so that it is below the digit that follows those before it. Then you multiply it by all the digits of the lower number, like the first time. Whenever you multiply by a number, you add the result with what is over the head of that number from the previous result, and you put it where it belongs. This procedure is universal for all problems of multiplication.**

96.3 For example, if we want to multiply forty-three by fifty-four, we write down the forty-three, the multiplicand, on a line, and the fifty-four, the multiplier, on another line, so that the first digit of the multiplier is below the last digit of the multiplicand, as mentioned. Here is the figure:

	4	3
5	4	

We multiply the last digit[43] of the multiplicand, which is the four, by the five, the last digit of the multiplier, giving twenty. We put a zero above the five, and the twenty, in the form of a two, after the zero. We also multiply it by the four below it, giving sixteen. We put the six in its place, and the ten, in the form of a one, in place of the zero. These are included with the line of the multiplicand. Then we shift the multiplier back one place, so the four is below the three, and the five is below the six. This is the figure:

2	1	6	3
		5	4

96.13 Then we likewise multiply the shifted three below it[44] by the entire multiplier, and we add that to what is above it. We first multiply it by the five, giving fifteen. We add it to the sixteen above it, giving thirty-one. We put the one in place of the six, and the thirty, in the form of a three, in place of the ten. Then we likewise multiply it by the four, giving twelve. We put the two in its place, and we add the ten, in the form of a one, to the one that is in the second rank, giving two. We replace it with it, and this completes the work. So the result is two thousand three hundred twenty-two, and its figure is 2322.

97.1 **The other type is known as vertical [multiplication]. You set up the two multiplicands in two vertical lines so that the first digit of the multiplier is opposite the last digit**

[43] We translate as "last digit" what is more literally "what is in the last rank".
[44] This is misstated. The multiplier, not the three, is shifted.

of the multiplicand. You proceed in multiplying them just as you did in the sleeper method, by shifting and erasing.

97.4 For example, suppose we want to multiply forty-two by thirty-seven. We put down the multiplicand in a column, as mentioned, and the multiplier as well, so that its first digit is next to the last digit of the multiplicand, and this is the figure:

2	
4	7
	3

97.7 Then we multiply the last digit of the multiplicand, which is four, by all the digits of the multiplier, as before. We first multiply it by the three, giving twelve. We put the two adjacent to the three in the column of the multiplicand, as before, and the ten, in the form of a one, below the two. Then we likewise multiply it by the seven next to it, giving twenty-eight. We put the eight in its place, and we add the twenty, in the form of a two, to the two that is below that rank, giving four. We replace it with it. Then we shift the multiplier back one place, so the seven is next to the two, and the three is next to the eight, as in this figure:

2	7
8	3
4	
1	

97.14 Next we likewise multiply the two by the whole multiplier, as before. We first multiply it by the three, giving six. We add it to the eight next to it, giving fourteen. We put the four in place of the eight, and we add the ten, in the form of a one, to the four in the rank below it, giving five. We replace it with it. Then we likewise multiply the two by the seven, giving fourteen. We put the four in its place, and we add the ten, in the form of a one, to the four that is in the rank below it, giving five. We replace it with it. This completes the work. The result is one thousand five hundred fifty-four, whose figure is 1554.

98.4 **The second kind is multiplication by half-shifting, which only works for two equal numbers. In this scheme you write down one of the equal numbers on a line, and you put marks in the form of dots between its digits. Then you multiply the last digit by itself and you put the result above it. Then you double it and you shift it by writing it down in place of the dot preceding it. Then you multiply the preceding digit by the shifted number and by itself, and you write the results of each multiplication above them. Then you double the digit that you just multiplied, as you did before. Then you shift it in place of the dot that precedes it. Then you shift the first doubled number, [adding] to its calculation. Then you multiply the digit preceding the dot that was replaced by all the doubled number, then by itself, as you did before. You continue**

in the same way, doubling, shifting, and multiplying until you reach the end of the line.[45]

98.15 For example, if we want to multiply four hundred sixty-three by itself, we put it down on a line and we separate the numbers with dots as mentioned, like in this figure: 4 • 6 • 3. We multiply the last four by itself, giving sixteen. We put the six above the four, and the ten, in the form of a one, after the six. Then we double it, giving eight, and we put it in place of the dot preceding it. So it yields this figure:

1	6				
	4	8	6	•	3

98.20 Then we multiply the preceding six facing it by the shifted eight, giving forty-eight. We put the eight above the eight of the multiplier, and we add the forty, in the form of a four, to the six in the next rank, giving ten. We put a zero in its place and we add the ten, in the form of a one, to the one in the next place, giving two. We replace it with it. Then we likewise multiply the six by itself, giving thirty-six. We put this six above it, and we add the thirty, in the form of a three, to the eight that is in the next rank, giving eleven. We put the one in place of the eight, and the ten, in the form of a one, in place of the zero. We then move the six back, also doubling, giving twelve. We put the two in place of the dot before it, and we add the eight to the ten, which resulted from doubling [the six], in the form of a one, giving nine. We put it down in place of the six, as in this figure:

2	1	1	6		
	4	8	9	2	3

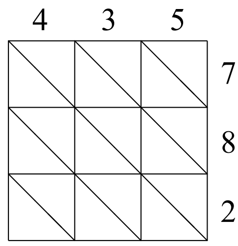
99.7 Then we likewise multiply the preceding three facing it by the whole shifted number and by itself, as before. We multiply it first by the nine, giving twenty-seven. We add to it the sixteen above it, giving forty-three. We put the three in place of the six which is above it, giving three, and the forty, in the form of a four, in place of the ten. Then we also multiply it by the two, giving six. We put it above it. Then we also multiply it by itself, giving nine. We put it above it. And this completes the work. The result is two hundred fourteen thousand three hundred sixty-nine, and its figure is 214369. So understand it and pursue similar problems the same way. Proceed by the power of almighty God.

99.14 **The third kind is multiplication without shifting, of which there are several types. One is table multiplication, in which you draw a quadrilateral surface, extending length and width according to the ranks in the two numbers to be multiplied. You draw diagonals through its squares from the lower right to the upper left, and you write the multiplicand above the quadrilateral, matching each digit with a column. Then you write the multiplier down the left or right side of the quadrilateral, so that each digit also corresponds to a row. Then you multiply digit after digit of the multiplicand by all the digits of the multiplier, and you put the digits of each rank in the intersecting square.** And the meaning of "intersection" is where they meet. **You put**

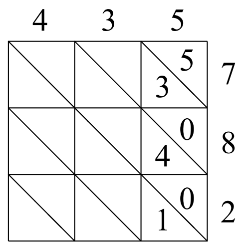
[45] Copied from Ibn al-Yāsamīn, (Zemouli n.d., 15.2-9).

the units above the diagonal and the tens below it. Then you begin adding from the upper right corner. You add what is between the diagonals, without erasing, and you write each number in its rank, and you add the tens of each sum to the next diagonal. You then put it together, and the sum that you obtain is the result.

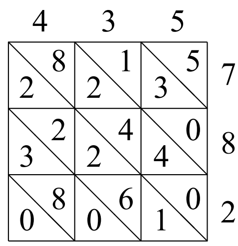
100.5 For example, suppose we want to multiply four hundred thirty-five by two hundred eighty-seven. We draw the quadrilateral as he mentioned, with the multiplier on the left or right of the quadrilateral, and the multiplicand above the quadrilateral. Each number is above a column of the quadrilateral, and similarly we write the multiplier on the right of the quadrilateral, as in this figure:



100.9 We multiply the five by the seven, giving thirty-five. We put down the five above the diagonal in the corresponding square, and the thirty, in the form of a three, below it. Then we likewise multiply the five by the eight, giving forty. We put the zero above the diagonal of the corresponding square, and the forty, in the form of a four, below it. Then we likewise multiply the five by the two, giving ten. We put the zero above the corresponding square, and the ten, in the form of a one, below it. This is the figure:



101.1 Then we do the same for the three. We multiply it by all the digits of the multiplier and we put the results of each of them in the corresponding square as before, and we do likewise for the remaining four of the multiplicand. Then we finish the multiplication. The numbers are situated in the quadrilateral in its entirety as in this figure:



101.5 Then we begin adding. We raise the five that is above the first diagonal in the upper right corner, as mentioned. Then we likewise add what is between the first and second diagonals, which are three and one, which add to four. We raise it after the five that we raised first. Then we likewise add what is between the second and third diagonals, which are four and four and two and eight. They add to eighteen. So we raise the eight after the raised four, and we add the ten, in the form of a one, with what is between the third and fourth diagonals, which are one and six and two and two and two. They add to fourteen. So we raise the four after the raised eight, and we add the ten, in the form of a one, with what is between the fourth and fifth diagonals, which are eight and three, giving twelve. We raise the two after the raised four, and the ten, in the form of a one, after it. This completes the work. All of this together is the required number, which is one hundred twenty-four thousand eight hundred forty-five, and its figure is 124845.

101.16 **Another [type] is vertical multiplication, in which you draw two vertical lines with some empty space between them, and you draw the two multiplicands beside them. Then you multiply one of them, digit by digit, by all the digits of the other, and you put the results in the space between the lines, taking into account the ranks of the indexes.**

102.1 For example, suppose we want to multiply one hundred eighty-three by three hundred forty-seven. We put down the multiplicand vertically in a line, and the multiplier parallel to the multiplicand, and a line after the multiplicand with another line before the multiplier as mentioned, as in this figure:

7		3
4		8
3		1

102.4 We multiply the three, which is the first [digit] of the multiplicand, by all the digits of the multiplier. We multiply it first by the seven, giving twenty-one. Then we put the one down next to the three beside the line, and the twenty, in the form of a two, below it. Then we multiply it likewise by the four, giving twelve. We add to it the two that is in the second rank of the result, giving fourteen. We put the four down next to the two, and the ten, in the form of a one, below it. Then we multiply it likewise by the three, giving nine, and we add it to the one that is in the third rank of the result, giving ten. We put the zero down next to the one, and the ten, in the form of a one, below it. This is the figure:

7		1	3
4	4	2	8
3	0	1	1
		1	

102.11 Then we similarly multiply the eight, which is in the second rank of the multiplicand, by all the digits of the multiplier. We multiply it first by the seven, giving fifty-six. We add it to the four that is in the second rank of the result, giving sixty. We put the zero down next to the four, and the sixty, in the form of a six, next to the zero that is after the second

[rank]. Then we multiply it also by the four, giving thirty-two. We add it to the six that is in the third rank of the result, giving thirty-eight. We put the eight down next to it, and we add the thirty, in the form of a three, to the one that is after that rank, giving four. We put it down next to it. Then we multiply it also by the three, giving twenty-four. We add it to the four that is in the fourth rank of the result, giving twenty-eight. We put the eight down next to it, and the twenty, in the form of a two, after that digit.

103.3 Then we multiply it also by the one, which is in the third rank of the multiplicand, again by all of the multiplier. We multiply it first by the seven, giving seven. We add it to the eight that is in the third rank of the result, giving fifteen. We put the five next to it, and we add the ten, in the form of a one, to the eight that is after that rank, giving nine. We put it down next to it. Then we multiply it also by the four, giving four. We add it to the nine that is in the fourth rank of the result, giving thirteen. We put the three down next to it, and we add the ten, in the form of a one, to the two that is after that rank, giving three. We put it down next to it. Then we multiply it also by the three, giving three. We add it to the three that is in the fifth rank of the result, giving six. We put it down next to it. This completes the work, and this is the figure:

7					1	3
4			0	4	2	8
3	5	8	6	0	1	1
	3	9	8	4	1	
			6	3	2	

103.12 The result is sixty-three thousand five hundred one. This is the vertical line adjacent to the multiplier. So know it.

103.14 **[Another type] is sleeper multiplication. You put the two multiplicands in two parallel lines. Then you multiply each digit of one of them by each digit of the other, and you put down the result taking into account the ranks of the indexes. You may start the multiplication from the first digit or the last. This type is also called multiplication by indexes.**

104.1 I say that we need here an introductory remark for this type that may also be useful for *rūmī* multiplication. The result of multiplying the units is units, since it comes from multiplying a number by a number. The index of the result is the sum of the indexes of the two multiplied numbers less one, as shown by the author, God be satisfied by him, in the section on addition in *Lifting the Veil*.[46] Certainly the index of the multiplicand is one and the index of the multiplier is one, so their sum is two. Dropping one leaves one, and the name of the one, according to what [was said] before, is units.[47] So the result of multiplying the units by the units is units. Likewise, their product by the tens is also tens, and by the hundreds is hundreds. And multiplying the tens by the tens is hundreds, and by the hundreds is thousands. And the hundreds by the hundreds is tens of thousands. This is the end of the introductory remark.

[46] The reference is to (Ibn al-Bannāʾ 1994, 215.12).
[47] Above at 71.6.

104.10 Suppose we want to multiply two hundred fifty-three by nine hundred eighty-seven. We put them down in two parallel lines, as mentioned. This is the figure:

2	5	3
9	8	7

104.12 We multiply the three, which is the units digit of the multiplicand, by all the ranks of the multiplier. We multiply it first by the seven, giving twenty-one. We put down the one in the units rank, as before, and the twenty, in the form of a two, after it. Then we multiply it also by the eight, giving twenty-four. We add it to the two that is in the tens rank of the result, since they are of the same species, as [mentioned] before, giving twenty-six. We put the six above the two, and the twenty, in the form of a two, after it. Then we multiply it also by the nine, giving twenty-seven. We add to it the two that is in the hundreds rank of the result, because they are also of the same species, giving twenty-nine. We put the nine above the two, and the twenty, in the form of a two, after it. This is the figure:

	9	6	
2	2	2	1
	2	5	3
	9	8	7

105.1 We likewise multiply the five that is in the tens rank of the multiplicand by all the ranks of the multiplier. We multiply it first by the seven, giving thirty-five. We add to it the six that is in the tens rank of the result, since it is also of the same species, giving forty-one. We put the one above the six, and we add the forty, in the form of a four, to the nine that is after that rank, giving thirteen. We put the three above the nine, and we add the ten, in the form of a one, to the two that is after that rank, giving three. We put it down above the two. Then we multiply it also by the eight, giving forty. We add to it the three that is in the hundreds rank of the result, giving forty-three. We put the three in its place[48] and we add the forty, in the form of a four, to the three that is after that rank, giving seven. We put it above the three. Then we multiply it also by the nine, giving forty-five. We add to it the seven that is in the thousands rank of the result, giving fifty-two. We put the two above the seven, and the fifty, in the form of a five, after it. This is the figure:

	2			
	7	3	1	
	3	9	6	
5	2	2	2	1
		2	5	3
		9	8	7

105.14 Similarly we multiply the two that is in the hundreds rank of the multiplicand by all the ranks of the multiplier. We multiply it first by the seven, giving fourteen. We add it to the

[48] The three is already there, so there is no need to write it.

three that is in the hundreds rank of the result, since they are also of the same species, giving seventeen. We put the seven above the three, and we add the ten, in the form of a one, to the two that is after that rank, giving three. We put it above the two. Then we multiply it also by the eight, giving sixteen. We add it to the three that is in the thousands rank of the result, giving nineteen. We put the nine above the three, and we add the ten, in the form of a one, to the five that is after that rank, giving six. We put it above the five. Then we multiply it also by the nine, giving eighteen. We add it to the six that is in the ten thousands rank of the result, giving twenty-four. We put the four above the six, and the twenty, in the form of a two, after it. This completes the work, and its figure is:

		9			
		3			
		2	7		
	4	7	3	1	
	6	3	9	6	
2	5	2	2	2	1
			2	5	3
			9	8	7

106.8 The result of the multiplication is two hundred forty-nine thousand seven hundred eleven, which is the required number. And if we wish, we can follow the method by starting from the last places. So know it.

106.11 **Another type [of multiplication] requires that the [numbers of] digits of the multiplicands be equal, and that the digits in each rank of the ranks in each line also be equal. The way to write it down is similar to the way [for the method] by erasing. Then you write a one below the first rank of the ranks of the upper line, and a two below the second [rank], likewise increasing by one until you reach the last rank of the multiplicand. What is below it is shared with the first rank of the multiplier.** From the second rank of the multiplier you begin decreasing one by one until you reach the last rank of the multiplier. **These written numbers are all on a third line. The indexes of the ranks of the multiplicand are in the correct order, but the indexes of the ranks of the multiplier are reversed. Then you multiply the number of the rank of the multiplicand by the number of the rank of the multiplier. The result is multiplied by what is in the written line, and the result is the required number. This type of multiplication is called "by repetition".**

107.6 For example, suppose someone said, "Multiply four hundred forty-four by three hundred thirty-three". We put them down on two lines, as mentioned. This gives the figure:

		4	4	4
3	3	3		

Then we write a one below the first four, a two below the second, and a three below the third. This is the first rank of the multiplier. Then we write a two below the second three, since from this point we begin decreasing one by one, as mentioned, and a one below the third three, as in this figure:

		4	4	4
3	3	3		
1	2	3	2	1

As mentioned, the indexes of the ranks of the multiplicand are in the correct order, and the indexes of the ranks of the multiplier are reversed, since the index of the three in the first [rank] of the multiplier is one, and we wrote it below the third three, which is its last rank.

107.15 Then we multiply the number of a rank of the multiplicand, which in this example is four, by the number of a rank of the multiplier, which in this example is three, giving twelve. We multiply it by the new line. We multiply it first by the one, which is the last in the line, giving twelve. We put the two above it, and the ten, in the form of a one, after it. Then we multiply it also by the two in the fourth [position] of the line, giving twenty-four. We put the four above the two, and we add the twenty, in the form of a two, to the two after that rank, giving four. We put it above it. Then we multiply it also by the three in the third [position] of the line, giving thirty-six. We put the six above the three, and we add the thirty, in the form of a three, to the four that is after that rank, giving seven. We put it above it. Then we multiply it also by the two in the second [position] of the line, giving twenty-four. We put the four above the two, and we add the twenty, in the form of a two, to the six that is after that rank, giving eight. We put it above it. Then we multiply it also by the one in the first [position] of the line, giving twelve. We put the two above it, and we add the ten, in the form of a one, to the four that is after that rank, giving five. We put it above it. That completes the work, and this is the figure:

	4	7	8	5	
1	2	4	6	4	2
			4	4	4
	3	3	3		
	1	2	3	2	1

108.7 The result of the multiplication is one hundred forty-seven thousand eight hundred fifty-two, which is the required number.

108.9 **Another [type] is multiplication by excess, in which you denominate the excess over ten of one of the two multiplicands with the ten. Then you take that ratio of the other [multiplicand]. You add it to it and you make it tens. If the ratio has a fractional part, you take it with respect to the ten[49] and you put it in the units place.**

108.13 For example, suppose we want to multiply twelve by fifteen. We denominate the two, the excess over the ten in the multiplicand, with the ten, yielding a fifth. We take a fifth of the fifteen, the multiplier, giving three. We add it to it, giving eighteen, and we make it tens.

[49] I.e., divide it by 10.

So it is one hundred eighty. This is the result, which is the required product, and its figure is 180. And if we denominate the five, the excess over the ten in the multiplier, with the ten, it yields a half. We take half of the twelve, giving six. We add it to it, again getting eighteen. We make it tens to get one hundred eighty, which is the result, as before.

109.1 Another example: suppose we want to multiply thirteen by seventeen. We denominate the three, the excess over the ten in the multiplicand, with the ten, yielding three tenths. We take three tenths of seventeen, the multiplier, giving five and a tenth. We add it to it, giving twenty-two and a tenth. We make it tens, and we take the fraction of the ten, which is a tenth, giving one. This we put it in the units place, as mentioned. That yields two hundred twenty-one, which is the required number, and its figure is 221.

109.7 **Another type is known as denomination. Here you add the two multiplicands, then you denominate one of them with the sum. Then you take that ratio of the other [multiplicand] and you multiply it by the sum, so the result is the required number.**[50]

109.10 For example, suppose we want to multiply six by twelve. We add them, giving eighteen. Then we denominate one of them with it. We denominate the six with it, giving a third. We take a third of the multiplier, which is the twelve, giving four. We multiply it by the sum. The result is the required number, which is seventy-two, and its figure is 72. And if we denominate the twelve with the sum, we get two-thirds. So we take two thirds of the six, giving four. We multiply it by the sum, and the result is the required number, as before.

109.16 And recall what the jurist Abū Muḥammad ibn Ḥajjāj, known as Ibn al-Yāsamīn, said about this method: "if we denominate one of them with the sum, then we drop that ratio of the number from itself and we multiply the remainder by the sum, it results in the required number".

109.19 **Another type is also known as denomination. You denominate the most convenient of the two multiplicands with whatever simple power of ten you wish, or you divide it by it. Then you multiply the result of the denomination or division by the other [multiplicand]. Then you raise each [digit] of the result by the power of ten of the divisor. The outcome is the required number.**

110.1 **If the result of dividing or denominating gives a whole number only by adding something to it or subtracting it from it, then do it. Then you multiply the added quantity by the number you did not add it to, and you subtract the outcome from the result. If you worked it by subtraction, then add the outcome to the result.**[51]

110.5 And by "simple power of ten" we mean that the first [non-zero] rank is equal to the ten or the hundred or the like.

110.6 Example problem: we want to multiply twenty-four by eight. So we denominate the eight, the multiplier, with whatever simple power of ten we wish. Supposing it is ten, it gives four-fifths. We multiply it by the twenty-four, resulting in nineteen and a fifth. We raise each [digit] by ten, the denominated power of ten, giving one hundred ninety, and a fifth of one [raised by] ten gives two. We add it to it, so the outcome is the required number, and that is one hundred ninety-two, and its figure is 192.

[50] Copied from Ibn al-Yāsamīn, (Zemouli h.d., 16.5-7).
[51] Copied from Ibn al-Yāsamīn, (Zemouli h.d., 16.8-13).

110.12 And another example: suppose we want to multiply twelve by fifteen. We divide the fifteen by the ten, resulting in one and a half. We multiply it by the twelve. The result of the multiplication is eighteen. We raise each [digit] by the ten, the divided power of ten, yielding one hundred eighty, which is the required number.

110.16 And if we wish, we can instead drop the five from the fifteen and divide the remaining ten by the ten, resulting in one. We multiply it by the twelve, giving twelve. We make it tens and we add to it the product of the subtracted five by the twelve, since it was not subtracted from it. The sum is one hundred eighty, which is the required number, as before.

111.1 Another example: we want to multiply three by fifteen. So we add two to the three, giving five. We denominate it with the ten, giving a half. We multiply it by the fifteen. The result of the multiplication is seven and a half. We raise each [digit] by ten, the power of ten, and half of the ten, to get seventy-five. We drop from it the product of the added two by the fifteen, since it was not added to it. The remainder is the required number, and that is forty-five, and its figure is 45.

111.7 **[Another type] is multiplication of "nines", with the requirements that the [numbers of] ranks in the two lines be equal, one of them consist of all nines, and the digits of the other be equal. A description of the procedure is that you write down the two parallel lines, one of them below the other, and you put dots above them, as many as there are places in them. You multiply the digit of the place of one of them by the digit of the place of the other. You put the units of the result in the first of the dots, and its tens in the middle of the remaining dots. You note the difference between the nine and the digit of the multiplier. Then with [this difference] you fill in what is between the two digits of the results, I mean the units and the tens, and you fill the remaining dots with the digit that is different from the nine. What this yields is the answer.**[52]

111.15 For example, we want to multiply four hundred forty-four by nine hundred ninety-nine. We put them down in two parallel lines, as mentioned, and we put dots above them, as many dots as there are places in both of them, and that is six. This is the figure:

•	•	•	•	•	•
			4	4	4
			9	9	9

111.18 We multiply the number of a rank of the multiplicand by the number of a rank of the multiplier, which is nine by four, giving thirty-six. We put down the six on the first of the dots, and five [dots] remain. We put down the thirty, in the form of a three, on the third of the five [remaining dots], since it is in the middle. Then we take the difference between the nine and the four, which is five. We fill in with it both of the dots between the units, which is six, and the tens, which is three, and we fill in the remaining dots with the four. This is what he meant when he said "with the number that is different from the nine". This is the figure:

[52] Copied from Ibn al-Yāsamīn, (Zemouli n.d., 16.14-17.2).

4	4	3	5	5	6
			4	4	4
			9	9	9

112.3 The result is the number that is above the two multiplicands, and that is four hundred forty-three thousand five hundred fifty-six. So know it, and solve similar examples the same way.

112.5 **Another type of multiplication of nines has no condition. Instead, the digits in one of the lines are nines, and the digits in the other line can be whatever they are, and the number of places can also be whatever they are. To work it out, you add as many zeros to the ranks of the other line as the number of ranks of the nines. Then you subtract from the total the number different from the nines, leaving the answer.**[53]

112.10 For example, we want to multiply nine hundred ninety-nine by nine thousand three hundred fifty-four. We add to the multiplier as many zeros as the number of ranks of the nines, which is three. So it comes to nine million three hundred fifty-four thousand, whose figure is 9354000. Then we drop the number that is not nines from it, which is the multiplier. The remainder is the required number, which is nine million three hundred forty-four thousand six hundred forty-six. Its figure is 9344646.

112.16 **Another type is known as squaring. You take half of the sum of the multiplicands and you square it. You subtract from the result a square of half of the difference between them. The remainder is the result of the multiplication.**[54]

113.1 For example, we want to multiply seventeen by nineteen. We take half of their sum, which is eighteen. We multiply it by itself, which is the meaning of "squaring", as [mentioned] before, giving three hundred twenty-four. We drop from it one, which is a square of half of the difference between the multiplicands, leaving three hundred twenty-three, which is the required number, and its figure is 323.

113.6 **Another type is also known as squaring, which is that you multiply a square of one of the two multiplicands by what results from the ratio of the other to the number you squared, or you divide a square of one of them by the result of dividing the squared one by the other one.**

113.9 For example, we want to multiply twenty-five by fifteen. So we multiply a square of twenty-five, which is six hundred twenty-five, by three fifths, which is the ratio of the fifteen to the twenty-five. The result is the required number, and that is three hundred seventy-five, and its figure is 375. Or we divide a square of the fifteen, which is two hundred twenty-five, by the three fifths, which is the result of dividing the fifteen that we squared by the twenty-five. The result is the required number, like before. So know it.

113.16 **Another type is that you multiply the difference between the two multiplicands by the greater of them, and you drop the result from a square of the greater. Or you**

[53] Copied from Ibn al-Yāsamīn, (Zemouli n.d., 17.3-7).
[54] Copied from Ibn al-Yāsamīn, (Zemouli n.d., 17.8-10).

multiply the difference by the smaller of them and you add the result to a square of the smaller. The result is the required number.[55]

113.19 For example, suppose someone said, "Multiply thirty-six by fourteen". We multiply the twenty-two, which is the difference between the multiplicands, by thirty-six, the greater of the multiplicands, resulting in seven hundred ninety-two. We drop that from a square of the greater, which is one thousand two hundred ninety-six. The remainder is the required number, which is five hundred four, and its figure is 504. Or we multiply the twenty-two, the difference, by the fourteen, the smaller multiplicand, resulting in three hundred eight. We add it to a square of the smaller, which is one hundred ninety-six. The sum is the required number, like before. So know it.

114.4 **And if you multiply a number with zeros by a number with zeros, multiply the parts of one by the parts of the other, stripped of the zeros. Then you dress the result with all the zeros. The outcome is the required number,**[56] SINCE MULTIPLYING THE NUMBER BY THE ZERO OR THE ZERO BY THE NUMBER IS IDENTICAL. IT COMES FROM VOIDING THE NUMBER OR DUPLICATING ZERO. NEITHER OF THESE GIVES A NUMBER, SO ITS SIGN IS ALWAYS A ZERO.

114.8 For example, suppose someone said, "Multiply thirty by one hundred forty". We remove the zero from the thirty and we set it aside, leaving three. Likewise we remove it from the one hundred forty and we set it aside, leaving fourteen. We multiply it by the three and we add the two zeros that were set aside to the result. The outcome is the required number, which is four thousand two hundred, and its figure is 4200.

114.12 **The upper limit on the [number of] ranks of the result is the sum of the ranks of the two multiplicands**, since the greatest [number] one can have in a single rank is a nine, and multiplying nine by nine is eighty-one. Thus multiplication in the units rank can give tens.

114.15 **To check it, you divide the result by one of the multiplicands. The result is the other [multiplicand].** For example, in the previous example we divide the eighty-one by the nine, which is one of the multiplicands. The result is nine, which is the other multiplicand.

115.1 **The student should memorize the following breakdown and master it, which is:**

115.2 **Multiplying a number by one or multiplying one by it leaves the number unduplicated,** like two by one gives two, or one by five gives five.

115.5 **And two by two gives four, and by each number that follows gives an additional two.** For successive numbers, multiplying two by three gives six, and by four gives eight, and by five gives ten, and so on up to the ten, since this is the traditional limit of the breakdown.

115.9 **And three by three gives nine, and by each number that follows gives an additional three.** He means "for successive numbers" here, too. Thus multiplying it by four gives twelve, and by five gives fifteen, and so on up to ten.

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[55] Copied from Ibn al-Yāsamīn, (Zemouli n.d., 17.11-13).

[56] Copied from Ibn al-Yāsamīn, (Zemouli n.d., 17.13-15).

115.12 **And four by four gives sixteen, and by each number that follows gives an additional four. And five by five gives twenty-five, and by each number that follows gives an additional five. And six by six gives thirty-six, and by each number that follows gives an additional six. And seven by seven gives forty-nine, and by each number that follows gives an additional seven. And eight by eight gives sixty-four, and by each number that follows gives an additional eight.** And for each of them for successive [numbers], just as the ones before. **And nine by nine gives eighty-one, and by ten gives ninety. And ten by ten gives a hundred.** So know it. Proceed by the power of almighty God.

117.1 **Section Five, on division.**

117.2 **Division is the decomposition of the dividend into equal parts in such a way that its number is equal to what is in the divisor in units.**[57] *This applies to discrete quantities.*[58]

117.5 **Division is also viewed as the ratio of one of two numbers with respect to the other.**[59] *And this concerns continuous quantities.*[60]

117.7 **Most people view division in all circumstances as knowing how many whole units of the divisor are in the dividend.**

117.9 ***Division has two meanings. One of them is described first, which concerns the division of a type by another type, like dirhams by men. The other is described second, and concerns the division of a type by the same type. So the word "division" has two meanings. And it should not be given a single description without regard to circumstances, as most people do. The meaning many people give it is specifically the first meaning, and they ignore the second meaning. Their description without regard to circumstances misleads the general public to lump the two meanings together, or to think that division truly has one meaning, and this is not so.***[61]

117.16 *Here is an example of division with the first meaning. Divide fifteen dirhams among three men. We decompose the fifteen into three equal parts, which is how many units are in the divisor. So each part consists of five dirhams, which is how many whole units there are of that three, the divisor.*

118.1 *Here is an example of the second meaning. Divide a piece of wood of fifteen spans by a piece of wood of three spans. The intention here is, how many copies of the divisor are in the dividend? So we cut up the dividend into copies of the divisor. There are five parts in the dividend* of copies of the divisor, *each part equal to the divisor.*

118.6 *So the result of working out the division in each of the two meanings is five. But the units of the result of five in the first meaning is different from the units of the result of five in the second meaning, since in the first meaning it is the number that is in a part of the parts of the dividend in units, and in the second meaning it is the number of parts in the dividend. So, in the first meaning the dividend is divided into a given number of parts, and*

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[57] Copied from Ibn al-Yāsamīn, (Zemouli 1993, 119.13-14).
[58] Copied from (Ibn al-Bannāʾ 1994, 263.3).
[59] Copied from Ibn al-Yāsamīn, (Zemouli 1993, 119.16).
[60] Copied from (Ibn al-Bannāʾ 1994, 263.4).
[61] Copied from (Ibn al-Bannāʾ 1994, 263.4-9).

what is in each of these parts comes to be known through the division. And, in the second meaning, what is in each of the parts of the dividend is given in units, and the number of parts into which it is divided is what comes to be known through the division. Thus the second meaning is the opposite of the first meaning.[62] So know it.

118.14 **Division comes in two types: dividing a greater number by a smaller number and dividing a smaller number by a greater number. Dividing the smaller by the greater specifically is called "denomination"**, AND THE WORD "DIVISION" IS RESERVED FOR DIVIDING THE GREATER BY THE SMALLER.

118.17 **The usual way of dividing the greater by the smaller is that you write the dividend on a line, and you write the divisor below it, making sure that the greater is not below the smaller. Then look for a number to put below the first digit of the digits of the divisor so that when you multiply it by all of its digits, it either exhausts the entire dividend or it leaves a remainder smaller than the divisor, in which case you denominate it with it.**

119.1 For example, suppose we want to divide two hundred forty-five by twelve. We put the dividend down on a line, and the divisor twelve on a line below it, with the two below the four, and the ten, in the form of a one, below the two, as in this figure:

2	4	5
1	2	

So, it happens that the divisor is below the twenty-four. If it were smaller, say for example eleven or ten or something similar, we would still put down the last digit of the divisor below the last digit of the dividend.

119.7 In our example, we look for a number to put below the two of the divisor, since it is the first digit. We multiply it by all of it, so that it either exhausts the twenty-four above it, or it leaves a remainder less than the twelve. We find that it is two. We multiply this two first by the one that is in the last rank of the divisor, giving two. This exhausts the two above it. Then we likewise multiply it by the two below it, giving four, and it exhausts the four above it. Then we shift the twelve back one place below the five, and we look for a number to put below the two of the divisor so that when we multiply them, it exhausts what is above it. We find nothing, so we put a zero in its place, and five remains from the twelve. We denominate it with it, to get two sixths and half a sixth. We add it to the whole number, which gives the required number, and that is twenty and two sixths and half a sixth, and its figure is $\frac{1\ 2}{2\ 6}$ 20. Similar problems are worked out the same way.

119.18 **If you want, you can divide the dividend into parts and add the quotients to get the result. Or you can decompose the divisor into the numbers of which it is composed and make them denominators, and divide the dividend by them. Or you can reconcile the dividend and the divisor and divide the reconciled divisor by the reconciled dividend.**

[62] Copied from (Ibn al-Bannāʾ 1994, 263.10-264.8).

120.5 An example of the first: suppose we want to divide forty-four equally among eleven men. We split the forty-four into twenty-two and twenty-two. If we wish, we can partition it differently. So we divide the twenty-two by eleven, resulting in two, and similarly for the other division. We add them, giving four, which is the result of dividing forty-four by eleven.

120.10 An example of the second: suppose someone said, "Divide ninety-six by twelve". We decompose the divisor twelve into what it is composed of, which is six and two, or three and four, which is similar. Suppose we decompose it the first way. We divide the ninety-six by the two first, resulting in forty-eight. We likewise divide it by the six, the other denominator, resulting in eight, which is the required number. And if we divide the dividend first by the six, then what results by the two, it still gives eight. So know it.

120.16 An example of the third: suppose someone said, "Divide thirty-five by fifteen". You rework each of them by a fifth, which reconciles them. For the dividend, its fifth is seven, and for the divisor, its fifth is three. We divide seven by three, resulting in two and a third, which is equal to the result of dividing the thirty-five by the fifteen. So know it.

120.20 **[Another] type of division is specifically called apportionment. The way you work it out is that you add the apportioned parts and you take it to be a denominator. Then you multiply each of the apportioned parts by the dividend, and you divide the results by the denominator, giving the required number.**

121.4 For example, a man is bankrupt. Some people give him ten dinars, which he accepts. Suppose there are three donors, and they divide it according to their own wealth. One of them has four dinars, the second five, and the third six. We add up these parts, which surpass them, that is, the apportioned parts, and their sum is fifteen dinars. We make it a denominator. Then we multiply what the first has in hand by the dividend, ten, giving forty. We divide it by the denominator, resulting in two dinars and two-thirds of a dinar, which is his obligation of the ten. Likewise we multiply what the second has in hand by the ten, giving fifty, and we divide it by the denominator, resulting in three dinars and a third of a dinar, which is his obligation of the ten. Then we likewise multiply what the third has in hand by the ten, giving sixty, and we divide it by the denominator, resulting in four dinars, which is his obligation of the ten. So if we add up these three results it gives ten.

121.14 There are four other ways to do this. One of them is that we denominate what each of them has in hand with the denominator, and we multiply the result by the dividend. This gives the required number. The second is that we denominate the dividend with the denominator, and what results is called the part of the share. We multiply it by what each one has in hand, giving the required number. In the third, we divide the denominator by what each one has in hand, and we divide the dividend by the results, giving the required number. In the fourth we divide the denominator by the dividend and we divide what each one has in hand by the results. This gives the required number.

121.23 There are other ways [of solving this problem] which involve combining the proportion, switching it, or some of the other conditions of the proportion, as will be seen later, almighty God willing.[63] What we have described is sufficient for anyone to understand.

[63] Manipulations of proportions are covered starting at 196.16.

122.2 **In the case that the apportioned parts are fractions, multiply everything in the problem by the smallest number divisible by the denominators. And if all of the parts have a common divisor, remove it by exchanging the parts or reconciling them.**

122.5 For example: a man is bankrupt, and he receives twelve dinars. He has three donors, the first of whom has four dinars and a third of a dinar. The second has five dinars and a fourth of a dinar, and the third has six dinars and a sixth of a dinar. We multiply each of the parts that each of them has in hand by the smallest number divisible by their denominators. Knowing the smallest number divisible by the denominators in this example and in others is reached by decomposition.

122.10 *We decompose each of the denominators into the numbers of which it is composed, and we drop numbers from the second that are repeats of numbers in the first, and from numbers of the third that are repeats of numbers in the first and remaining in the second, and from numbers in the fourth that are repeats in the previous numbers, and so on to the last. Then we compose the remaining numbers by multiplication. If there is no repeated number, then they are all different, so multiplying them together gives the smallest number divisible by the denominators.*[64]

122.15 If we want to apply this in the present example, we find that the denominator of a third, which is three, cannot be decomposed. The denominator of a fourth, which is four, decomposes into two and two, and there is no repetition, so we keep them with the three. The denominator of a sixth decomposes into three and two. Both are repeats, so we discard them and we compose the rest, and that is three by two, giving six, and six by two, giving twelve, which is the least number divisible by the denominators.

122.19 We multiply everything in the problem by it. We multiply it first by what the first has in hand, resulting in fifty-two. We exchange it with what he has in hand. Then we likewise multiply it by what the second has in hand, resulting in sixty-three, and we also exchange it with what he has. Then we likewise multiply it by what the third has in hand, resulting in seventy-four, and we also exchange it with what he has in hand. Then we see if these parts have a common divisor, and we remove it and replace each one with its reconciled number, as mentioned.

123.3 To know how to remove common divisors in this example and others, *we decompose the numbers according to how they are composed, and we drop the repeated numbers in each of them from the total. Then we compose what is left of each of them by multiplication to get the reconciled amounts. Whenever no number remains, exchange it with one, since multiplying a number that has vanished involves no duplication. Thus the exchanged one is the reconciled amount. The common divisor of these numbers can always be obtained by means of the denominated part of the repeated number that was dropped.*[65]

123.9 If it is two, the common divisor is a half, and if it is five, then it is a fifth, and if it is ten, then it is a tenth, and if it is eleven, then it is a part of eleven, and so on.

123.11 After testing the parts mentioned in the example above by this procedure, we find them to be different, so we add them. This is the denominator, and it is one hundred eighty-

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[64] Copied from (Ibn al-Bannāʾ 1994, 266.15-20).
[65] Copied from (Ibn al-Bannāʾ 1994, 267.1-5).

nine. So we multiply what each of them has in hand by the twelve and we divide by the denominator. The result for the first [donor] is three dinars and two ninths of a dinar and five sevenths of a ninth of a dinar, which is his obligation of the twelve. For the second it is four dinars, which is his obligation of the twelve. And for the third it is four dinars and six ninths of a dinar and two sevenths of a ninth of a dinar, which is his obligation of the twelve dinars. If we add these, they likewise result in twelve. And, if we wish, we can also work this out by the other methods mentioned above.

123.18 **For denomination, the most well-known way is that you decompose the denominating number into the numbers of which it is composed, and you take them as denominators. Then you divide what you want to denominate by them, resulting in the answer. One obtains its value by means of the ratio of these parts to the denominators serving as divisors.**

123.22 For example, suppose someone said, "Denominate eleven with fifteen". We decompose the fifteen, the denominating number, into the numbers from which it is composed, which are five and three. We put them below a line and we divide the eleven first by the three, and we put the remainder above it. And we divide the quotient by the five, which is smaller than it. We put it above it, so it yields this figure: $\frac{2\ \ 3}{3\ \ 5}$. Thus we form the ratio of the three to the five that is below it, and the two to the three that is below it, and we attach the ratio to the five. This gives the size of eleven with respect to fifteen, which is three fifths and two thirds of a fifth. So know it.

124.8 ***A lesser known way is that you divide the denominating number by the denominated number, and you denominate one with the result. Or you denominate one with the denominating number and you take this ratio of the denominated number. Or you multiply the denominated number by some [convenient] number and you divide the result by the denominating number, and [you divide] the result by that multiplied number.***[66]

124.12 An example of the first [way]: suppose someone said, "Denominate four with twelve". We divide the twelve by the four, and we denominate one with the result. This gives the required number, which is a third.

124.14 An example of the second: suppose someone said, "Denominate nine with fifteen". We denominate one with the fifteen, to get a third of a fifth. We take from the nine a third of its fifth. The result is the required number, which is three fifths.

124.17 And an example of the third: suppose someone said, "Denominate ten with sixteen". We multiply the ten by whatever number we wish, such as eight, to get eighty. We divide it by the sixteen, and the result of that by the eight, to get the required number, which is five-eighths. So know it.

124.20 **To decompose numbers requires that some preliminary remarks be learned. They are:**

125.1 **Every number that does not begin with units has a tenth and a fifth and a half, [the latter] being a characteristic of every even number.** For example, fifty and similar numbers. Its half is twenty-five, its fifth is ten, and its tenth is five.

[66] Copied from (Ibn al-Bannāʾ 1994, 267.11-14).

125.5 **If it begins with a five, it has a fifth. If it begins with units, and if it is even, then one casts out numbers using one of the three moduli,** IN THE SENSE OF WHAT IS SAID ABOVE ON CASTING OUT. **If it is cast out entirely by nines, then it has a ninth and a sixth and a third.** For example, thirty-six is cast out entirely by nines. Its ninth is four, its sixth is six, and its third is twelve.

125.11 **If the remainder is three or six, then it has a sixth and a third.** An example with a remainder of three is sixty-six, and similar numbers. Its sixth is eleven and its third is twenty-two. An example with a remainder of six is forty-two, and similar numbers. Its sixth is seven and its third is fourteen.

125.16 **If the remainder is something else, then cast out eights. If it is cast out entirely, then it has an eighth and a fourth.** For example, sixty-four is cast out entirely by eights. Its eighth is eight and its fourth is sixteen.

125.18 **If the remainder is four, then it has a fourth.** For example, sixty-eight. Its fourth is seventeen.

125.20 **And if the remainder is something else, then cast out sevens. If it is cast out entirely, then it has a seventh.** For example, fourteen is cast out entirely by sevens, and its seventh is two.

126.2 **If it is not cast out entirely, then it only has a half and its half is odd, [and you then] look for deaf parts.** For example, twenty-six is not cast out entirely by any of the three moduli, so it only has a half, and its half is thirteen, which is an odd number. So know it.

126.5 **If [the number] is odd, cast it out by two numbers, nine and seven. If it is cast out entirely by nines, then it has a ninth and a third.** For example, eighty-one is cast out entirely by nines. Its ninth is nine and its third is twenty-seven.

126.9 **If the remainder is three or six, then it has a third.** An example with a remainder of three is thirty-nine. Its third is thirteen. An example with a remainder of six is one hundred twenty-three. Its third is forty-one.

126.12 **And if the remainder is some other number, then cast out sevens. If it is cast out entirely, then it has a seventh.** For example, seventy-seven is cast out entirely by sevens, and its seventh is eleven.

126.14 **If it is not cast out entirely, then look for deaf parts by dividing by them; and you continue dividing the required number to be decomposed by deaf parts until you arrive at a number that divides into it, or you get a number whose square is greater than your given number, or [equivalently, that] the result from the division is equal to or less than the divisor and leaves a remainder after the division. At this point you know that it is one of the deaf parts, and the denomination is formed from it.**

127.1 For example, suppose we want to decompose two hundred twenty-one. We find that it is not cast out entirely by the two moduli, so we need to divide it by deaf parts, as mentioned. We find that it is not divisible by eleven, which is the first of them. But it is divisible by thirteen, and the result of the division is seventeen. So it is composed from thirteen by

[67] The text mistakenly has “a square of the dividend”.

seventeen. If a square of the thirteen had been greater than the dividend,[67] we would have also known that it is deaf. If the result had been equal to the divisor thirteen, or if it had been less, with a remainder from the dividend, we would also have known that it is deaf. We denominate it with separate pieces, saying something like “three parts of this” or “a hundred twenty parts of that” or something similar.

127.9 **Subsection on finding deaf parts.**

127.10 **This process is called the sieve. Here you write the odd numbers beginning with three. Then, for each of these numbers, you count off its successors by the number of units in it. When this is done, the next number is composite, and it counts that [first] number. You continue to do this until you reach a number whose square is greater than the last number in the sieve, at which point you know that the work is finished. Each marked number is composite and each unmarked number is deaf.**

127.15 For example, we write down the odd numbers from three consecutively, as mentioned, in a table, as in this figure:

19	17	$\overline{15}$	13	11	$\overline{9}$	7	5	3
37	$\overline{35}$	$\overline{33}$	31	29	$\overline{27}$	$\overline{25}$	23	$\overline{21}$
$\overline{55}$	53	$\overline{51}$	$\overline{49}$	47	$\overline{45}$	43	41	$\overline{39}$
73	71	$\overline{69}$	67	$\overline{65}$	$\overline{63}$	61	59	$\overline{57}$
$\overline{91}$	89	$\overline{87}$	$\overline{85}$	83	$\overline{81}$	79	$\overline{77}$	$\overline{75}$
109	107	$\overline{105}$	103	101	$\overline{99}$	97	$\overline{95}$	$\overline{93}$
127	$\overline{125}$	$\overline{123}$	$\overline{121}$	$\overline{119}$	$\overline{117}$	$\overline{115}$	113	$\overline{111}$
$\overline{145}$	$\overline{143}$	$\overline{141}$	139	137	$\overline{135}$	$\overline{133}$	131	$\overline{129}$

128.1 If we want to know which among them is composite with three, we count it off from its cell, ending at the cell of the seven, so the nine that is next is composite with the three. So we put a mark above it. Similarly, we count beginning with the cell of the nine, ending with the cell of the thirteen. So the fifteen that is next is also composite with the three, and we put a mark above it. Do this until the end of the sieve. Likewise, do this with the five and the seven, but not with the nine, since it is composite, and not with any other composite number. This ends in our example with counting by thirteen. We know that the work is finished because its square is one hundred sixty-nine, which is greater than one hundred forty-five, which is the last number in the prescribed sieve.

128.9 And if we wish, we can work with a sieve greater or smaller than this one, since the method is the same for all of them. Every number in this sieve that is marked is composite, and every unmarked number is deaf, as we noted. These deaf parts are counted only by one, as we recall from the first part of the book.[68] We cannot find a number which, when multiplied by a number gives, for example, thirteen or one hundred fifty-one and the like.

[68] On prime numbers, at 66.7.

So if someone said to you, “Of which numbers is thirteen composed”, you would say thirteen by one. Multiplication by one is not duplication, as mentioned before. And this is the answer to similar [questions], so know it.

129.1 **Section Six, on restoration and reduction.**

129.2 **Restoration is reconstitution (*iṣlāḥ*), and reduction is its opposite. The purpose of restoration and reduction is to know what to multiply by a number to produce the required outcome. Restoration is only for taking the smaller to the greater, and reduction is for the opposite**, THAT IS, FROM THE GREATER TO THE SMALLER.

129.6 An example of restoration: suppose someone said, “By how much must one restore eight, for example, so that it gives nineteen?” An example of reduction: suppose someone said, “By how much must one reduce fifty, for example, so that it gives six?”

129.8 **To work out restoration, you divide the restored number by the number to be restored, which gives the required number.** For example, suppose someone said, “By how much must one restore three so that it gives six?” You divide the six, the restored number, by the three, the number to be restored, resulting in two. If you multiply that by the three it becomes six, which is the required number.

129.12 **To work out reduction, denominate the reduced number with the number to be reduced. This results in the answer.** For example, suppose someone said, “By how much must one reduce eight so that it yields three?” You denominate the three, the reduced number, with the eight, the number to be reduced. This gives three eighths. If you multiply the three eighths by the eight, it becomes three, which is the answer. This is the end of the first chapter, with God’s benediction and His good guidance.

131.1 **Chapter Two, on fractions.**

133.1 **The fraction is the ratio between two numbers that are a part or parts. The ratio between the part and its name is called a fraction.**

133.3 For example, three and six. The relation that arises with the ratio of the smaller to the greater is called the fraction. It cannot be named in terms of the three alone because of the separation, and likewise it cannot be named in terms of the six alone; nor in terms of [the two words] taken together, since the ratio is not a sensible object but is the idea of a specific intelligible object. It is called a fraction, like terrain with fractures that ascends and descends; or also like the surface, the solid, and the line, and there is nothing of this in discrete quantities except similar abstractions. Furthermore, the fraction consists of names, like “the half” and others that will be presented below, almighty God willing.

133.10 **We intend to present calculation with fractions in six sections.**

134.1 **Section One, on the names of fractions and numerating them.**

134.2 **Ten fractions have simple names. The first is the half, which is the greatest**, WHOSE FIGURE IS $\frac{1}{2}$. **Then the third**, WHOSE FIGURE IS $\frac{1}{3}$. **Then the fourth**, WHOSE FIGURE IS $\frac{1}{4}$. **Then the fifth**, WHOSE FIGURE IS $\frac{1}{5}$. **Then the sixth**, WHOSE FIGURE IS $\frac{1}{6}$. **Then the seventh**, WHOSE FIGURE IS $\frac{1}{7}$. **Then the eighth**, WHOSE FIGURE IS $\frac{1}{8}$. **Then the**

ninth, WHOSE FIGURE IS $\frac{1}{9}$. **Then the tenth**, WHOSE FIGURE IS $\frac{1}{10}$. **Then the "part"**, OF WHICH THERE ARE MANY TYPES. YOU CAN SAY A PART OF ELEVEN, WHOSE FIGURE IS $\frac{1}{11}$, AND A PART OF SEVENTEEN, WHOSE FIGURE IS $\frac{1}{17}$, AND SO ON.

134.8 **You can form the dual and plural of these fractions. Adding terminates for each fraction when you have [one] less than the name of the part.** THE NAME IS THE GREATER OF THE TWO NUMBERS IN THE RATIO OF ONE OF THEM TO THE OTHER. For example, we say a fourth and two fourths and three fourths, but we do not say four fourths. Likewise we say a seventh and two sevenths and three sevenths and four sevenths and five sevenths and six sevenths, but we do not say seven sevenths. So know it.

135.1 **You can attach these simple names to one another so that the resulting name is a combination of two or more names.** For example, we say two eighths and a seventh of an eighth, whose figure is $\frac{1\ 2}{7\ 8}$, and likewise for similar examples. Another example is eight ninths and four sevenths of a ninth and two sixths of a seventh of a ninth and a third of a sixth of a seventh of a ninth, whose figure is $\frac{1\ 2\ 4\ 8}{3\ 6\ 7\ 9}$, and likewise for similar examples.

135.8 **Numeration is that you reduce all of what is given to you in a particular problem to a finer fraction in it…**[69] *Know that the finer fraction in it is the part named with respect to all the denominators of the problem.*

135.10 *When the fraction is related, like five-sixths and four-fifths of a sixth and two-thirds of a fifth of a sixth,* whose figure is $\frac{2\ 4\ 5}{3\ 5\ 6}$, *we multiply what is above the first denominator by the denominator that follows it. That means to relinquish its name, which is sixths, in favor of fifths of sixths, since the second denominator is a number of what is in the one of the first denominator in units. So we add it with the four that is above it, since they are fifths of sixths, then we multiply that by the three, the third denominator, which is the number of what is in the one of the second denominator in units. This results in thirds of fifths of sixths. We add it with the two, since it is thirds of a fifth of a sixth. This gives the numerator of the problem, which is how many thirds of fifths of sixths there are in it,* and that is eighty-nine. *The multiplication of the denominators one by the other*, which is ninety, *is what is in the whole one of these parts, and a part of one of them is a finer fraction in the problem.*

136.8 *If the fraction is distinct, like five-sixths and four-fifths*, whose figure is $\frac{4}{5}\,\frac{5}{6}$, *we multiply the five-sixths by five, the denominator of the fifths, and that is the number of what is in the one sixth of fifths, so it becomes fifths of sixths. And we multiply the four-fifths by six,* the denominator of the sixths, *and that is the number of what is in the one fifth of sixths. So it becomes fifths of sixths. The sum of these two numerators are parts of thirty, which is sixths of fifths or fifths of sixths*, both being equal.

137.1 *The fraction may be portioned, in which the fractions are taken one of another, and which is how they are described in the expression. An example is three fourths of five sixths,* whose figure is $\frac{5 \bullet 3}{6 \bullet 4}$. *We multiply the three by the five, giving fifteen fourths of a sixth, or sixths of a fourth, which are parts of twenty-four parts of the unit. This is because five of the sixths, of which we want three of its fourths, [is found] by taking its fourth, which is five fourths of a sixth, and multiplying it by three, giving fifteen fourths of a sixth, or by taking a fourth of three times it. And three times it is its multiplication by three, giving*

[69] This sentence continues at 137.11.

fifteen sixths, and a fourth of that is fifteen fourths of a sixth, which is also fifteen sixths of a fourth. So three fourths of five sixths is five sixths of three fourths, since multiplying three by five is like multiplying five by three.[70] So know it.

137.11 **…and this is different for different [kinds] of fractions, that is, the numeration. There are five kinds [of fractions]: simple, related, distinct, portioned, and excluded.**

137.13 **The numerator of a simple fraction is what is above it.** For example, we said "a seventh": its numerator is the one that is above the line. Likewise for combined fractions, like if we said, "a third of a seventh": its numerator is also the one that is above the line.

138.1 **The numerator of a related fraction is what is above the first denominator multiplied by the next denominator, then adding to the end of the line; or, it is what is above the first denominator multiplied by the denominators after its denominator, and what is above the second denominator also multiplied by the denominators after its denominator, and so forth until the line is completed, then adding them together.**

138.5 For example, suppose someone said, "Numerate five eighths and four sevenths of an eighth and three fifths of a seventh of an eighth and two thirds of a fifth of a seventh of an eighth", whose figure is $\frac{2\ 3\ 4\ 5}{3\ 5\ 7\ 8}$. We multiply the five that is above the first denominator by the seven, the second denominator, and we add the four that is above it, giving thirty-nine. We also multiply that by the five, the third denominator, and we add the three that is above it, giving one hundred ninety-eight. We likewise multiply it by the three, the fourth denominator, and we add the two that is above it to it, giving five hundred ninety-six, which are eighths of sevenths of fifths of thirds. This is the numerator, and its figure is 596.

138.12 For the second way, we multiply the five that is above the first denominator by the other denominators except its denominator, as mentioned. This results in five hundred twenty-five. Keep it in mind. Then we multiply the four that is above the second denominator by the denominators after it, resulting in sixty. Keep this in mind as well. Then we multiply the three that is above the third denominator by the three, the fourth denominator, and we add what is above it, since there is nothing left after that denominator, giving eleven. We add it with the two remembered numbers, which gives five hundred ninety-six, which is the numerator, as before.

139.1 **The numerator of a distinct fraction is found by multiplying the numerator of each part by the other denominators, then adding them together.**

139.2 For example, suppose someone said, "Numerate five sevenths and half a seventh and four sixths". We put them below two lines as in this figure: $\frac{4}{6}\ \frac{1\ 5}{2\ 7}$. We find that the first part is related, so we take its numerator in the manner shown above, to get eleven. We multiply it by the six, the denominator of the second part, resulting in sixty-six. Keep it in mind. We find that the second part is simple, so its numerator is the four that is above the denominator. We likewise multiply it by the denominators of the first part, giving fifty-six. We add this with the remembered number to get a total of one hundred twenty-two, which is sevenths of halves of sixths. This is the numerator, and its figure is 122.

[70] Copied from (Ibn al-Bannāʾ 1994, 272.14-273.19).

139.10 **The numerator of a portioned fraction is found by multiplying what is above the line, each one by the next.**

139.11 For example, suppose someone said, "Numerate seven-ninths of five-sixths of three-tenths". We put them on a line and we always separate them with marks, as in this figure: $\frac{3 \bullet 5 \bullet 7}{10 \bullet 6 \bullet 9}$.We multiply the seven that is above the first denominator by the five that is above the second denominator, and their product by the three that is above the third denominator, to get the numerator. That comes to one hundred five sevenths of sixths of tenths, and its figure is 105.

140.1 **The numerator of an excluded fraction of disconnected [type] is found the same way as for distinct fractions, by subtracting the smaller from the greater.**

140.2 For the disconnected [type], what follows the "less" is not taken from what precedes it, but rather is taken from one, and then it is removed. For example, if we said "a half less a third", we mean less a third of one. So, after taking a third of the one, it is removed from the half. And when he said "the same way as for distinct fractions" he meant that you multiply what is before the "less" with what is after it in two different steps. We multiply the numerator of each part by the other denominator, and then we subtract the smaller from the greater.

140.8 For example, suppose someone said, "Numerate six eighths less a ninth of one". We put them on a line as in this figure: $\frac{1}{9}$ ﻻ $\frac{6}{8}$. We multiply the six, the numerator of what is before the "less", by the nine, the denominator after it, giving fifty-four. From that we drop the product of the one, the numerator after the "less", by the eight, the denominator before it, which is eight. The remainder is forty-six eighths of ninths, which is the numerator. Work out similar problems the same way, as explained above.

140.14 **For the connected [type], multiply the numerator of the diminished fraction by the numerator of the excluded fraction, and likewise multiply by their denominators, and then subtract the smaller from the greater.**

140.16 The connected [type] occurs when what follows the "less" is taken from what precedes it, without mediation. For example, suppose we said "a half less its third". A third of the half is a sixth, so it is as if someone had said "a third", or "a half less a sixth", so it becomes disconnected.

141.1 Suppose someone said, "Numerate six-sevenths and half a seventh less its third". We put them on a line, as in this figure: $\frac{1}{3}$ ﻻ $\frac{1\ 6}{2\ 7}$. We take the numerator of what is before the "less", which is the diminished fraction. We multiply it by the denominators of what is after the "less", which is the excluded fraction, to get thirty-nine. Keep it in mind. Then we also multiply the numerator of the diminished fraction by the numerator of the excluded fraction, similarly giving the complement thirteen. We drop it from the remembered number. The remainder is twenty-six sevenths of halves of thirds, and it is the numerator.

141.7 Supplementary remark. ***If there are repeated deletions in which each of them has the conjunction "and", it is with respect to the first [term]. And in relation to the first [term],***

[71] Copied from (Ibn al-Bannāʾ 1994, 275.8-9).

whether they are all connected to or detached from the diminished fraction, [treat them as] distinct fractions excluded from the diminished fraction.[71]

141.11 A first example: suppose someone said, "Numerate five and a third less its fourth and less its seventh and less its fifth". We put it down on a line like this figure: $\frac{1}{5} \searrow \frac{1}{7} \searrow \frac{1}{4} \searrow \frac{1}{3}\,5$. The fourth and the seventh and the fifth are distinct fractions excluded from the five and a third, which is the diminished part, and they are all connected to it. Removing the second and third particles of exclusion, the figure in this problem becomes $\frac{1}{5}\,\frac{1}{7}\,\frac{1}{4} \searrow \frac{1}{3}\,5$. We work it out as before, with the result that the numerator is nine hundred twelve thirds of fourths of sevenths of fifths, and its figure is 912.

142.6 A second example: suppose someone said in the previous problem, "less a fourth of one and less a seventh of one and less a fifth of one". Then these are distinct fractions removed from the diminished part, which are all detached. Write them also as if they were connected and work it out as before. The result is that the numerator is one thousand nine hundred ninety-one thirds of fourths of sevenths of fifths, and its figure is 1991.

142.10 If the exclusions are repeated without a coordinating conjunction,[72] in such a way that each [term] is excluded from the one before it, whether they are detached or connected, *then we take the [last] excluded fraction and the diminished fraction [before it], as in the other problem, and we work it out as shown above*, whether they are connected or detached. *The numerator we get is excluded from the one before it. Then we work that out similarly. The result is the numerator excluded from the one before it. Continue like this to the first [fraction].*[73] If we wish, we can work it out for disconnected fractions as in the two preceding cases.

143.1 *Whenever some of them are disconnected and others are connected, then the connected fractions must be transformed to their disconnected form when writing it down, so that all of them become disconnected. For instance, five sixths less three of its fourths is five sixths less three fourths of five sixths. Now what is written after the "less" is a disconnected fraction.*[74]

143.5 **And if there is a whole number before the fractions in a problem, it is multiplied by the denominators and then added with the numerator to become fractions.** For example, suppose someone said, "Numerate five and five sixths and three fourths of a sixth". So we put it down on a line like this: $\frac{3\ 5}{4\ 6}\,5$. We multiply the whole number five by the six, the first denominator, and what results by the four, the second denominator. This gives the result one hundred twenty. We add it with the numerator of the fraction, which is twenty-three. This gives a total of one hundred forty-three sixths of fourths, and its figure is 143, which is the numerator.

143.13 **And if it is after them, multiply the numerator by it,** SINCE THE FRACTIONS ARE PORTIONS OF IT. For example, suppose someone said, "Numerate four sevenths and six eighths of ten". We put it down on a line like this: $10\,\frac{6}{8}\,\frac{4}{7}$. The numerator of the fractions, as shown before, is seventy-four. We multiply it by the whole number ten to get seven hundred forty sevenths of eighths, which is the numerator, and its figure is 740.

[72] The coordinating conjunction is the word "and" (*wa*).
[73] Copied from (Ibn al-Bannāʾ 1994, 275.14-16).
[74] Copied from (Ibn al-Bannāʾ 1994, 276.3-5).

144.4 **And if it is in the middle, it is either attached to what is before it, so it is on the end, or it is attached to what is after it, so it precedes it. You numerate it according to one of the two cases, with the remaining [fractions] as distinct in the latter case, and in the former case one multiplies by the numerator of the remaining [fractions].**

144.7 "Attached to what is before it" means that only the first fraction is taken of the whole number, so it is one part. The remaining fraction is the distinct part. You multiply the numerator of each part by the denominator of the other and you add the results.

144.10 For example, suppose someone said, "Numerate four ninths of five, and three sixths". We put down the problem on a line like this: $\frac{3}{6}\,5\,\frac{4}{9}$. We multiply the four that is above the nine by the whole number five, giving twenty, which is the numerator of the first part. Then we multiply it by the six, the denominator of the second part, giving one hundred twenty. Keep it in mind. Then we multiply the three, the numerator of the second part, by the nine, the denominator of the first part, giving twenty-seven. We add it with the remembered number to get one hundred forty-seven sixths of ninths, which is the numerator. Its figure is 147.

145.1 And the "attached to what is after it" [means] that the first fraction is taken of the whole number and of the fraction after it. So the whole number is attached to what is after it, and it precedes it. So you numerate it with it and you multiply that by the numerator of the remaining [fraction], which is the first fraction, since it is a portion of it.

145.4 For example, suppose someone said, "Numerate two thirds of seven and four sevenths". We put it down like this: $\frac{4}{7}\,7\,\frac{2}{3}$. We multiply the whole number seven by the seven, the denominator of the fraction, and we add the four that is above it and we multiply that by two, the numerator of the remaining [fraction], which is the first fraction. This gives one hundred six thirds of sevenths, which is the numerator, and its figure is 106.

145.8 *One understands from this that for every fraction, or more than one fraction, that is taken only of a whole number, the whole number comes after it. So we numerate it with it as one part. And the other fraction which is not taken of it we regard as distinct. And for every fraction, or more than one fraction, together with the whole number, if the whole number precedes it, then we numerate it with it as one part, and one multiplies its numerator by the numerator of the fraction taken of that whole number and the fraction that is with it.*

145.13 *Any remaining fractions in the problem other than those taken of the whole number and what is with it are distinct fractions. So the whole number and what is with it and what is taken of that is one part, and every fraction among the distinct fractions is a part. So one multiplies the numerator of each fraction by the denominator of the other and we add the results.*[75]

146.3 **Common divisors between the numerator and denominators should be removed.** We mentioned the way to do this by decomposition in the section on division.[76] *For portioned fractions in particular one should remove the common divisors before the numeration,*

[75] Copied from (Ibn al-Bannāʾ 1994, 276.16-277.2).
[76] At 123.3 above.

since the numbers ascribed to the numerator are above the line and the numbers ascribed to the denominators are below the line. So we drop the repeated [factors] from both of them.[77]

147.1 **Section Two, on adding and subtracting fractions.**

147.2 **To work out the addition, you multiply the numerator of each line by the denominators of the other, and you divide the sum by the denominators. And for subtraction you drop the smaller from the greater before dividing by the denominators.**

147.4 An example of addition: suppose someone said, “Add three and four fifths and six eighths to four tenths and three eighths of a tenth and half an eighth of a tenth”. We put the addend on a line and the augend on another line below it, like in this figure:

$$\frac{6}{8}\,\frac{4}{5}\,3$$

$$\frac{1\;\;3\;\;\;4}{2\;\;8\;\;10}$$

We numerate the upper fraction as before. Its numerator is one hundred eighty-two. We multiply it by the denominators of the lower [fraction], to get twenty-nine thousand one hundred twenty. Keep it in mind. Then we multiply the numerator of the lower [fraction], which is seventy-one, by the denominators of the one above, resulting in two thousand eight hundred forty. We add it with the remembered number to get thirty-one thousand nine hundred sixty, and its figure is 31960. We divide it by the denominators of the two lines. The result is the required number, and that is four and nine tenths and seven eighths of a tenth and half an eighth of a tenth, and its figure is $\frac{1\;7\;\;9}{2\;8\;10}4$. And its answer is given by five.

147.15 Another example, of subtraction: suppose someone said, “Subtract seven tenths of two less a third of one from four and three fourths of five sixths”. We put the minuend down on a line and the subtrahend on another line below it, like with the addend, as in this figure:

$$\frac{5\;\bullet\;3}{6\;\bullet\;4}\,4$$

$$\frac{1}{3}\setminus 2\,\frac{7}{10}$$

148.3 We numerate the upper [fraction] as before. Its numerator is one hundred eleven. We multiply it by the denominators of the lower [fraction], resulting in three thousand three hundred thirty. Keep it in mind. Then we multiply the numerator of the lower [fraction], which is thirty-two, by the denominators of the upper [fraction], resulting in seven hundred sixty-eight. We drop it from the remembered number, leaving the required number, which is two thousand five hundred sixty-two, and its figure is 2562. We divide it by the denominators

[77] Copied from (Ibn al-Bannāʾ 1994, 277.4-6).

of the two lines. The result is the answer, which is three and five tenths and three sixths of a tenth and half a sixth of a tenth, and its figure is $\frac{1\ 3\ \ 5}{2\ 6\ 10}3$. And its answer is given by subtraction.

149.1 **Section Three, on multiplying fractions.**

149.2 **This is the portioning of one of two fractions by the amount of the other.** This contrasts with whole numbers or the duplication of fractions by the amount of a whole number. If the multiplier is a whole number and the multiplicand is a fraction, or vice versa, then either one extracts the whole number by the amount of the fraction, or one duplicates the fraction by the amount of the whole number.

149.6 **To work it out,** that is the multiplication of fractions**, you multiply the numerator of each line by the numerator of the other, and you divide the result by the denominators.**

149.8 For example, suppose someone said, "Multiply three fourths and a third by three ninths and four sixths of a ninth and a fifth of a sixth of a ninth". We put the multiplier on a line and the multiplicand below it on another line, as in this figure:

$$\frac{1}{3}\,\frac{3}{4}$$

$$\frac{1\ \ 4\ \ 3}{5\ \ 6\ \ 9}$$

We multiply the numerator of the upper [fraction], which is thirteen, by the numerator of the lower [fraction], which is one hundred eleven. It results in one thousand four hundred forty-three, and its figure is 1443. We divide it by the denominators, resulting in the required number, which is four ninths and a fourth of a fifth of a sixth of a ninth, and its figure is $\frac{1\ \ 0\ \ 0\ \ 4}{4\ \ 5\ \ 6\ \ 9}$. And its answer is given by one.

150.2 Another example: suppose someone said, "Multiply a third of four and an eighth by a fifth of two thirds of ten". We put the problem down on two lines as before, like in this figure:

$$\frac{1}{8}\,4\,\frac{1}{3}$$

$$10\,\frac{2\ \bullet\ 1}{3\ \bullet\ 5}$$

We multiply the numerator of the upper [fraction], which is thirty-three, by the numerator of the lower [fraction], which is twenty, resulting in six hundred sixty, and its figure is 660. We divide it by the denominators, resulting in the required number, which is one and five sixths, and its figure is $\frac{5}{6}1$. And its answer is given by two.

151.1 **Section Four, on division and denomination.**

151.2 **To work these out you multiply the numerator of each line by the denominators of the other, and you divide the result for the dividend by the result for the divisor, or you denominate.**

151.4 An example of division: suppose someone said, “Divide six and a third by four fifths of seven eighths of three”. We put the dividend on a line and the divisor on another line below it, as in this figure:

$$\frac{1}{3}\,6$$

$$3\,\frac{7 \bullet 4}{8 \bullet 5}$$

We multiply the numerator of the dividend, which is nineteen, by the denominators of the divisor, resulting in seven hundred sixty. Its figure is 760, which is the result for the dividend. Keep it in mind. Then we multiply the numerator of the divisor, which is eighty-four, by the denominators of the dividend, resulting in two hundred fifty-two. Its figure is 252, which is the result for the divisor. We divide the remembered number by it. The result is the required number, which is three and a seventh of a ninth, and its figure is $\frac{1\;\,0}{7\;\,9}\,3$. And its answer is given by four.

151.14 Another example, of denomination: suppose someone said, “Denominate three and a fourth less two ninths of it with six and two eighths and three fifths”. We put down the problem as in this figure:

$$\frac{2}{9}\;\text{ﻻ}\;\frac{1}{4}\,3$$

$$\frac{3}{5}\,\frac{2}{8}\,6$$

We multiply the numerator of the denominated number, which is ninety-one, by the denominators of the denominating number to get the result of three thousand six hundred forty, and its figure is 3640. Keep it in mind. Then we multiply the numerator of the denominating number, which is two hundred seventy-four, by the denominators of the denominated number to get the result of nine thousand eight hundred sixty-four, and its figure is 9864. We denominate the remembered number with it to get the required number, which is fifty parts of one hundred thirty-seven parts and five ninths of a part of one hundred thirty-seven parts, and its figure is $\frac{5\;\;50}{9\;137}$. And its answer is given by subtraction.

152.9 **When the denominators of the two lines are equal, you divide the numerator by the numerator or you denominate with it, without multiplying by the denominators.**

152.11 An example of division: suppose someone said, “Divide eight and nine tenths and two thirds of a tenth by five tenths and a third of a tenth”. We put down the problem like this:

$$\frac{2\ \ 9}{3\ \ 10}\,8$$

$$\frac{1\ \ 5}{3\ \ 10}$$

We divide the numerator of the dividend, which is two hundred sixty-nine, by the numerator of the divisor, which is sixteen. The result is the required number, which is sixteen and six eighths and half an eighth, as in this figure: $\frac{1\ 6}{2\ 8}\,16$.

153.3 Another example, with denomination: suppose someone said, “Denominate two and a third with six and two thirds”. We put down the problem like this:

$$\frac{1}{3}\,2$$

$$\frac{2}{3}\,6$$

We denominate the numerator of the denominated number, which is seven, with the numerator of the denominating number, which is twenty. This results in the required number, which is three tenths and half a tenth, and its figure is $\frac{1\ \ 3}{2\ \ 10}$.

153.7 **When the two numerators are equal, you divide the denominators of the divisor by the denominators of the dividend, or you denominate, without multiplying by the numerator**, since, if we multiply by the denominators, the dividend will be composed from its numerator and the denominators of the divisor, and the divisor will be composed from its numerator and the denominators of the dividend. The two numerators vanish when the common divisors are removed. This is also the cause for the previous method.

153.12 For example, *suppose someone said, “Divide five by five sixths”. We divide the six,* the denominator of the divisor, *by one.*[78] It gives six, since whenever the dividend or the divisor is a whole number it is its numerator, and its denominator is one.

153.15 Similarly, suppose someone said, “Denominate five sixths with five”. You denominate one with six, giving a sixth. So know it.

154.1 **Section Five, on restoration and reduction.**

154.2 **To work these out you divide the restored number by the number to be restored, or you denominate the reduced number with the number to be reduced, to get the required number.** When he said “required number” he meant what is multiplied by the number to be restored in order to restore, or by the number to be reduced in order to reduce.

[78] Copied from (Ibn al-Bannāʾ 1994, 279.15).

154.6 There are six problems of restoration. One of them is restoration of a fraction to a fraction. For example, suppose someone said, “By how much must we restore a half so that it gives nine tenths?” We divide the nine tenths, the restored number, by the half, the number to be restored. The result is the required number, which is one and eight tenths.

154.10 The second is restoration of a fraction to a whole number and a fraction. For example, suppose someone said, “By how much must we restore two sevenths and half a seventh so that it gives five and a half?” We work this out as shown above to get the required number, which is fifteen and four tenths.

154.13 The third is restoration of a fraction to a whole number. For example, suppose someone said, “By how much must we restore two thirds of five sevenths so that it gives ten?” We work it out as shown above to get the required number, which is twenty-one.

154.16 The fourth is restoration of a whole number to a whole number and a fraction. For example, suppose someone said, “By how much must we restore five so that it gives ten and four sixths?” We work it out as shown above to get the required number, which is two and a tenth and a third of a tenth.

155.2 The fifth is restoration of a whole number and a fraction to a whole number. For example, suppose someone said, “By how much must we restore four and three tenths and half a tenth so that it gives eight?” We work it out as shown above to get the required number, which is one and seventy-three parts of eighty-seven parts.

155.6 The sixth is restoration of a whole number and a fraction to a whole number and a fraction. For example, suppose someone said, “By how much must we restore three and three fifths less a third of one so that it gives twelve and three fifths?” We work it out as shown above to get the required number, which is three and six sevenths. So understand it and you will succeed, almighty God willing.

155.11 And for reduction there are also six problems. One of them is reduction of a fraction to a fraction. For example, suppose someone said, “By how much must we reduce seven tenths so that it becomes a third?” We work it out as described. You denominate the third, the reduced number, with the seven tenths, the number to be reduced. This results in the required number, which is three sevenths and a third of a seventh.

155.16 The second is reduction of a whole number to a whole number and a fraction. For example, suppose someone said, “By how much must we reduce eight so that it becomes two and a half?” We work it out as shown above to get the required number, which is two eighths and half an eighth.

155.19 The third is reduction of a whole number to a fraction. For example, suppose someone said, “By how much must we reduce ten so that it becomes three fourths?” We work it out as shown above to get the required number, which is three fourths of a tenth.

156.3 The fourth is reduction of a whole number and a fraction to a whole number and a fraction. For example, suppose someone said, “By how much must we reduce seven and a fourth so that it gives three and four sixths?” We work it out as shown above to get the required number, which is fourteen parts of twenty-nine parts and four sixths of a part of twenty-nine parts.

156.7 The fifth is reduction of a whole number and a fraction to a whole number. For example, suppose someone said, “By how much must we reduce eleven and nine tenths and four sevenths of a tenth so that it becomes five?” We work it out as shown above to get the required number, which is thirty-eight parts of ninety-three parts and eight ninths of a part of ninety-three parts.

156.12 The sixth is reduction of a whole number and a fraction to a fraction. For example, suppose someone said, “By how much must we reduce two and a third so that it becomes a ninth?” We work it out as shown above to get the required number, which is three sevenths of a ninth. So know it and manage it.

157.1 **Section Six, on converting.**[79]

157.2 ***This section covers two kinds. The intent of one kind concerns only the name,*** *such as when it is said “five sixths and three fourths: how many tenths is it?” We want to denominate these two fractions by naming the fraction in tenths. So we work it out as described, resulting in one and five tenths and five sixths of a tenth, which is what we get from gathering these two fractions together after turning them into tenths. So we shifted the problem from naming in terms of sixths and fourths to naming in terms of tenths and its fractions. This is shifting one kind of fraction to another kind, and it is this kind that is intended in the book.*

157.9 ***The intent of the second kind is: how many of that name, taken as units, are in the whole [fraction]? Work out this kind the same way as for whole numbers. Whenever we want to convert it, we look back to see how it was described in the second type of multiplication.***[80]

157.12 *Suppose someone said, “Five sixths and three fourths: how many tenths are in it?” We multiply them by the whole number ten, resulting in one and five tenths and five sixths of a tenth, which is the answer. This is the amount that comes about from the tenth, which is fifteen tenths and five sixths of a tenth.*[81]

158.1 *Unlike for the first kind, this kind [of conversion] does not require division by the denominator of the converted fraction, since it is similar to the case in which someone said, “Five dirhams, how many tenths are in it?” We multiply the five by the ten to get the result of fifty, which is the answer.* ***The same [rule] can be applied to fractions.*** *Since this kind belongs to the section on multiplication, it is not mentioned* by the author *in the book. He mentioned only the kind specific to the section [on fractions].*[82]

158.7 **To work this out,** that is, the intended kind**, you multiply the numerator of the fraction to be converted by the denominator of the converted fraction. One divides the result first by the denominators of the fraction to be converted and then the result**

[79] Everything from 157.2 to 158.6 is slightly condensed from (Ibn al-Bannāʾ 1994, 279.18-280.20).
[80] At 95.6 above.
[81] Copied from (Ibn al-Bannāʾ 1994, 279.18-280.12).
[82] Copied from (Ibn al-Bannāʾ 1994, 280.16-20). Al-Hawārī added “by the author” and changed Ibn al-Bannāʾ’s “I mentioned” to “he mentioned”.

by the denominator of the converted fraction. The advantage of this kind is that one can convert a fraction into a finer fraction.

158.11 For example, suppose someone said, “Six eighths and four tenths, how many ninths are in it?” We put the fraction to be converted on a line, and the denominator of the converted fraction on a line below it, as in this figure:

$$\frac{4}{10}\ \frac{6}{8}$$

$$\frac{\cdot}{9}$$

We multiply the numerator of the fraction to be converted, which is ninety-two, by the nine, the denominator of the converted fraction, resulting in eight hundred twenty-eight, and its figure is 828. We divide it first by the denominators of the fraction to be converted, then what is left by the denominator of the converted fraction, resulting in the required number. This is one and a ninth and three tenths of a ninth and four eighths of a tenth of a ninth, and its figure is $\frac{4}{8}\frac{3}{10}\frac{1}{9}1$. The same works for similar [examples].

159.5 This completes the chapter, with God’s blessing.

161.1 **Chapter Three, on roots. Related to this, we cover what we intend on this topic in four sections.**

163.1 **Section One, on taking a root of a whole number and a root of a fraction.**

163.2 **These are divided into two varieties, rational and surd.** ***A rational [root] is any number whose ratio to one is known. This can be a whole number, a fraction, or a whole number and a fraction.***

163.4 ***A surd [root] is one whose ratio to one is unknown.*** *For example, a root of ten, a root of a half, and a root of ten and a half. Surds come in two varieties: those that are expressed with [the word] “root” once, like those just mentioned, and which are called rational in square, and those expressed by “root” more than once, like a root of a root of ten, and which are called medial.*[83]

163.9 **And the root is any number, which multiplied by itself, results in the number whose root is sought.** Examples have been given above.

163.11 In language, it is the origin of everything. He said “root” (*jadhr*) with a “dh” and either an “a” or an “i” as the vowel for the “j”. Our professor al-shaykh Abū l-ʿAbbās [Ibn al-Bannāʾ], God be satisfied with him, told me that the “a” is more appropriate.

163.14 *The ranks in a whole number “have a root” and “do not have a root” for each successive place. This is evident by examination for the units and tens. The hundreds have a root because they come from multiplying the tens by themselves, and the thousands do not have a root because they are in relation to the hundreds in the position that the tens are*

[83] Copied from (Ibn al-Bannāʾ 1994, 283.3-7).

in relation to the units, and similarly for what comes after that. A place is said to have a root if there is a number in it that has a root.[84]

164.3 *Some conditions for a number may indicate that it does not have a root. But if they do not hold it only implies that it may have a root. These are:*

164.5 *Any number that begins with a two or a three or a seven or an eight does not have a root.*

164.6 *Any number that begins with a one, and half of its tens is different from the number of hundreds in the even and the odd,*[85] *does not have a root*, like three hundred forty-one and four hundred sixty-one and the like.

164.10 *Any number that begins with a five, and its tens is not twenty, does not have a root*, like seventy-five and one hundred eighty-five and the like.

164.12 *Any number that begins with a six, and its tens are even, does not have a root*, like forty-six and three hundred twenty-six and the like.

164.14 *Any number that does not begin with six, and its tens are odd, does not have a root.*[86]

164.15 Indications for non-square numbers, including those given above, mean the same as knowing that the number does not have a root.

164.17 Indications for [the first digit of] a square are five numbers: a one, with the condition that its tens [place] has a half; a five, with the condition that its tens are twenty; and a six, with the condition that its tens are odd. This leaves nine and four. If one of these is the first digit, as mentioned, and its tens are odd, then it does not have a root, and if it is even then it might.

165.3 *Any number beginning with an odd number of zeros does not have a root*, like ten, twenty-one thousand, three thousand, and the like.

165.5 *Any number beginning with an even number of zeros, and for which the number [remaining after the zeros are deleted] does not have a root, [also] does not have a root*, like five hundred, thirty thousand, and the like.

165.8 *Any number that is not exhausted after casting out nines, and the remainder is not a one or a four or a seven, does not have a root,*[87] like four hundred twenty-five and the like.

165.11 Any number that is not exhausted after casting out eights, and the remainder is not a one or a four, does not have a root, like two hundred sixty-six and the like.

165.14 Any number that is not exhausted after casting out sevens, and the remainder is not a one or a two or a four, does not have a root, like three hundred forty-nine and the like.

166.1 **The way to take a root of a whole number is to count off the ranks with "root", "no root" to the end of the line. Once you arrive at the last "root", put a number**

[84] Copied from (Ibn al-Bannāʾ 1994, 283.8-11).
[85] I.e., the parity of half of the tens digit is different from that of the hundreds digit.
[86] Copied from (Ibn al-Bannāʾ 1994, 284.14-285.1).
[87] Copied from (Ibn al-Bannāʾ 1994, 285.2-285.6).

below it so that if you multiply it by itself, it cancels what is above it or it leaves the smallest possible whole number as remainder. Then you back up, double it, and put it below the place [marked] "no root". Now you look for a number to put below the "root" preceding it, so that when you multiply it by the doubled number one step back and then by itself, it either cancels what is above it or it leaves the smallest possible remainder. Then if it did not cancel, you do it again. You continue doubling the backed-up number and shifting until you have covered all the line. What you then have in the second line, before doubling, is the root.

166.9 **If something remains, denominate it with double the whole part of the root if it is equal to or smaller than the root; and if it is greater than the root, always add one to it, and two to double the root, then you denominate it with it and you add the denomination to the whole part. This gives the root that you multiply by itself, and it is an approximation of a root of the given number.**

166.13 For example, suppose someone said, "How much is a root of six hundred twenty-five?" We put it on a line like this: 625. The first place has a root, the second does not have a root, and the third has a root, as mentioned above. We then look for a number to put below the six, which is in a rank that has a root, such that when we multiply it by itself it either cancels the six or it leaves the smallest possible whole number as remainder. We find that it is two. We multiply it by itself, giving four. We drop it from the six, leaving two, which we put in its place. We back up the doubled two and put it below the place [marked] "no root", which is below the two. Then we look for a number to put below the [rank that] has a root that is before the backed up number below it, and that is below the five. We find it is five. It cannot be anything else.

166.21 If it were a six, you could only put a four or a six below it, and if it were a one, you could only put a one or a nine below it, and if it were a four, you could only put a two or an eight below it, and if it were a nine, you could only put a three or a seven below it. So know it.

167.3 So we multiply the five by the doubled four, giving twenty. The twenty-two above its head cancels the twenty, leaving two in its place. Then we likewise multiply it by itself, giving twenty-five, and above its head is twenty-five, which cancels it. The meaning of the five and the doubled two after it is twenty-five, which is the required number. Work it out the same way if there are more digits.

167.8 Another example: suppose someone said, "How much is a root of twenty?" We know that it does not have a rational root since it begins with one zero. We work it out as explained above, and we divide the remainder according to the calculation described above. We look for a number such that when we multiply it by itself, it cancels it or it leaves the smallest possible remainder. We find it is four. We multiply it by itself, giving sixteen. The remainder is four, which is equal to the root. We denominate it with its double, which gives a half. We add it to the whole part of the root, giving four and a half, which is a root of twenty by approximation.

167.14 Another example: suppose someone said, "How much is a root of fifty-four?" We work it out as before, to get seven for the whole part of the root and five for the remainder, which is smaller than the root. We denominate it with its double, giving two sevenths and half a seventh. We add it to the root, to get the sum of seven and two sevenths and half a seventh, which is the required root by approximation.

168.1 Another example: suppose someone said, “How much is a root of ninety-two?” We work it out as before to get nine for the whole part of the root and eleven for the remainder, which is greater than the root. So we add one to it, and two to double the nine, which is the root. We denominate the smaller with the greater and we add it to the root. The result is the required root, which is nine and three fifths. If we multiply the nine and three fifths, which is the approximated root, by itself, it results in ninety-two and four fifths of a fifth. The approximation is found with the added fraction.

168.8 **And if you want to refine the approximation, denominate it with double the root. Drop the result from the root, leaving a root whose square is closer to the number whose root is required than the first square.**

168.10 He said “denominate it”, by which he meant the fraction added to the square that was found by approximation; because taking the root by approximation can be obtained from a previous close smaller square as seen above, and it can also be obtained from a previous close greater square, which is what he meant when he said “denominate it, etc.”. The description of how to work it out is that we drop the number whose root is required from the square, and we denominate the remainder with double a root of the square, and we also subtract the result from a root of the square. The remainder is a root of the number by approximation.

168.16 If we want to take a root of ninety-two using the previous greater square, we make the greater square ninety-two and four-fifths of a fifth. If we drop the number from it, the remainder will be the fraction found by approximation. So we denominate it with double the root, as mentioned, to get half a sixth of a tenth. We drop it from the root, leaving nine and five-tenths and five-sixths of a tenth and half a sixth of a tenth. A square of this remainder is closer than the first square. So know it.

169.1 **There is another method of approximation, which is that you multiply the number whose root is required by a greater square number, and you take a root of the result by approximation, and you divide by a root of the multiplied square. The result is the approximated root.**

169.4 For example, suppose someone said, “How much is a root of twelve?” We multiply it by sixteen, for instance, resulting in one hundred ninety-two. We take its root to get thirteen and six-sevenths. We divide it by a root of sixteen. The result is a root of twelve by approximation, and that is three and three-sevenths and a fourth of a seventh.

169.8 *The condition “the smallest possible whole number”*[88] *[is needed] because if one diverges from the well-known method by working with fractions, then the remainder will be smaller than the remainder with whole numbers.*[89]

169.10 *And suppose someone said, “How much is a root of six hundred twenty-five?”* This example is from the earlier problem. *We put two and a half below the six, so its square is six and a fourth. This exceeds [the six]. So you take away the six with the six, and take away the fourth with the twenty-five, which is a fourth of a hundred. So all the numbers vanish, and the remainder begins with the last two zeros. We take one of them. Thus the two and a*

[88] From 166.3 and 166.16 above.
[89] Copied from (Ibn al-Bannāʾ 1994, 283.11-12).

half are tens, which is twenty-five. We double the two and a half to get five, and we place it below the tens. And we look for something to multiply by the doubled number. We find it to be nothing, since there are zeros above it. We put down a zero and we halve what we doubled, so it yields half of fifty.[90]

169.17 Another *example: suppose someone said, "How much is a root of seven hundred twenty-nine?" If we work it out with whole numbers, then the remainder from the seven, which is seven hundred, in its rank is three. And if we work it out with fractions, then the remainder is smaller. So if we put two and a half below it, three fourths of one remains in that rank, and three fourths of a hundred is seventy-five. We add it to the twenty-nine that is with it, to get one hundred four. Then we back up the doubled two and a half, which is five, and we look for a number to multiply by the five and by itself. We find it is two, and nothing remains of the number. So we halve what we doubled, which is fifty. Its half is twenty-five, so the total root is twenty-seven. Or we double the two, so it becomes fifty-four, and we take half of it.*[91]

170.6 Another example: suppose someone said, "How much is a root of a hundred?" The basic rule for this and similar problems is that we always take half of the number of zeros, and we add them to a root of the remaining number to get the root. The hundred begins with two zeros. We take one of them and we add it to a root of the remaining one, giving ten, which is a root of the hundred. So know it.

170.10 **To take a root of a fraction, you multiply the numerator by the denominator and you divide a root of the result by the denominator. If the numerator has a rational root and the denominator does too, then divide a root of the numerator by a root of the denominator.**

170.14 WITH REGARD TO TAKING A ROOT, I MEAN OF A FRACTION, THERE ARE FOUR TYPES. IN ONE OF THEM THE NUMERATOR HAS A RATIONAL ROOT AND THE DENOMINATOR DOES TOO. WORK IT OUT AS DESCRIBED.

170.16 For example, suppose someone said, "How much is a root of four-sixths and a sixth of a sixth?" Its figure is $\frac{1\ 4}{6\ 6}$. We take a root of the numerator, giving five. We divide it by a root of the denominator, which is six. The result is the required root, and that is five-sixths.

171.1 Another example: suppose someone said, "How much is a root of twelve and a fourth?" We take a root of the denominator, giving two. We divide seven, a root of the numerator, by it. The result is the required number, which is three and a half. And if we wish, we can work it out by the first method, since it is general, while this one is particular. So know it.

171.5 IN THE SECOND [TYPE] NEITHER OF THEM HAS A RATIONAL ROOT, SO WORK IT OUT BY THE FIRST METHOD.

171.6 For example, suppose someone said, "How much is a root of four-ninths and three sixths of a ninth?" Its figure is $\frac{3\ 4}{6\ 9}$. We multiply the numerator by the denominator to get one thousand four hundred fifty-eight. We take its root, which is thirty-eight and three parts of nineteen parts and half a part of nineteen parts. We divide it by the denominator. The result is the required number, which is thirteen parts of nineteen parts and three ninths of a

[90] Copied from (Ibn al-Bannāʾ 1994, 284.4-9).
[91] Copied from (Ibn al-Bannāʾ 1994, 283.13-284.3).

part of nineteen parts and five-sixths of a ninth of a part of nineteen parts and half a sixth of a ninth of a part of nineteen parts, and its figure is $\frac{1\ 5\ 3\ 13}{2\ 6\ 9\ 19}$.

171.13 In the third [type] the denominator has a rational root but the numerator does not have a rational root. For this type, if we wish, we can work it out by the first [method] or by the second.

171.15 For example, suppose someone said, "How much is a root of ten and seven-eighths and half an eighth?" Its figure is $\frac{1\ 7}{2\ 8}10$. If we want, we can work it out by the first method. We multiply the numerator by the denominator, resulting in two thousand eight hundred. We take its root, which is fifty-two and forty-eight parts of fifty-three parts and half a part of fifty-three parts. We divide it by the denominator to get the required number, which is three and sixteen parts of fifty-three parts and two eighths of a part and a fourth of an eighth of a part of fifty-three parts, and its figure is $\frac{1\ 2\ 16}{4\ 8\ 53}3$.

172.6 By the second method, we take a root of the numerator, which is thirteen and three parts of thirteen parts. We divide it by a root of the denominator. The result is the required number, which is three and four parts of thirteen parts, and its figure is $\frac{4}{13}3$. This method is closer than the first [method].

172.10 In the fourth [type] the numerator has a rational root and the denominator does not have a rational root, so work it out by the first method.

172.12 For example, suppose someone said, "How much is a root of four sevenths and half a seventh?" Its figure is $\frac{1\ 4}{2\ 7}$. We multiply the numerator by the denominator and we take a root of the result. This is eleven and two parts of eleven parts and half a part of eleven parts. We divide it by the denominator. The result is the required number, which is eight parts of eleven parts and five sevenths of a part and three fourths of a seventh of a part of eleven parts, and its figure is $\frac{3\ 5\ 8}{4\ 7\ 11}$.

173.2 Of the four types, a root of the first one is found exactly, and the other three are found by approximation. If we wish to approximate [more accurately] the root like we did for whole numbers, then follow the same procedure.

173.4 **To take roots of binomials and apotomes, you drop a fourth of a square of the smaller of the two terms from a fourth of a square of the greater one; you take a root of the remainder, you add it to half of the greater term, and you also subtract it from half of the greater term; and you drop a root on each of them. If the number whose root is required is a binomial, then its root is the sum of these two roots, and if it is an apotome, then its root is the difference between these two roots.**

173.10 I say we need to begin with an introduction to clarify binomials and apotomes and how to find them. After that we will take their roots, almighty God willing.

173.12 We say that *there are six binomials and six apotomes.*

173.13 *A binomial is a number and a root of a number, or a root of a number and a root of a number in which the two can only be joined with the coordinating conjunction.*[92] *For example, five and a root of three, and a root of five and a root of three.*

[92] The coordinating conjunction is the word "and" (*wa*).

173.16 *An apotome consists of two terms in which the smaller term is removed from the greater by the particle of exclusion.*[93] *For example, five less a root of three, and a root of five less a root of three.*

174.1 *Roots of the first three binomials and apotomes are closer to being rational in rank than roots of the other three. The first three can be distinguished from the others by multiplying the difference between the squares of the terms by a square of the greater one. If the result is a square, then it is one of the first three, and if it is not a square, then it is one of the second three.*

174.5 *The greater term is rational in the first and the fourth.* An example of the first is five and a root of twenty-one, and an example of the fourth is two and a root of two.

174.8 *The smaller is rational in the second and the fifth.* An example of the second is five and a root of forty-five, and an example of the fifth is five and a root of seventy-two.

174.11 *Neither of them is rational in the third and the sixth.* An example of the third is a root of ten and a root of eighteen, and an example of the sixth is a root of seven and a root of eight.

174.14 *It is necessary to recall their characteristics if we want to find them. If we subtract a square from a square and the remainder is not a square, and we join a root of the remainder with a root of the greater square, this gives the first binomial.* Or we make whatever number we wish the greater term, and [we make] the smaller a root of a surface whose two sides are in numerical ratio,[94] with the condition that it is not rational. However, this procedure is rare.

175.1 *We subtract a non-square number from a square such that the remainder is not a square, and we join a root of the remainder with a root of the square to get the fourth binomial.*

175.3 *We multiply two squares by their difference, which gives non-squares, and we join a root of the greater of the two results with a root of their difference to get the second binomial.*

175.5 *We multiply two squares by something other than their difference, which gives non-squares, and we join a root of the greater of the two results with a root of their difference to get the third binomial.*

175.7 *We add a square to a square so that the sum is not a square, and we join a root of the sum with a root of one of the two squares to get the fifth binomial.*

175.9 *We add a non-square number to a square so that the sum is not a square, and we join a root of the sum with a root of the added number to get the sixth binomial.*[95]

175.11 These are the six binomials. Now let us return to examples of their roots. Example: suppose someone said, "Eight and a root of sixty: how much is a root of that?" We work this out as described. We drop a fourth of a square of a root of the sixty, since it is the smaller, and that is fifteen, from a fourth of a square of the eight, since it is the greater, and that is sixteen. The remainder is one. We take its root, giving one. We add it to half of the eight,

[93] The particle of exclusion is most often the word "less" (*illā*).
[94] I.e., they are commensurable.
[95] Copied from (Ibn al-Bannāʾ 1994, 287.19-288.19).

since it is the greater, giving five, and we drop it likewise from its half. The remainder is three. We drop a root on the five and the three. This gives a root of five and a root of three, which is the required number. The same rule applies to other [examples].

175.19 And suppose someone said, "Eight less a root of sixty: how much is its root?" We work this out as before. We drop a root of the three from a root of the five. A root of their difference is the required root, and that is a root of the five less a root of the three.

176.1 Another way to do it is that we drop a square of the smaller term from a square of the greater one, and we take a root of the remainder. We add it to the greater term and we take a root of half of the sum. And likewise, we subtract it from the greater term and we take a root of half of the sum. If the number whose root is required is a binomial, then its root is the sum of these two roots. If it is an apotome, then its root is the difference between the two roots.

176.6 Suppose someone said, "Eight and a root of fifty-five: how much is its root?" This is the first binomial. We take its root as before, to get a root of five and a half and a root of two and a half. It is called "one of the binomials". Its apotome is the first apotome, and an apotome of its root is a root of its apotome, and it is called "one of the six apotomes".

176.10 Suppose someone said, "Seven and a root of one hundred twelve: how much is its root?" This is the second binomial. We take its root as before to get, after the addition and subtraction, a root of a root of eighty-five and three fourths and a root of a root of one and three fourths. This is called "the first bimedial", and its apotome is the second apotome. An apotome of its root is a root of its apotome, and it is called "a first apotome of a medial".

176.15 Suppose someone said, "A root of thirty-two and a root of fourteen: how much is its root?" This is the third binomial. We take its root as before. This gives, after the addition and subtraction, a root of a root of twenty-four and a half and a root of a root of a half. This is called "the binomial of the second bimedial", and its apotome is the third apotome. An apotome of its root is a root of its apotome, and it is called "a second apotome of a medial".

176.20 Suppose someone said, "Seven and a root of thirty: how much is its root?" This is the fourth binomial. We take its root as before, to get three and a half and a root of four and three fourths, taking its root, and three and a half less a root of four and three fourths, taking its root. It is called "the major", and its apotome is the fourth apotome. And an apotome of its root is a root of its apotome, and it is called "the minor".

177.5 Suppose someone said to you, "Three and a root of twenty: how much is its root?" This is the fifth binomial. We take its root as before, to get a root of five and a root of two and a half and a fourth, taking its root, and a root of five less a root of two and a half and a fourth, taking its root. It is called "[the number whose] power is a rational and a medial". Its apotome is the fifth apotome, and an apotome of its root is a root of its apotome. It is called "the joining with a rational to become a whole medial".

177.11 Suppose someone said, "A root of ten and a root of eleven: how much is its root?" This is the sixth binomial. We take its root as before, to get a half and a root of two and three fourths, taking its root, and a root of two and three fourths less a half, taking its root. It is called "[the number whose] power is a bimedial". Its apotome is the sixth apotome, and an apotome of its root is a root of its apotome. It is called “the joining with a medial to become a whole medial”.

179.1 **Section Two, on adding and subtracting roots of numbers.**

179.2 **You multiply the two numbers, one of them by the other, when you want to add or to subtract their roots. If the result is a square, then the roots of the two numbers can be added and subtracted. If it is not a square, then they cannot be added or subtracted. If you know they can be added, take two roots of the result and add it to the sum of the two numbers. Take a root of this sum to get the required number.**

179.7 For example, suppose someone said, “Add a root of three to a root of twenty-seven”. We multiply the three by the twenty-seven, giving eighty-one, which is a square. We take two of its roots, giving eighteen. We add it to the sum of the two numbers, and a root of that sum is the required number, which is a root of forty-eight.

179.11 Another way is that we divide one of the two addends by the other, we add one to the result, and we multiply the sum by the divisor. The result is their sum.[96] So here we divide a root of the twenty-seven by a root of the three, resulting in three. We add one to it, and we multiply the sum by a root of the three, the divisor, as explained [in the section on] multiplying roots.[97] This results in a root of forty-eight, which is the required number, as shown above.

179.16 Another example: suppose someone said, “Add a root of two to a root of eight”. We multiply the two by the eight, giving sixteen, and we take two of their roots, giving eight. We add it to the sum of the two numbers, and a root of that sum is the required number, and that is a root of eighteen. If we wish, we can work it out the second way to get the required number.

179.20 Another example: suppose someone said, “Add half a root of twenty to two roots of five”. Half a root of twenty is less than one root, so we transform it to one root, as described in the section on division.[98] This gives, according to what was explained on working out the multiplication, a root of five. And two roots of five are more than one root, so we transform them to one root to again get a root of twenty. It is as if someone had said, “Add a root of five to a root of twenty”. We work it out as before to get the required number, which is a root of forty-five.

180.6 Likewise, whenever the roots are of different ranks, we transform them to one rank. For example, suppose the addend is a root of a number rational in square and the augend is a root of a root of a number, that is, a medial. We transform [the number] rational in square to a medial in ratio[99] with the other [addend], at which point we add them.[100]

180.10 Another example: suppose someone said, “Add a root of three to a root of fifteen”. We find that their surface is not a square, so they are incommensurable. We thus add them with the

[96] Judging by the corresponding passage at 181.10, this sentence should have been attributed to Ibn al-Bannāʾ. It is not in the *Condensed Book*, so it should be in SMALL CAPS.
[97] At 183.1 below.
[98] At 187.1 below.
[99] I.e., commensurable.
[100] Judging by the passage at 182.1, this passage should have been attributed to Ibn al-Bannāʾ. It is not in the

coordinating conjunction to get a root of three and a root of fifteen. Any examples like this, that can only be added with the coordinating conjunction, are called binomials.

180.15 Another example: suppose someone said, "Add half a root of a root of eighty to a third of a fourth of a root of six hundred eighty-four". It is known, according to what is explained in the section on multiplication,[101] that half a root of a root of eighty is a root of a root of five, and that a third of a fourth of a root of six hundred eighty-four is a root of four and three fourths. It is as if someone had said, "Add a root of a root of five to a root of four and three fourths". We transform them to the same rank, as mentioned before. So the problem becomes as if someone had said, "Add a root of a root of five to a root of a root of twenty-two and four eighths and half an eighth". Their surface is likewise not a square, so we add them with the coordinating conjunction, and that is a root of a root of five and a root of a root of twenty-two and four eighths and half an eighth. So know it.

181.4 **For subtraction, you subtract two of the roots that result from multiplying the two numbers from the sum of the two numbers, and you take a root of the remainder to get the required number.**

181.6 For example, suppose someone said, "Subtract a root of eight from a root of thirty-two". We multiply the eight by the thirty-two, giving two hundred fifty-six. We drop two of its roots, which are thirty-two, from the sum of the two numbers, and we take a root of the remainder. It is the required number, which is a root of eight.

181.10 ANOTHER WAY TO DO THIS IS TO DIVIDE ONE OF THE SUBTRAHENDS BY THE OTHER. THEN TAKE THE DIFFERENCE BETWEEN THE RESULT AND ONE, AND MULTIPLY BY THE DIVISOR TO GET THE REQUIRED NUMBER.

181.12 For example: suppose someone said, "Subtract a root of twelve from a root of twenty-seven". We divide a root of the twenty-seven by a root of the twelve, resulting in one and a half. We take the difference between it and the one, which is a half. We multiply it by a root of the twelve, which is the divisor, resulting in a root of three, which is the required number.

181.16 If we denominate a root of the twelve with a root of twenty-seven, it results in two thirds. We take the difference between it and the one, and that is a third. We multiply it by a root of the twenty-seven, the denominating number, and the result is the required number, which is a root of three. So know it.

182.1 IF THE SUBTRAHEND OR THE MINUEND IS MORE THAN ONE ROOT OR LESS, OR IF THE RANKS OF THEIR ROOTS ARE DIFFERENT, THEN IT IS NECESSARY TO TRANSFORM THEM TO ONE ROOT OR THE SAME RANK, AS WITH ADDITION.

182.4 Another example: suppose someone said, "Subtract a root of eight from a root of ten". We find that their surface is not a square, so they are incommensurable. So subtract it using the particle of exclusion, and that is a root of ten less a root of eight. Work out [other problems] similarly. So know it. Here, too, any examples like this one that can only be subtracted using the particle of exclusion are called apotomes. So understand it.

Condensed Book, so it should be in SMALL CAPS.

[101] At 186.11 below.

183.1 **Section Three, on multiplying roots.**

183.2 **To work this out you multiply one of the numbers by the other and you take a root of the result. This is the result of multiplying a root of one of them by a root of the other.**

183.4 For example, suppose someone said, “Multiply a root of eight by a root of nine”. We multiply the eight by the nine, and a root of the outcome is the required number, which is a root of seventy-two.

183.7 Another example: suppose someone said, “Multiply a root of a root of five by a root of a root of seven”. We multiply the five by the seven, and we drop a root of the root, which the two multiplicands had, on the result. This gives the required number, which is a root of a root of thirty-five. The rule is similar for other medials, no matter how far from rational they are.

183.11 Another example: suppose someone said, “Multiply a root of a root of a root of three by a root of a root of a root of eight”. We multiply the eight by the three, and we drop a root of a root of the root on the result to get the required number, and that is a root of a root of a root of twenty-four.

183.15 Another example: suppose someone said, “Multiply three by a root of seven less two”. To work this out we multiply the three by a root of the seven, and we subtract from the outcome its multiplication also by the excluded two. The remainder is the required number, which is a root of sixty-three less six. The principle behind multiplying appended and deleted terms will be covered in [the chapter on] algebra,[102] with the power of almighty God.

183.20 **If you want to multiply a number by a root of a number, square the number and work with the two numbers as shown above.**

184.1 For example, suppose someone said, “Multiply three by a root of seven”. We multiply the three by itself, giving nine. Then it is as if someone had said, multiply a root of nine by a root of seven. We work it out as before, resulting in the required number, which is a root of sixty-three.

184.4 Another example: suppose someone said, “Multiply two by a root of a root of three”. We multiply the two by itself, and the result by itself. This is multiplied by the three. A root of a root of the outcome is the required number, which is a root of a root of forty-eight. The rule is similar if there are more [roots] than these. So know it.

184.8 From this principle, I mean his saying “if you want to multiply a number by a root of a number, etc.”, one knows how to work it out by transforming the problem to one root when there is a term with more than one root or less than one root, and by duplicating a root of a number or partitioning it.

184.11 Here is an example with a root that is more than one root. Suppose someone said, “Multiply two by two roots of seven”. It is necessary that we focus on what [makes] this number of a root of the seven a [single] root. To do this we multiply the two, which is the number

[102] At 228.9 below.

of the root, by itself, and the outcome by the seven. We take a root of the result, which is the required number, and that is a root of twenty-eight. So it is as if someone had said, “Multiply two by a root of twenty-eight”. We work it out as before, resulting in the required number, which is a root of one hundred twelve.

184.18 Another example: suppose someone said, “Multiply five by three roots of a root of two”. So we likewise focus on what number makes three roots of a root of two a [single] root of a root. To do this we multiply the three, the number of roots, by itself, and the outcome by itself, and the outcome of that by the two, and we take a root of a root of the result. This is the required number, which is a root of a root of one hundred sixty-two. So it is as if someone had said, “Multiply five by a root of a root of one hundred sixty-two”. We work it out as before to get the required number, which is a root of a root of one hundred one thousand two hundred fifty. Work it out similarly if there are more [roots] than these.

185.6 Here is an example with less than one root: suppose someone said, “Multiply two thirds by half a root of twenty”. So we focus on half a root of twenty, and we change it into what [single] root it is, as before. To do this we multiply the half by a root of the twenty, so it gives, as explained, a root of the five. It is then as if someone had said, “Multiply two thirds by a root of five”. We again work it out as before. The result is the required number, which is a root of two and two ninths.

185.12 Another example: suppose someone said, “Multiply a root of five by half a root of a root of forty”. We focus on half a root of a root of forty, and we change it into what root of a root it is, as before. To do this we multiply the half by itself, and the outcome by itself, and the outcome of that by the forty. We get, as we have explained, a root of a root of two and a half. It is then as if someone had said, “Multiply a root of five by a root of a root of two and a half”. We work it out again as before, resulting in the required number, which is a root of a root of sixty-two and a half. Work out similar [examples] the same way.

186.1 Here is an example of duplicating roots: suppose someone said, “Duplicate a root of three twice”. It is as if someone had said, “Multiply two by a root of three”. Work it out as before to get the required number, which is a root of twelve.

186.4 Another example: suppose someone said, “Duplicate a root of seven five times”. It is as if someone had said, “Multiply five by a root of seven”. Work it out as before to get the required number, which is a root of one hundred seventy-five. Work out similar [examples] the same way.

186.8 An example of partitioning roots: suppose someone said, “How much is half of a root of ten?” It is as if someone had said, “Multiply a half by a root of ten”. Work it out as before to get the required number, which is a root of two and a half.

186.11 Another example: suppose someone said, “How much is a third of four eighths of a root of a root of sixty?” It is as if someone had said, “Multiply a third of four eighths by a root of a root of sixty. Work it out as before to get the required number, which is a root of a root of two sixths of a ninth and half a sixth of a ninth. So know it.

187.1 **Section Four, on dividing and denominating roots of numbers.**

187.2 **You divide the number by the number or you denominate it with it, and you take a root of the result. What you get is the result of dividing a root of the dividend by a root of the divisor.**

187.4 For example, suppose someone said, “Divide a root of twenty by a root of three”. We divide the twenty by the three, and we drop the root on the result to get the required number, which is a root of six and two thirds.

187.7 Another example: suppose someone said, “Divide a root of three by a root of eight”. We denominate the three with the eight, and we drop the root on the result to get the required number, which is a root of three eighths.

187.10 Work it out similarly for medials. For example, suppose someone said, “Divide a root of a root of six by a root of a root of two”. We divide the six by the two, and we drop a root of the root on the result to get the required number, which is a root of a root of three.

187.14 Another example: suppose someone said, “Divide a root of a root of eighteen by a root of a root of thirty-two”. We denominate the eighteen with the thirty-two, and we drop a root of the root on the result to get the required number. This is a root of a root of four eighths and half an eighth. The same rule applies to other medials, no matter how far from rational they are.

188.1 **In these [preceding] three sections,** ON ADDITION, MULTIPLICATION, AND DIVISION, **whenever you encounter a term that is more than one root or less than one root, or roots of different ranks, transform them to one root or make them the same rank.**

188.4 I say I have already introduced examples of these and how to work them out in the sections on addition and multiplication, but let us also present some examples in this section.

188.6 Along these lines, suppose someone said, “Divide a root of a root of fourteen by a root of two”. The divisor is rational in square and the dividend is medial. So we transform a root of the two so that it is medial like the divisor, and then we divide. We get a root of a root of four. If we divide the dividend by it and we drop a root of the root which the two dividends had on the result, it gives the required number, and that is a root of a root of three and a half.

188.11 Another example: suppose someone said, “Divide two roots of fifteen by two”. We already knew that two roots of fifteen are the same as a root of sixty, and that the two is a root of four. So it is as if someone had said, “Divide a root of sixty by a root of four”. We work it out as before to get the required number, which is a root of fifteen.

188.15 Another example: suppose someone said, “Divide half a root of twenty-four by a root of two”. We already knew that half a root of twenty-four is the same as a root of six. So it is as if someone had said, “Divide a root of six by a root of two”. We work it out as before to get the required number, which is a root of three.

188.18 **For division by binomials and apotomes, you multiply the dividend and the divisor by an apotome of the divisor if it is a binomial, or by its binomial if it is an apotome. Then you divide the result of the dividend by the result of the divisor.**

189.1 For example, suppose someone said, “Divide twelve by five and a root of three”. We multiply the twelve, the dividend, by the five and a root of three, an apotome of the divisor, and we distribute across the appended and the deleted terms, as will be made clear in [the chapter on] algebra,[103] to get sixty less a root of four hundred thirty-two. This is the result for the dividend. Then we divide it by the result of multiplying the five and a root of three, the divisor, by the five less a root of three, its apotome, and that gives twenty-two, since any binomial multiplied by its apotome, or vice versa, results in the difference between the squares of the terms. The result from the division is the required number, which is two and eight parts of eleven parts less a root of nine parts of eleven and nine parts of eleven parts of eleven.

189.11 Another example: suppose someone said, “Divide ten by three less a root of seven”. We multiply the ten, the dividend, by the three and a root of seven, the binomial of the divisor, to get thirty and a root of seven hundred. We divide it by the result of multiplying the three less a root of seven, the divisor, by its binomial, and that is two. The result of the division is the required number, which is fifteen and a root of one hundred seventy-five. So know it.

189.16 This completes Chapter Three, with the blessing and help of God.

191.1 **Part Two, on the basic rules by which one arrives at knowledge of the required unknown from the posited known.**

191.4 **This is divided into two chapters: a chapter on working out [problems] with proportion, and a chapter on algebra.**

193.1 **Chapter One, on working out [problems] with proportion. This can be done in two ways, with four proportional numbers and with scales.**

195.2 I say there are different kinds of proportions. Among them are arithmetical, harmonic, harmony, and ex-aequali proportion. For arithmetical [proportion], its principles have already been covered,[104] and for the other three, they are not described.[105] As mentioned, they are dispensable, since they originate from it, and it is the foundation of calculation. All three derive from it and it does not derive from them, as was shown in *Lifting the Veil*.[106]

195.7 **Four proportional numbers are those for which the ratio of the first to the second is as the ratio of the third to the fourth, and the product of the first by the fourth equals the product of the second by the third.**

195.9 **Whenever you multiply the first by the fourth and you divide by the second, it results in the third; or by the third, it results in the second. And whenever you multiply the second by the third and you divide by the first, it results in the fourth; or by the fourth, it results in the first. Whichever is unknown can be worked out from the other three known numbers.**

[103] As in the example at 228.11.
[104] At 73.17 above.
[105] I.e., in Ibn al-Bannāʾ’s *Condensed Book*.
[106] (Ibn al-Bannāʾ 1994, 293.17). By “as mentioned” he means “as mentioned in *Lifting the Veil*”. See our commentary at 195.2.

195.14 **The way to do this is that you multiply the isolated number different in kind from the others by the number in ratio with the unknown, and then you divide by the third number, resulting in the unknown.**

195.16 For example, the ratio of three to six is as the ratio of four to eight. So the three with respect to the six is a half, and the four with respect to the eight is a half. The product of the first, which is the three, by the fourth, which is the eight, equals the product of the six, which is the second, by the four, which is the third.

195.19 If we multiply the three by the eight and we divide it by the six, it results in the four; or by the four, it results in the six. And similarly, if we multiply the six by the four and we divide by the three, it results in the eight; or by the eight, it results in the three.

196.3 Suppose in this example that the fourth is unknown, which is the eight, and we want to find it. So we work it out as mentioned. We multiply the isolated number – different in kind from the other two, namely in its classification, which is what is intended, which is discussed in this book,[107] and which in this example is six, since it is related to it, and thus the remaining two are related – by the number in ratio with the unknown, which is the four, since the fourth proportional is unknown. This gives twenty-four. We divide it by the third of the three remaining [numbers], which is the first, resulting in eight, which is the unknown fourth [number]. If the first is unknown, we divide the twenty-four by the eight to get the three.

196.10 If the second is unknown, which is the six, then we multiply the eight – which is the isolated number, since it is the one related to it, thus the remaining two are related – by the number in ratio with the unknown, which is the first, since the second is the unknown in this proportion, giving twenty-four. We divide it by the four, which is the third of the remaining three. The result is six, which is the unknown second. If the third is unknown, which is the four, we divide the twenty-four by the six, resulting in the four. So know it.

196.16 For the four proportional numbers, *if one switches to get the ratio of the first to the third and the second to the fourth; or reverses them to get the ratio of the second to the first and the fourth to the third; or combines to get the ratio of the sum of the first and the second to one of them is as the ratio of the sum of the third and the fourth to one of them; or separates to get the ratio of the difference between the first and the second to one of them is as the ratio of the difference between the third and the fourth to one of them; or combines after switching them; or separates after switching them; or switches after combining them; or switches after separating them; or reverses after any one of these, they will remain proportional.*[108]

197.4 There are four ways to find the unknown other than those mentioned. First, if the fourth, for example, is unknown, we divide the second by the first, and we multiply the result by the third, to get the fourth. Second, we divide the third by the first, and we multiply the result by the second, to get the fourth. Third, we divide the first by the second, and we divide the third by the result, to get the fourth. Fourth, we divide the first by the third, and we divide the second by the result, to get the fourth. These are not presented by the author because what he presented is the source of all of them. They derive from it, and it does

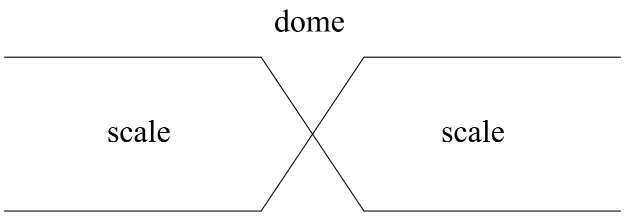
[107] Just above, at 195.14. Ibn al-Bannāʾ does not discuss it in *Lifting the Veil*.
[108] Copied from (Ibn al-Bannāʾ 1994, 294.3-9).

not derive from them. As the jurist Abū Muḥammad ʿAbd al-Ḥaqq ibn Ṭāhir reported, the first way is called the procedure, and the other four are called deductions by analogy.

198.2 **The [method of] scales comes from the art of geometry.** It comes from the art of geometry because ***the ratio of the error of each scale to the difference between the scale and the unknown number is as the ratio of the assigned number to the unknown.***[109]

198.4 You switch, you separate, you switch. The ratio of the portion to its scale is as the ratio of the assigned number to the unknown, as was made clear in *Lifting the Veil*.[110]

198.7 **You represent this by drawing a balance, as in this figure:**[111]



198.8 **You place the given assigned number above the dome. You choose any number you wish for one of the scales, and you perform the prescribed additions, reductions, and so forth from among the operations. Then you confront it with what is above the dome. If you got it right, then this scale is the unknown number. If you got it wrong, then write the error above the scale if it exceeds, or below if it falls short.**

198.13 **Then you choose for the other scale any number you wish other than the first [number], and you work it through as you did with the first [scale]. Then multiply the error of each scale by the posited number**[112] **in the other [scale]. Then look. If the two errors both exceed or both fall short, then subtract the smaller from the greater, and the smaller of the two products from the greater. Then divide the remainder from the two products by the remainder from the two errors. And if one of them exceeds and the other falls short, you divide the sum of the products by the sum of the errors,** resulting in the required number.

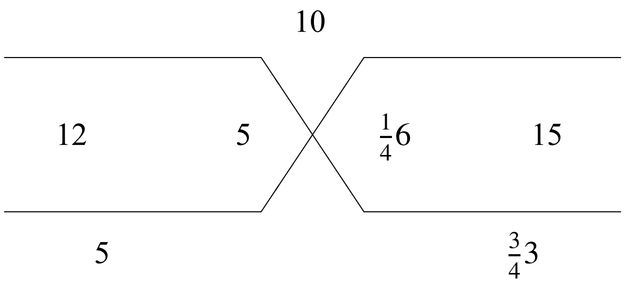
199.1 For example, suppose someone said, "A quantity: taking away its third and its fourth leaves ten. How much is the quantity?"

199.2 We draw the balance as described, and we place the given ten above the dome. Then we choose fifteen for the first scale. We place it between the lines of the balance. Then we take its third and its fourth, which are eight and three fourths, and we drop it from it, leaving six and a fourth, which is the portion that we confront with what is above the dome. So we place it inside the balance, next to the scale. The remainder from the ten above the dome,

[109] Copied from (Ibn al-Bannāʾ 1994, 297.17-18).
[110] (Ibn al-Bannāʾ 1994, 301.4ff).
[111] This is the only figure in Ibn al-Bannāʾ's *Condensed Book*.
[112] Literally "the whole [number]", i.e., the number in its entirety before performing operations and taking the difference with the number above the dome. We also translate this as "posited number" in the passages

after the confrontation, is three and three fourths. This is the error for the scale, which falls short. We place it below the scale as mentioned. If the portion confronted with it were, for instance, ten, then the scale would be the unknown.

199.9 We choose twelve for the second scale. We also place it inside the balance, on the other side, and we take its third and its fourth, giving seven. We subtract it from it, leaving five, which is also the portion we confront it with. We place it inside the balance, next to the scale. The remainder from the ten after the confrontation is likewise five, which is the error for the second scale, and it falls short. We place it below the scale, as in this figure:

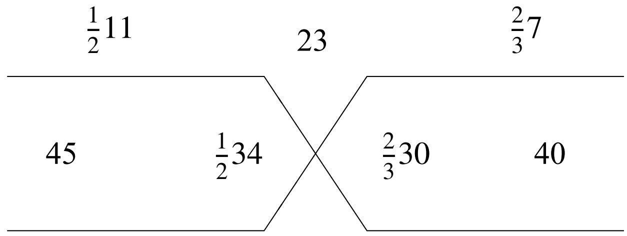


199.14 Then we multiply the three and three fourths, the first error, by the twelve, the second posited number, to get forty-five, which is one of the two products. Then we multiply the five, the second error, by the fifteen, the first posited number, to get seventy-five, which is the second product. From this we drop the first product, which is smaller, since the two errors fall short, leaving thirty. We keep it in mind. Then we drop the first error, since it is smaller, from the second error, since it is greater. The remainder is one and a fourth. We divide the remembered number by it. The result is the unknown quantity, which is twenty-four.

200.1 Another example: suppose someone said, "A quantity: we take the sum of its third and its fifth, and we add to it half of the remainder, so it comes to twenty-three. How much is the quantity?"

200.3 We draw the balance similarly and we place the twenty-three above its dome. We choose forty for one of the scales. We take its third and its fifth, and we add half of the remainder to it, as mentioned, to get thirty and two-thirds. This is the portion we confront with what is above the dome. The error is seven and two-thirds, which is an excess. We choose forty-five for the second scale. We take its third and its fifth and half of the remainder to get thirty-four and a half, which is the portion we also confront. The error is eleven and a half, which exceeds. So we place it and the first [error] above the scales, to get this figure:

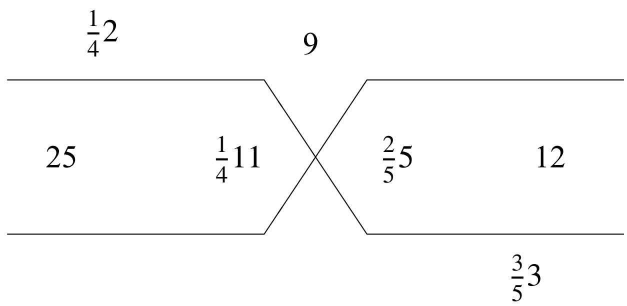
at 199.14, 200.10, 201.10, and 201.14.



200.10 Then we multiply the seven and two-thirds, the error of the first scale, by the forty-five, the second posited number, to get three hundred forty-five, which is the first product. Doing the reverse[113] gives four hundred sixty, which is the second product. From this we drop the first product, which is the smaller, since the two errors exceed. The remainder is one hundred fifteen. We divide it by the difference between the two errors, which is three and five sixths. The result is the unknown quantity, which is thirty.

201.1 Another example: suppose someone said, “A quantity: we add its tenth to the difference between its fourth and three of its fifths, so it comes to nine. How much is the quantity?”

201.3 We draw the balance similarly, and we choose twelve for one of the scales, and we drop its fourth from three of its fifths. The remainder is four and a fifth. To this we add a tenth of the scale, to get the sum five and two-fifths, which is the portion that we confront with what is above the dome. The error is three and three fifths, falling short. We put it below the scale. Then we choose twenty-five for the second scale. We again drop its fourth from three of its fifths, leaving eight and three fourths. To this we add a tenth of the scale to get the sum eleven and a fourth, which is the portion that we also confront. The error is then two and a fourth, which exceeds. We put it above the scale to get this figure:



201.10 Then we multiply the first error by the second posited number, giving ninety. Doing the reverse gives twenty-seven. We add these products, since one of the errors exceeds and the other falls short, to get the sum one hundred seventeen. We divide it by the sum of the two errors, which is five and four fifths and a fourth of a fifth. The result is the unknown quantity, which is twenty. So know it.

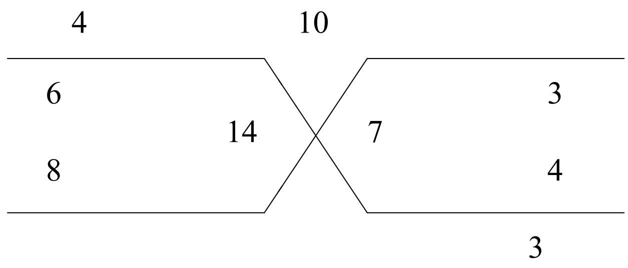
201.14 **If you wish, you can choose the first number or any other number for the second scale, then figure its portion, which you confront with what is above the dome. Multiply it by the first posited number, and multiply the first error by the second posited number. If the first error falls short, you add the two products. If it exceeds, you take**

[113] That is, the same calculation with the scales reversed.

their difference. You divide the result by the portion of the second scale to get the required number.

202.3 This second method is used only when there is a proportional relation. For example, suppose someone said, “Ten: you divide it into two parts so that a third of one of them is a fourth of the other”.[114]

202.5 We draw the balance similarly, and we put the ten above the dome. Then we choose two numbers for the scale so that a third of one of them is a fourth of the other. They are, for example, three and four. We confront their sum, since it is the portion, with the ten. The error is three, which falls short. Then we choose [numbers for] the second scale the same way, such as six and eight. We figure its portion, which is the sum of the numbers, to get fourteen. This gives the following figure:

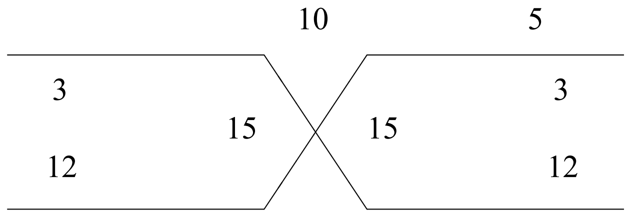


202.10 Regardless of which part we want to find, we multiply the first error by the upper number of the second, and the portion of the second by the upper number of the first. Then we continue with the procedure. The result is the required part, and the remainder from the ten is the other part. Suppose in this example that we want the smaller part. We multiply the three, the first error, by the six, the upper number of the second, to get eighteen. Then we multiply the fourteen, the second portion, by the three, the upper number of the first, to get forty-two. We add it with the first product, since the error falls short, to get sixty. We divide it by the fourteen, the second portion. The result is the required number, which is four and two-sevenths. The remainder from the ten is the greater part, which is five and five sevenths.

203.1 Another example: suppose someone said, “Ten: you divide it into two parts. You divide the greater by the smaller, so the result is four. How much is one of them?”

203.3 One of the parts is necessarily four times the other. So we draw the balance similarly, and we place the ten above the dome. Then we choose for one scale two numbers so that one of them is a fourth of the other. They are, for example, three and twelve. We confront their sum with the ten, since it is the portion. The error is five, which exceeds. Then we make our choice for the second scale. Here we choose the same as the first. We figure its portion, which is fifteen. This gives the following figure:

[114] Copied from (Saidan 1986, 557.7).



203.8 We work it out as described in the previous example, to find that one of the parts is two, and the other is the remainder from the ten, which is eight. So know it. If we wish, we can work out these two problems by the first method.

203.11 We follow the second method *in any problem where the assigned numbers are one of the scales and its error. For example, I say, "A quantity: we subtract its third and its fourth from a third of sixty and its fourth, leaving fourteen. How much is the quantity?"*

203.15 *The sixty is one of the scales, and the fourteen is its error, which exceeds. We choose whatever number we wish for the other scale, and we find its third and its fourth, which is the portion that we would confront with what is above the dome, if there were a number there. We work it out as mentioned, so the quantity is thirty-six.*

204.1 *Thus, the method of scales derives from the four proportional numbers. You know that approaches for working it out come from relating the two errors and from the differences of the two scales when combined and separated.*[115]

204.3 Regarding this, my professor, the jurist and great erudite Abū l-ʿAbbās [Ibn al-Bannāʾ], God bless him, dictated to me while I was studying with him on Wednesday, the twenty-eighth of the month of *rajab* in this year:

204.6 "There are three approaches. One of them is that you multiply the difference between the scales by one of the errors. If the two errors both exceed or both fall short, you divide the product by their difference. If one of them exceeds and the other falls short, you divide the product by their sum. You add the result of this to the scale whose error you multiplied if it falls short, and you subtract it from it if it exceeds. This gives the required number.

204.12 "The second approach is that you multiply the difference between the scales by the sum of the errors if they both exceed or both fall short, and you divide by their difference. If one of them exceeds and the other falls short, you multiply the difference between the scales by the difference between the errors, and you divide by the sum of the errors. Keep this result in mind. If you wish, you add this remembered number to the difference between the scales and you take half of the sum. You add it to the scale whose error is greater if it falls short, and you subtract it from it if it exceeds, to get the required number.

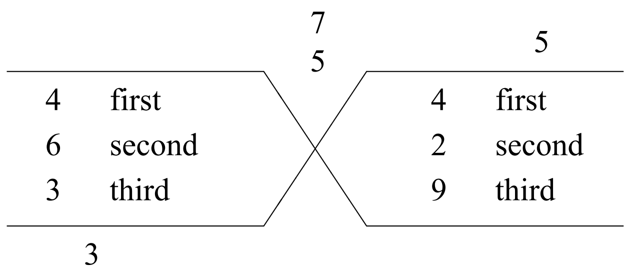
204.18 "And, if you wish, take the remembered number and the difference between the scales, subtract the smaller from the greater, and take half of the remainder. Add it to the scale whose error is smaller if it falls short, and subtract it from it if it exceeds, to get the required number.

[115] Copied from (Ibn al-Bannāʾ 1994, 298.13-19).

205.1 “The third approach is that you multiply the difference between the scales by your assigned number. If the errors both exceed or both fall short, you divide the product by their difference, and if one of them exceeds and the other falls short, you divide the product by their sum, to get the required number. So know it.”

205.5 *Scales can also be set up to find the unknown in cases without a proportional relation, as in this problem: Three men want to buy a horse.*[116] *The first says to the second, give me half of what you have, and, together with what I have, I will have the price of the horse. The second says to the third, give me a third of what you have, and, together with what I have, I will have the price of the horse. And the third says to the first, give me a fourth of what you have, and, together with what I have, I will have the price of the horse.*

205.11 *We choose one scale for the three men. We assign to the first man whatever we wish, so we make it four. The second also has whatever we wish, so we make it two. Then the price of the horse is calculated to be five. We put it above the dome, and it is what we will confront. The third man has, also by calculation, nine. If we add a fourth of what the first has, it amounts to ten. The error of the first scale with these three numbers is five, which exceeds. Then we choose [the numbers in] the other scale. We assign four for the first man, the same as we assigned for him in the first [scale], and we make the second whatever we wish. We see that it should not be eight or more*, since it would lead to the third having nothing. *So know it. So we make it six. This makes the price of the horse, which we confront, seven. We put it above the dome also. The third necessarily has three, and adding this to a fourth of what the first has gives four. The error of the second scale is then three, which falls short.* This is the figure:



206.1 *We multiply the error of each scale by what each one has in hand in the other scale, and we divide the sum of the two products as mentioned, to get what each one has in hand and the price of the horse.*[117] This gives us what the first has in hand, which is the four, the second four and a half, the third five and a fourth, and the price of the horse is six and a fourth.

206.5 If we wish the result to be without fractions, we multiply all of the problem by the smallest number divisible by their denominators. We knew [how to do] this before,[118] and it is four. By this calculation the first has sixteen, the second has eighteen, the third has twenty-one, and the price of the horse is twenty-five. So understand it.

[116] The word *dābba* means a “riding animal”, such as a horse, mule, or donkey. We translate it as “horse”.
[117] Copied from (Ibn al-Bannāʾ 1994, 299.1-14).
[118] At 122.2 above.

206.9 *If we wish, we can change what is assigned for the first [man] in the first scale, and leave unchanged what is assigned for the second, since the condition is that one of them should have the same number in both scales.*

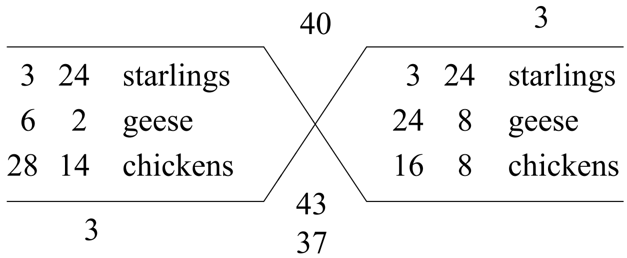
207.1 *If we [choose to] assign the price of the horse, we put it above the dome, and we put some portion of it for the first [man], and twice its remainder for the second. Then we drop a fourth of what we put for the first from the price of the horse, leaving what the third has. Then we take what the second has and a third of what the third has, and we confront it with the assigned price. We then do this likewise for the other scale: we first make the price of the horse whatever we wish, as long as it is not the first number. Then we work it out as before.*

207.6 *Another problem: forty birds, among which are geese, chickens, and starlings, [all] for forty dirhams. The starlings are eight for a dirham, the chickens are one for two dirhams, and the geese are one for three dirhams. How many were chosen of each kind of bird?*

207.9 *Not all examples of this type can be worked out. Two conditions must be satisfied. One is that the numbers must be whole numbers and not fractions. Second, the price of one of the cheapest [birds], if multiplied by the number of birds, must be smaller than the total price; and the price of one of the most expensive [birds], if multiplied likewise, must be greater than the [total] price.*

207.13 *It is obvious in this problem that the number of starlings must be eight or sixteen or twenty-four or thirty-two, and no other [number]. If there are eight, then there remain thirty-two birds and thirty-nine dirhams. If we check the second condition, the product of the remaining individual birds by the price of the cheapest one is greater than the number of the [total] price, which is not valid. If we make the starlings sixteen, and we again check the remaining birds, then the remaining price is also not valid.*

208.1 *If we make the starlings twenty-four, and we again check what remains, the two conditions are met. So we suppose there are twenty-four starlings, and we assign the chickens to be whatever we wish. Suppose they are eight. Then the geese are eight, the remaining number. The error in the price is three dirhams, which exceeds. Then we choose [the numbers in] the other scale. We make the starlings twenty-four, as in the first [scale], since it is a condition for it to work out that a number be repeated in the two scales. And we make the chickens be whatever we wish that is different from the first, so let them be fourteen. Then the number of geese is two. The error is three dirhams falling short.* Here is the figure:



[119] Copied from (Ibn al-Bannāʾ 1994, 299.15-300.17).

208.8 *We work it out as before to get the required number of each kind of bird and the price of each kind, whichever we want to find first.*[119]

208.10 The price of the starlings is three, and their number is twenty-four. The geese are five and their price is fifteen, and the chickens are eleven and their price is twenty-two.

208.12 *If we make the starlings thirty-two, it is not valid since the condition for the remaining [birds] is not met. Thus this problem has only one answer. For similar problems, look to these two problems. Problems like this cannot be worked out by the second method of solving by scales, since that is specific to a proportional relation, as said above. Problems involving multiplication are not based in proportion, so they cannot be solved by the scales. So know it.*[120]

208.17 Chapter One is completed, with the praise of God and His help.

209.1 **Chapter Two, on algebra. We present this work in five sections.**

211.1 **Section One, on the meaning of algebra (restoration and confrontation) and an explanation of its types.**[121]

211.2 **Restoration is the reconstitution we mentioned in Part One of this book.**[122] **Confrontation is subtracting each species from like [species] until there are no longer two species of the same kind in the two parts.**[123] **And equalization is the restoration of the deleted to the appended, and the subtraction of the appended from the appended and the deleted from the deleted of things of the same kind.** This will be covered with clear examples in the section on addition and subtraction,[124] almighty God willing.

211.8 **Algebra depends upon three species: the number, the things, and the *māl*s. The things are roots,** since every unknown in numbers is a thing, and a root of its square when it is known. **The *māl* is what one gets from multiplying the root by itself.** The term *māl* serves to distinguish it from the other [species].

211.13 **These three species are equated to one another, in either a simple or composite way. This results in six types, three simple and three composite.**

211.15 **The first of the simple [types], following convention, is: some *māl*s equal some roots.** For example, three *māl*s equal seven things. **The second is: some *māl*s equal a number.** For example, five *māl*s equal twenty. **The third is: some roots equal a number.** For example, three roots equal twelve.

212.6 **In the first of the three composite [types], which is the fourth type, the number is isolated.** For example, a *māl* and ten roots equal twenty-four. **In the fifth, the roots are isolated.** For example, a *māl* and four equal five roots. **And in the sixth, the *māl* is isolated.** For example, a *māl* equals four roots and five.

[120] Copied from (Ibn al-Bannāʾ 1994, 300.18-22).
[121] The "types" are the six types of equations, described beginning at 211.13.
[122] At 129.1 for whole numbers, and at 154.1 for fractions.
[123] I.e., in the two parts of a subtraction, or in the two sides of an equation.
[124] At 219.1 below.

213.1 **Section Two, on working out the six types.**

213.2 **For the three simple [types], you divide by the *māl*s what they are equal to, and by the roots if there are none. The result of the division in the first and third types is the root, and in the second it is the *māl*. If the root is known, then the *māl* is known by multiplying the root by itself. And if the *māl* is known, then the root is known from it.**

213.7 For example, suppose someone said, "Three *māl*s equal fifteen things". The meaning of this problem is: what quantity (*māl*), if we take its root fifteen times, gives a sum equal to three times the quantity? This is of the first type, and it is worked out as explained above. If we divide the fifteen, the number of things, by the three, the number of *māl*s, it results in five, which is a root of the unknown *māl*, the latter being twenty-five.

213.13 Another example: suppose someone said, "Two *māl*s equal eighteen". The meaning of this problem, likewise, is: what quantity (*māl*), if we add itself to it, becomes equal to eighteen? This is of the second type. To work it out we divide the eighteen by the two, the number of *māl*s, as mentioned, to get nine. This is the unknown *māl*, and its root is three.

214.1 Another example: suppose someone said, "Five things equal twenty". The meaning of this problem, likewise, is: what quantity (*māl*), if we take five of its roots, gives a result equal to the twenty? This is of the third type. To work it out we divide the twenty by the five, the number of things, since there are no *māl*s. It results in four, which is a root of the unknown *māl*, and that is sixteen.

214.7 **To work out the fourth type, you halve the number of roots, you square the half, you add it to the number, you take a root of the result, and you drop the half from it, leaving the root.**

214.9 For example, suppose someone said, "A *māl* and two things equal fifteen". Its meaning is, what quantity (*māl*), if we add two of its roots to it, becomes equal to fifteen? We take half of the things, giving one, and we square it, giving one, and we add the number to it, giving sixteen. We take its root, giving four. From this we drop the one, which is the half, leaving three. This is the unknown thing, and the *māl* is nine.

214.14 If we wish to find the *māl* first, before the root, we add half a square of the number of roots to the number, then keep the sum in mind. Then we subtract a square of the number from a square of the remembered number, and we take a root of the remainder. We subtract this from the remembered number. The remainder is the *māl*.

214.17 If we want to work it out in the example, we add half a square of the number of things, which is two, to the fifteen, giving seventeen. We keep it in mind. Then we square the number to get two hundred twenty-five. We subtract it from a square of the remembered number, which is two hundred eighty-nine, leaving sixty-four. We take its root, giving eight. We subtract it from the seventeen, leaving nine, which is the required *māl*.

215.3 **For the fifth [type], you subtract the number from a square of half of the number of roots, and you take a root of the remainder. If you add it to the half, it gives a root of the greater *māl*, and if you subtract it, it gives a root of the smaller *māl*.**

215.6 For example, suppose someone said, “A *māl* and eight equal six things”. Its meaning is, what quantity (*māl*), if we add eight to it, gives a result equal to six of its roots? We take half of the things and we square it, giving nine. We drop the number from it, leaving one. We take its root, giving one. If we add it to the half, it gives four, which is a root of the greater *māl*, so the greater *māl* is sixteen. If we subtract it from it, it leaves two, which is a root of the smaller *māl*, and the *māl* is four.

215.12 KNOW THAT **whenever a square of the half is equal to the number, then the half is the root, and the *māl* is the number.**

215.14 For example, suppose someone said, “A *māl* and nine equal six things”. We square half of the things, giving nine, and this is equal to the number. So the number is the *māl*, and the half is the root. And if we continue the solution, we subtract the number from the square, leaving nothing. We take its root, giving nothing. We add nothing to the half, or we subtract it from it, leaving the half. It is the root, and its square is the number. So know it.

216.1 If we wish to find the *māl* first, before the root, we subtract the number from half of a square of the number of roots, and we keep the remainder in mind. Then we subtract a square of the number from a square of the remembered number. If we add a root of the remainder to the remembered number, it gives the greater *māl*; and if we subtract it from it, then the remainder is the smaller *māl*. Note that this is what it is when the number is smaller than a square of half of the number of roots.

216.6 If we want to work this out in the preceding example, we subtract the number from half of a square of the number of roots, which is eighteen, leaving ten. We keep it in mind. Then we subtract a square of the number, which is sixty-four, from a square of the remembered number. The remainder is thirty-six. We take its root, giving six. If we add it to the remembered number, we get sixteen, the greater *māl*. And if we subtract it from the remembered number, then the remainder is four, the smaller *māl*. So know it.

216.11 **The sixth [type] is solved like the fourth, except that you add the half at the end to a root of the sum to get the root.**[125]

216.13 For example, suppose someone said, “A *māl* equals two of its roots and three”. Its meaning is, what quantity (*māl*) is equal to two of its roots and three? So we square half of the things and we add it to the number, giving four. We take its root, giving two. We add the half to it to get three, which is the unknown thing, and the *māl* is nine.

216.17 If we wish to find the *māl* first, before the root, we add double the number to a square of the roots, and we keep half of the sum in mind. Then we drop a square of the number from a square of the remembered number, and we add a root of the remainder to the remembered number. This gives the *māl*.

216.20 If we want to work it out in the example, we add double the number, which is six, to a square of the roots, giving ten. We keep in mind its half, which is five. Then we drop a square of the number from a square of the remembered number. The remainder is sixteen.

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[125] In the *Condensed Book* this is placed just after the passage at 214.7.

We add its root, which is four, to the remembered number. This gives the unknown *māl*, which is nine.

217.1 **For each of the three composite types, whenever there is more than one *māl*, reduce it to one *māl*, and reduce by that term all of the equation. And for each of them, whenever there is less than one *māl*, restore it to one *māl*, and restore by that term all of the equation. The way to work out restoration and reduction was described earlier.**[126] **And if you wish, divide [each of] the names**[127] **in the problem by the number of *māl*s. This reduces the problem.**[128] **Then confront the two sides.**

217.8 He indicated this only for the composite [equations], since the simple [equations] do not require restoration if there is less than one *māl*, nor reduction if there is more than one *māl*.

217.10 For example, suppose someone said, "Two *māl*s and six things equal thirty-six". We reduce the two *māl*s to one *māl* by multiplying them by a half, as mentioned above. And we reduce by this [multiplication] the things and the number. So the problem becomes: a *māl* and three things equal eighteen, which is the fourth type. This is also how to work out the fifth and sixth [types].

217.15 And if we work it out the other way, we divide the two, the number of *māl*s, by itself, resulting in one. And we also divide the number of things by it, resulting in three. And we also divide the number by it. So the problem becomes: a *māl* and three things equal eighteen, just as before.

218.1 Another example: suppose someone said, "Half a *māl* and two things equal six". We restore the half to one *māl* by multiplying it by two. We multiply it also by the things and the number. So the problem becomes: a *māl* and four things equal twelve, which is also the fourth type. This is also how to work out the fifth and sixth [types].

218.6 And if we work it out the other way, we divide the half by itself, resulting in one. And we also divide the number of things by it, resulting in four. We also divide the number by it, resulting in twelve. So the problem becomes: one *māl* and four things equal twelve, just as before. So understand it.

219.1 **Section Three, on addition and subtraction.**

219.2 **Adding different species is done with the conjunction "and",** like: a *māl* and six things and ten dirhams.[129]

219.4 **…and the different excluded [species] cannot be subtracted.**

219.5 *For example, suppose someone said, "Add a* māl *less a thing to ten dirhams". The sum is: a* māl *and ten dirhams less a thing. So the exclusion remains as it was, since [the thing] cannot be subtracted from anything.*[130]

126 At 129.1 and 154.1 above.
127 By "names" (here *alqāb*) he means the terms of the equation.
128 Literally, "what results is a returning to the problem".
129 Copied from (Ibn al-Bannāʾ 1994, 313.3).
130 Copied from (Ibn al-Bannāʾ 1994, 313.3-5).

219.7 **Whenever appropriate, subtract the smaller from the greater, that is, in the problem.** *For example, suppose someone said, "Add two* māl*s less a* māl *to ten dirhams". So we subtract the excluded* māl *from the two* māl*s, so the remaining sum is a* māl *and ten dirhams.*

219.10 *Another example: suppose someone said, "Add a* māl *less two things to ten things". We subtract the two things from the ten things. The remaining sum is a* māl *and eight things.*[131]

219.12 *Another example: suppose someone said, "Add a* māl *less five things to ten things". We subtract the five things from the ten things. The remaining sum is a* māl *and five things.*[132]

220.1 **Subtracting different species is done using the particle of exclusion.** For example, suppose someone said, "Subtract a thing from a *māl*", then the remainder is a *māl* less a thing.

220.3 Another example: suppose someone said, "Subtract ten dirhams from two *māl*s and three things". The remainder is two *māl*s and three things less ten dirhams.

220.5 **The exclusion can be in both parts[133] or in one of them, and it might be one species or two different species. To work it out, you add the exclusion in each part to both parts at once, and then you subtract. Work out the two sides of an equation similarly whenever they have exceptions.**

220.9 For example, suppose someone said, "Subtract two and a thing from a *māl* less three things". We find the things on the side of the minuend to be deleted, and they are three. We add them to the two parts at once. This is what is meant by equalization with restoration, since we added to the *māl* what it is associated with, which are the three things. Thus we restore it from deleted to appended. It becomes greater, as is desired for the minuend. We [also] equalize by restoring the subtrahend by the amount we added in the minuend, which is by the three things, since subtracting two numbers is the same as subtracting them after having added a number to both of them, or after subtracting a number from both of them. The problem is as if someone had said: "Subtract two and four things from a *māl*". Then we work it out as before. The remainder after that is the required number, which is a *māl* less four things and less two. So know it.

221.1 Another example of this: suppose someone said, "Subtract fifty-two dirhams less five things from two cubes and thirty dirhams". We find the things on the side of the subtrahend to be deleted, and they are five. We add them to the two sides at once, as before. The problem becomes as if someone had said: "Subtract fifty-two dirhams from two cubes and five things and thirty dirhams". Then we find dirhams on both sides. We remove the smaller quantity from the two sides at once. The problem becomes as if someone had said, "Subtract twenty-two dirhams from two cubes and five things".

221.8 This is confrontation with equalization, since when we removed the thirty from the subtrahend it becomes smaller than the original subtrahend. So we equalize, by which we remove from the minuend the same amount we removed from the subtrahend. It is worked out like

131 Copied from (Ibn al-Bannāʾ 1994, 313.6-9).
132 Copied from (Ibn al-Bannāʾ 1994, 313.10-11), except that Ibn al-Bannāʾ solves "add a *māl* less ten things to five things".
133 That is, in the minuend or the subtrahend.

before. We work the remainder as before. What remains after that is the required amount, which is two cubes and five things less twenty-two dirhams.

221.13 Another example: suppose someone said, "Subtract twelve dirhams less four things from three *māl*s less two things". We find two deleted things in the minuend, and also four deleted things in the subtrahend. We add them. Since they are of the same species, they amount to six things. We add them to the two parts at once, or we restore both the subtrahend and the minuend by what is excluded from them, and we add their equals to the other [part], as before. The problem becomes as if someone had said, "Subtract twelve dirhams and two things from three *māl*s and four things". So we confront and subtract. The remainder after that is the required number, and that is three *māl*s and two things less twelve dirhams.

222.1 Another example: suppose someone said, "Subtract a cube less two *māl*s from thirty dirhams less four things". We restore and subtract. The remainder is the required number, which is thirty dirhams and two *māl*s less a cube and less four things.

223.1 SUBSECTION CONCERNING EXAMPLES WITH THE TWO SIDES OF AN EQUATION.

223.2 Along these lines, suppose someone said, "A *māl* less three things equal two and a thing". We find the things on the side of the *māl* to be deleted, and they are three. We restore the *māl* by them, and we also add them to the two sides of the equation[134] as we did before, since whenever one adds equals to equals or subtracts equals from equals, the outcomes are equal. So the problem becomes: a *māl* equals two and four things, which is the sixth type.

223.7 Another example: suppose someone said, "A *māl* less three things equal twenty-four dirhams less five things". We restore and confront as before, so the problem becomes: a *māl* and two things equal twenty-four dirhams, which is the fourth type. If we wish, we can first subtract the three things which are deleted on the side of the *māl* from the five things deleted on the side of the number. This leaves two things deleted on the side of the number. And we finish working it out, again arriving at the fourth type.

223.14 Another example: suppose someone said, "A *māl* less ten dirhams equal a *māl* less two things and a half". We restore and confront, so the problem becomes: ten dirhams equal two things and half a thing, which is the third type.

223.17 Another example: suppose someone said, "A *māl* and ten dirhams equal fifty-one dirhams less four things". We restore, and the problem becomes: a *māl* and four things equal forty-one dirhams, which is the fourth type.

224.1 Another example: suppose someone said, "A *māl* and five things equal ten dirhams and two *māl*s less a thing". We restore and confront, so the problem becomes: six things equal a *māl* and ten dirhams, which is the fifth type.

224.4 Another example: suppose someone said, "Two *māl*s and ten dirhams less two things equal three *māl*s and seven dirhams less seven things". We restore and confront, so the problem

[134] He should add them only to the side with two and a thing, because the act of restoration already adds them to the *māl*.

becomes: a *māl* equals three dirhams and five things, which is the sixth type. So know it. Work out similar problems the same way, with the help of God.

225.1 **Section Four, on multiplication and knowing the power and the term.**

225.2 **For the power, know that the power of the things is one, the power of the *māl*s is two, and the power of the cubes is three. As for the term, the term for one is "things", the term for two is "*māl*s", and the term for three is "cubes". After that, there are three for each cube and two for each *māl*.** The cube is the surface of the thing by the *māl*. It is called this way because it is a cube, although you do not know its amount.

225.8 Suppose someone said, "What is the power of a *māl māl*?" We say four. And suppose someone said, "What is the power of a *māl* cube?" We say five. Suppose someone said, "What is the power of a *māl māl māl*?" We say six. And suppose someone said, "What is the power of a *māl* cube *māl* cube?" We say ten. To work these out, we always count each *māl* as two, since that is its power, as mentioned. Likewise, each cube is always three, since that is its power, as mentioned. The result is the associated power.

225.13 And suppose someone said, "What is the power of a *māl* cube *māl māl*?" We would say nine. Suppose someone said, "What is the power of a cube *māl* cube cube *māl māl*?" We would say fifteen. Work it out this way for other combinations, too.

226.1 Conversely, suppose someone said, "What is a term for four?" We would say a *māl māl*. And if someone said, "What is a term for seven?" We would say a cube *māl māl*. And if someone said, "What is a term for six?" We would say a *māl māl māl*, or a cube cube.

226.3 We can also work this out by separating the powers by twos or threes or by grouping them: first the *māl*, and then the cube, and then the *māl* and the cube. Suppose someone said, for example, "What is a term for eight?" We would say a *māl māl māl māl*. Or if we wish, we can say a cube *māl* cube, or a cube cube *māl*, or a *māl* cube cube, or anything else that is allowable.

226.7 Suppose someone said, "What is a term for nine?" We would say a cube cube cube, or if we wish, we can say a cube *māl māl māl*, or if we wish, we could continue or stop. [It can be] anything that is allowable. So know it.

226.9 **To multiply these species, add the power of the multiplicand and the power of the multiplier. Then the sum of the powers is the power of the result. To multiply a number by one of these species, the result is the same species.**

226.12 For example, suppose someone said, "Multiply five things by seven things". We multiply the number of the multiplicand, which is five, by the number of the multiplier, which is seven, giving thirty-five. Then we add the powers of the two multiplicands, giving two. This two is the power of the thirty-five resulting from the multiplication, so it is *māl*s. So the result of the multiplication is thirty-five *māl*s, which is the required number.

226.17 Another example: suppose someone said, "Multiply ten things by six *māl*s". We multiply the number of the multiplicand by the number of the multiplier, which gives sixty. Then we add the powers of the two multiplicands, giving three. We make it the power of the result, which you recall is sixty. This is the required number, which is sixty cubes.

227.4 Another example: suppose someone said, “Multiply a thing by a cube”. The result of multiplying the number of the multiplicand by the number of the multiplier is one. The sum of their powers is four, which is the power of the one, the result of the multiplication. So it is a *māl māl*.

227.7 Another example: suppose someone said, “Multiply six by four *māl*s”. We multiply the multiplicand by the number of the multiplier, giving twenty-four. The outcome of the multiplication in this example is *māl*s, so it is then twenty-four *māl*s.

227.10 Another example: suppose someone said, “Multiply seven by three *māl*s cube”. We multiply the multiplicand by the number of the multiplier, giving twenty-one. The outcome of the multiplication in this example is likewise *māl* cube, so it is then twenty-one *māl*s cube. So know it.

227.14 **Whenever you equate between the *māl*s *māl*s and the cubes and the *māl*s, or the cubes and the *māl*s and the things, or any combination in which you do not have a number, subtract the smaller of the powers from the power of each of them. Then you equate what remains, one with the others, in the same way as the equation.**

227.17 For example, suppose someone said, “Three *māl*s *māl* equal four cubes and ten *māl*s”. We find the power of the *māl*s to be the smallest power in the equation, and it is two. If we drop it from the power of the *māl*s *māl*s, which is four, there remains two, and its term is *māl*s. We also drop it from the power of the cubes, which is three, leaving one, and its term is a thing. And we reduce the *māl*s to a number. The problem becomes: three *māl*s equal four things and ten dirhams, which is the sixth type.

228.1 Another example: suppose someone said, “Three cubes equal ten *māl*s and twenty things”. We find the power of the things to be the smallest power in the equation, and we work it out as before. The problem becomes: three *māl*s equal ten things and twenty dirhams, which is also the sixth type.

228.4 Another example: suppose someone said, “A cube and ten *māl*s equal thirty-nine things”. We work it out as before. The problem reduces to: a *māl* and ten things equal thirty-nine dirhams, which is the fourth type. Work out similar examples the same way.

228.8 If it is not possible to transform it to [one of] the six types, then do not work it out since it would be of no use for anything.

228.9 **The multiplication of two appended or deleted [terms], one of them by the other, is appended, and the multiplication of an appended [term] by a deleted [term] is deleted.**

228.11 For example, suppose someone said, “Multiply five things by thirteen less four things”. We multiply the five things by the appended thirteen, and we drop from the result the product of the five things again by the deleted four things. The remainder, then, is the required number, which is sixty-five things less twenty *māl*s.

228.15 Another example: suppose someone said, "Multiply eight less two things by seven less four *māl*s". We multiply the appended eight by the appended seven, to get an appended [amount]. We add to it the product of the two deleted things by the four deleted *māl*s, since it is appended. And we drop from the sum the product of the appended eight by the deleted four *māl*s, since it is deleted, and the product of the deleted two things by the appended seven, since it is deleted. The remainder, then, is the required [number], and that is eight cubes and fifty-six less fourteen things and less thirty-two *māl*s. So know it.

229.1 **Section Five, on division.**

229.2 **To divide one of these species by a lower species, drop the power of the divisor from the power of the dividend. What remains is the power of the species of the result of the division.**

229.4 For example, suppose someone said, "Divide ten *māl*s by two things". We divide the number of *māl*s by the number of things, and we call the resulting five by the name for the difference between the power of the things and the *māl*s, and that is a thing. The result of the division is five things, which is the required number.

229.8 Another example: suppose someone said, "Divide fifteen cubes by three things". We divide the number of cubes by the number of things, and we call the resulting five by the name for the difference between the power of the things and the cubes, and that is a *māl*. The resulting five is then five *māl*s, which is the required amount. So know it.

229.12 **And whenever you divide a species by itself, the result is a number.**

229.13 For example, suppose someone said, "Divide twelve *māl*s by three *māl*s". We divide the number of the divisor by the number of the dividend, resulting in four. There is no difference between the powers,[135] so we call the term[136] of the resulting four "number", most certainly.

229.16 **Whenever you divide one of these species by a number, the result is the same species.**

229.17 For example, suppose someone said, "Divide twelve things by four dirhams". We divide the number of the dividend by the number of the divisor, resulting in three. And the number of the divisor has no power, so that dropping from the power of the dividend leaves the power of the dividend as the power of the result, so we call it with its term. The result is three things, which is the required number.

230.1 **If there is an exclusion in the dividend, divide both the excluded [term] and the diminished [term] by the divisor, and exclude the result of the excluded [term] from the result of the diminished [term]. The outcome of this is the result of the division.**

230.4 For example, suppose someone said, "Divide twelve cubes less three *māl*s by two things". We divide the twelve cubes, which is the diminished [term], by the two things, which is the divisor. And the exclusion in this division is the three *māl*s, which is the excluded [term that we] likewise [divide] by the two things. The remainder is the required number, which is six *māl*s less a thing and half a thing.

[135] Literally, "there is nothing between the powers".
[136] Literally, "we name the name", with *ism*.

230.8 Another example: suppose someone said, “Divide ten *māl*s less three things by two dirhams”. We work it out as before. The result is five *māl*s less a thing and half a thing, which is the required number.

230.11 **One cannot divide the lower of two species by the higher** EXCEPT BY CANCELLING THEIR COMMON DIVISOR, BY WHICH YOU SUBTRACT THE POWER OF THE SMALLER FROM THE POWERS OF BOTH OF THEM.

230.13 For example, suppose someone said, “Divide six *māl*s by three cubes”. We divide the number by the number. Then the result is divided by the difference between the two ranks, which is a thing. The result is two divided by a thing.

230.16 **And one cannot divide by a diminished [term].** SO UNDERSTAND IT AND FIGURE IT OUT WITH THE HELP OF GOD.

230.17 For example, suppose someone said, “Divide ten *māl*s by three less a thing”. So we say the result is ten *māl*s divided by three less a thing. The answer is the same as the question. So understand it.

231.1 SECTION [ON SECRET NUMBERS].

231.2 Let us conclude this work with three problems of witty reckoning, like those with which arithmeticians still conclude their compositions.

231.4 One of them is that we tell someone to drop his [secret] number from ten, then to drop a square of the remainder from a square of his number. If [a square of] the remainder is smaller, we ask for the remainder. We divide it by ten, and we add ten to the result and then take half of the sum[137] to get the secret number. And if a square of the remainder is greater, he drops a square of the secret number from it, and we ask for the remainder. We divide it by the ten, and we drop the result from ten. Half of the remainder is the secret number. If we wish, we can ask that he drop his secret number from something other than the ten. Following the procedure will give the required number.

231.13 The second problem. We tell someone to divide the ten into two parts secretly. Then we tell him to divide a square of one of them by their surface, and we ask for the outcome. If we knew this, it is the ratio of one of the two parts to the other. So we divide the ten according to this ratio. And similarly, we can work with whatever number we wish other than the ten, dividing it secretly. This results in the two secret parts.

232.1 The third problem. A secret number is divided into two secret numbers. How much is it? And how much are each of its parts? We tell him to multiply one of the parts by the other and to square each of them, then to drop a square of the smaller from a square of the surface, and we ask for the remainder; and to drop the surface from a square of the greater, and we ask for the remainder. We take a root of the difference between the two asked numbers, which is the difference between the parts. Then we divide the sum of the

[137] This sentence is our reconstruction of what appears to be a corrupt passage. Literally, it reads “We divide it by ten, and what results, we add to it half of its remainder to the ten”. Only the Oxford manuscript attempts to correct it, by writing “We divide it by ten, and what results, we add to the remainder from the ten”. This is mathematically correct, but we cannot add the remainder from the ten because that is not revealed to us. This MS also has “square” scratched in above “remainder”.

asked numbers by it to get the secret number, which is the sum of the two parts. If we add to it a root of the difference between them, it gives double the greater, and if we subtract it from their sum, it leaves double the smaller. So know it.

232.9 These three problems were dictated to me by my professor, the great jurist Abū l-ʿAbbās [Ibn al-Bannāʾ], may God be pleased with him.

233.2 As the servant [of God], the named, the tempted, ʿAbd al-ʿAzīz ibn ʿAlī ibn Dāwud al-Hawārī al-Miṣrātī, may God have mercy on him, said: With the praise of God for His excellent assistance, I have achieved what I set out to accomplish to the extent of my ability, and not absolving myself of my errors and of anything leading to wrong ideas, I beg Almighty God's protection and to preserve me from loosing His favors. He is sufficient for us, and He is the best disposer of our affairs.

This work was completed on Saturday, the eighteenth of Dhū al-Qaʿda, 704. May God's generosity benefit the author. Full prayers upon His prophet and servant, our Master and Lord Muḥammad, and upon his family and companions and followers.

Commentary on al-Hawārī's Commentary

59.1 [Introduction]

A copyist wrote the introductory words "In the name of God… said:". Al-Hawārī first praises God and Muḥammad, and then Abū Yaʿqūb, Marinid ruler of Morocco from 1286 to 1307. Last, praise is given to Abū Muḥammad ʿAbdallah Ibn Abū Madyan, an intellectual and an important government minister from 1302 to 1307.

59.15 Al-Hawārī asked permission from Ibn al-Bannāʾ to write this commentary, which Ibn al-Bannāʾ granted. The latter had already written a commentary on his own book, titled *Lifting the Veil from the Face of the Operations of Arithmetic*. What the *Condensed Book* and *Lifting the Veil* lacked, according to al-Hawārī, were ample numerical examples of the rules.

61.1 Four of the manuscripts we consulted show the phrase "may God forgive him", which agrees with manuscripts of the *Condensed Book*. The Medina manuscript has instead "may God preserve his splendor, his reputation, and keep his memory whole". This version implies that Ibn al-Bannāʾ is already deceased.

61.9 Part I. Known numbers.

63.1 Chapter I.1. Whole numbers.

65.1 Section I.1.1. The divisions of numbers and their ranks.

65.2 Euclid defined "number" in the beginning of Book VII of the *Elements* as "a multitude composed of units".[1] Because these units are indivisible, Euclidean numbers are restricted to positive integers. This is in contrast with the numbers of practical Arabic arithmetic, which include fractions and irrational roots. Al-Fārisī, a Persian contemporary of Ibn al-Bannāʾ, gives the definition of number of "the arithmeticians" as "a quantity you obtain from one by repetition or partition or both, and it is clear by this meaning that the type is divided into whole numbers and their fractions".[2] Ibn a-Bannāʾ condenses and combines the Greek and Arabic definitions: "A number is a collection of units, and it is divided according to how it is produced into two kinds: whole and fractional". Later, he made a philosophical apology for fractional numbers in *Lifting the Veil* in which he claimed that Euclid's definition is not really a definition at all, but merely an expression of "what

[1] Translated in (Euclid 1956, vol. 2, 277).
[2] (al-Fārisī 1994, 71.8).

is in the soul".[3] Later commentators entered into the discussion, including al-Mawāḥidī (ca. 1382), Ibn Qunfudh (1370) and Ibn Ghāzī (1483).[4]

Al-Hawārī will explain the various ways of representing fractions in Chapter 2, beginning at 131.1. The examples he gives here are, in the notation described later in the book, $\frac{1}{2}$, $\frac{3\ 1}{8\ 2}$ (equivalent to our $\frac{7}{16}$), $\frac{1}{4}\frac{1}{9}$ (our $\frac{13}{36}$), $\frac{6 \bullet 7}{7 \bullet 8}$ (our $\frac{3}{4}$), and $\frac{1}{9}$ ↄ $\frac{5}{6}$ (our $\frac{13}{18}$).[5]

65.6 Reading right to left, the first place is the units place. Zero was not considered to be a digit, but was instead a sign indicating an empty place in the representation of a number.

65.10 Evenly-even numbers are the powers of two starting with 2, and evenly-odd numbers are the double of an odd number ≥ 3. Evenly-evenly-odd numbers are numbers in between: they are the product of some power of two ≥ 4 by an odd number ≥ 3. The classification of even numbers into evenly-even, evenly-odd, and evenly-evenly-odd comes from Greek number theory. These definitions are taken from Nicomachus's *Arithmetical Introduction*, translated into Arabic by Thābit ibn Qurra in the late ninth century.[6]

Euclid's definitions of these terms in *Elements* Book VII are different, and do not correspond to the standard Greek definitions. He defines a number to be "even-times even" if it is the product of two even numbers, and "even-times odd" if it is the product of an even number by an odd number. This way a number can be both, like $28 = 14 \cdot 2 = 7 \cdot 4$. Also, Euclid has nothing corresponding to the "evenly-evenly-odd".

Note the wording "each of the sixteens". When we take half of 32, the result is a single number, 16. For al-Hawārī, halving 32 means to partition it into a pair of 16s. Numbers in medieval Arabic arithmetic admit multiplicity. See our comments at 163.2 below.

66.7 The word we translate as "prime" is *awwal*. This word also means "first, foremost", etc., and so is close to our "prime" or "primal". Ibn al-Bannāʾ and al-Hawārī also use the word *aṣamm* ("deaf") to mean "prime", though other arithmeticians, like al-Uqlīdisī and al-Baghdādī, use that word to mean "irrational". The use of the word "deaf" for "prime" has to do with Arabic ways of expressing fractions, which we explain below at 134.2. We describe the words for "irrational" at 163.2. A prime number can also be called *basīṭ* ("simple"), in contrast to composite numbers, which are formed from more than one prime.

An "oddly-odd" number is the product of two odd numbers, both ≥ 3. This time Euclid, not Nicomachus, is probably the source. Definition VII.10 in *Elements* reads "An odd-times odd number is that which is measured by an odd number according to an odd number", i.e., it is the product of two odd numbers. Nicomachus has no such definition.

[3] (Ibn al-Bannāʾ 1994, 207); (Aballagh 1988, 142-143).
[4] (Aballagh 1988, 142-143).
[5] See below at 86.1 for an explanation of the "ↄ".
[6] Book I, Chapters 7-10 (Nicomachus 1959, 19-28); (Nicomachus 1938, 190-201). Ibn al-Bannāʾ's terms are precisely those in Thābit ibn Qurra's late ninth century Arabic translation of Nicomachus's work (Nicomachus 1959, 20.18). D'Ooge translated Nicomachus's Greek terms as "even-times even", "even-times odd", and "odd-times even".

He calls composite odd numbers simply "composite" (*sýnthetós*, translated as *murakkab* by Thābit).[7]

The word for "composite" in Ibn al-Bannāʾ and al-Hawārī is similarly *murakkab*, or some related form. (Sometimes we translate these words as "composed" or "composition".) The associated verb *rakiba* serves to multiply the factors together to produce the number composed of those factors. See below at 196.16 and 211.13 for related forms of this word in the context of proportions and algebra.

66.13 Al-Hawārī presents another classification of numbers. All numbers are either prime or composite, and composite numbers come in three types: (a) perfect squares, (b) products of two or more different numbers, and (c) perfect cubes. He does not mention that some numbers fall into more than one category. The number 36, for example, is a square (6^2) and a product of two different numbers ($4 \cdot 9$). The classification of composite numbers is explained for even numbers starting at 66.17, then again for odd numbers at 67.17.

We have translated the adjective *majdhūr* as "has a root". A number "has a root" if its square root is rational, like 9 or $\frac{16}{25}$. We could have translated it as a single word like "rootable", but that seemed too awkward. Later, first at 163.14, we will encounter the same concept for ranks. A rank is a *majdhūra* if a number of that rank can have a root. This is true for the units, the hundreds, ten thousands, and every other rank after that.

66.17 The words *ḍilʿ* ("side"), *musaṭṭaḥ* ("surface" or "plane"), and *murabbaʿ* ("square") are geometric terms used in arithmetic in a metaphorical sense. They derive from the corresponding Greek words in the number theory books VII to IX in Euclid's *Elements*, and are first encountered in Definition 16 at the beginning of Book VII. They do not imply any underlying geometric conception of number.

67.12 The word *mukaʿʿab* can mean a geometrical cube, or, as here, it can be an arithmetical term meaning "[perfect] cube", like 27 or $\frac{8}{125}$. The related word *kaʿb* in the present passage means "cube root", but it is also the name given to the third power of the unknown in algebra (first encountered at 221.1), which we translate as "cube". The meanings are clear by the context.

68.11 Perhaps al-Hawārī is thinking of the fact that "root" and "cube root" are particular to dimensions 2 and 3 respectively, while "side" is the term for the n-th root for any particular n.

To "decompose" (*ḥalla*) a number means to express it as the product of two or more numbers. The opposite operation is to "compose" (*rakiba*) two or more numbers into their product.

Ibn al-Bannāʾ and al-Hawārī explain a rule for extracting square roots numerically beginning at 166.1. Similar rules for extracting cube roots were well known in their time, but al-Hawārī remarks here that going through the work is "of little benefit". Instead, he briefly explains how to find the cube root of a perfect cube by factoring. For example, to

[7] (Nicomachus 1866, 27.12); (Nicomachus 1938, 202, Chapter XII); (Nicomachus 1959, 30.4).

find the cube root of 216, one can break it down as 3×72, then to $3 \times 9 \times 8$, and then to $2 \times 2 \times 2 \times 3 \times 3 \times 3$. Then one can piece it together as $6 \times 6 \times 6$, so 6 is the cube root.

68.18 Ibn al-Bannāʾ uses a few different words to explain the base ten system for writing numbers with the Indian figures:

- Rank (*martaba*). This word indicates the position of a digit in a number. The Arabic word suggests a ranking of the digits, as al-Hawārī explains at 68.18. Ranks are sometimes designated by ordinal numbers, like first, second, third, etc., or by the names "units", "tens", "hundreds", etc. For example, the "7" in 17,285 is the fourth rank, or the rank of thousands.
- Place, position (*manzila*). Like *martaba*, this word indicates the position of a digit in a number. But here the Arabic word evokes the image of a place or a home, "because the numbers reside in them", as remarked at 68.18. Again, the "7" in 17,285 is in the fourth place, or the thousands place. Sometimes the word *mawdhiʿ*, also meaning "place", is used with the same meaning (at 108.12, 109.4).
- Index (*uss*). This is the number indicating the position of the digit. The index of units is 1, of tens is 2, of hundreds is 3, of thousands is 4, etc. The Arabic word *uss* is used today to indicate the exponent in mathematics. We could have translated it as "power", but the numbers would be off by one. For us 10 is the first power of ten, while the corresponding index is 2. Later, in the chapter on algebra, the word *uss* is used to mean the power of the unknown, and there it matches our exponents. The *uss* of the second degree unknown is 2, for instance. Thus we translate it as "power" there.
- Name (*ism*). The name of a digit is "units", "tens", "hundreds", "thousands", "ten thousands", etc., depending on its place. For instance, the name of the 7 in 17,285 is "thousands".
- Species (*nawʿ* or *jins*). There are three species of number: units, tens, and hundreds. These are repeated for the thousands, for the thousand thousands (i.e., millions), etc. So the species of the "7" in 17,285 is units, and the species of the "1" is tens. The same two words *nawʿ* and *jins* are used for the different "species", or what we would call the powers of the unknown, in algebra.

At 68.18 Ibn al-Bannāʾ limits "rank" and "place" to units, tens, and hundreds, these being repeated for the thousands, thousand thousands, etc. But at 70.2, 70.23, 72.4, and 105.1 al-Hawārī regards these terms as progressing indefinitely instead of repeating.

One word absent in the book is *ḥarf*, meaning "digit". Other medieval books, such as *Principles of Indian Reckoning* of Kūshyār ibn Labbān, use *ḥarf* to mean "digit". The word is also used by our authors, but with the meanings of "letter", "particle", or "conjunction". In many instances, beginning in the section on addition, we translate *martaba* as "digit". This is not technically correct since the rank is a location and the digit is a number that is placed there, but it makes the reading easier. We also often translate *manzila* as "digit", too, first at 74.9; and in the passages at 111.7 and 112.5, we render *ʿadad* ("number") as "digit" where appropriate.

There were several Arabic words that played the roles of our words “type”, “kind”, “species”, “division”, or “variety”. The ones appearing in our book are *nawʿ*, *jins*, *ḍarb*, *qism*, and *ṣanf*. These words were more or less interchangeable in Arabic mathematics, whether for types of number, of fraction, of addition (and other operations), of proportion, of equation, etc. It is partly from the ways numbers were classified, and from the descriptions of the two types each of multiplication and division (95.3, 117.9), that we can recognize numbers in Arabic mathematics as being numbers *of something*. See our commentary at 95.3 below.

69.2 By the thirteenth century CE, and perhaps much earlier, two distinct styles of writing the Indian numerals had developed in the Islamic world.[8] The Western forms, written in the Maghreb and al-Andalus, ultimately led to the European forms 123456789 and 0, while the Eastern forms led to the current Arabic forms of the numerals, ١٢٣٤٥٦٧٨٩ and ٠. There are of course variations within each style.

Naturally, al-Hawārī wrote the Western forms, and it is these that are described by the poet. The poem makes use of the similarities between the shapes of the numerals and the shapes of letters of the Arabic alphabet to teach the student how to write the numerals. The copyist of the Medina manuscript was not familiar with the Western way of writing the numerals, so he followed the instructions in the poem to the letter (pun intended). Figure 1 below shows the numerals 987654321, with the Eastern forms written underneath.

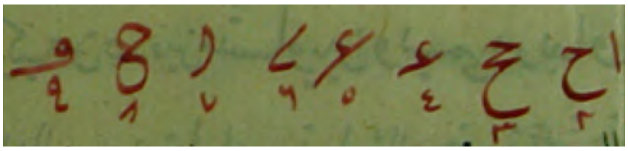


Figure 1: Digits in the Medina MS.

Starting from the right, the “1” looks like the letter *alif*, which is just a vertical line. The “2” is written as the letter *ḥā*, and the “3” is the ligature of the letters *ḥā* and *jīm*, pronounced *ḥajja* as if it were a word. The “4” should be the ligature of *ʿayn* and *wāw*, pronounced *ʿuw*, but the copyist forgot to add the *waw*. See below in Figure 2 for the “4” in the Istanbul manuscript. The “5” is shown as an *ʿayn* alone, so it looks like the “4” in this figure. The “6” is shaped like the letter *hā* (the copyist failed to close the loop), the “7” like an anchor, and the “8” is said to be a couple zeros (small circles), one above the other, connected by a vertical line (the *alif*). The line is not evident in the figure, and is generally not written at all. The Tunis manuscript writes the “8” the normal way, as , but just below it follows the instructions with the vertical line:  (and afterward the copyist wrote the Eastern forms of the digits). Finally, the “9” is written as the letter *wāw*.

The copyist of the Istanbul manuscript was already familiar with the Western forms. These are shown in Figure 2.

[8] (Kunitzsch 2003).

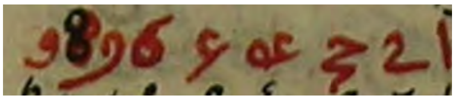


Figure 2: Digits in the Istanbul MS.

For comparison, Figure 3 shows the number 9367184225 from the same manuscript, with the Eastern forms above the Western forms. The Eastern “5”, on the right in black, is mistakenly written as a “4”.

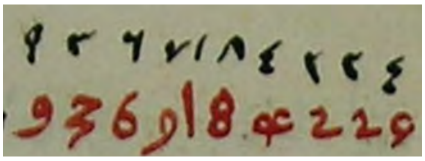


Figure 3: More digits, from the Istanbul MS.

70.2 There were no words for “million”, “billion”, etc. in Arabic arithmetic. They wrote instead “thousand thousand”, “thousand thousand thousand”, etc.

70.8 The places/ranks of a number were spoken in Arabic in a different order than they are in English. We preserve the Arabic order in our translation from here to the end of the chapter (through page 72 of our edition) to give the reader a feel for how the numbers were expressed.

Arabic numbers less than one hundred were spoken with the units first, like “four and sixty” instead of “sixty-four”. So “four and sixty thousands” is 64,000, and “four and sixty thousands and three hundred thousands” is 364,000, though this was often spoken as “three hundred thousands and four and sixty thousands”. In Arabic, the hundreds (like “three hundred” and “five hundred”) are compound words, so we translate “three hundred” instead of “three hundreds”. The word for "Thousands" is stated separately from its number, and is made plural when there is more than one of them. They wrote “a thousand” for one, and “two thousands”, “three thousands”, etc. for more than one.

The plural becomes more complicated when we get to the millions. When there are more than ten of something, the plural form of an Arabic noun is written the same as the singular form. So we read “four trees”, but fourteen of them reads like “fourteen tree”. Thus, what appears to be “ten thousands thousand” should be understood as “ten thousands thousands”. If there are 10,000 of the second word “thousand”, it must be plural.

Also, where we say “one hundred”, “one thousand”, etc., in Arabic the “one” is not written. They expressed these numbers with the implied indefinite article, as “[a] hundred”, “[a] thousand”, etc.

71.12 **Subsection on knowing the index of the repeated number.**

72.1 A 1, 2, or 3 is needed for the rule, so if the number is divisible by three, then three is regarded as the remainder.

73.1 **Section I.1.2. Addition.**

73.2 The act of adding two numbers, according to al-Hawārī's definition, should result in a single expression. Sometimes, though, the sum can only be expressed with more than one expression. For example, adding 978 to 456 yields one expression, 1434 (at 75.9), as does adding "a root of two to a root of eight" ($\sqrt{2}+\sqrt{8}$), which gives "a root of 18" ($\sqrt{18}$, at 179.16). The example at 180.10 shows an addition that results in two expressions: adding a root of three ($\sqrt{3}$) to a root of fifteen ($\sqrt{15}$) can only be expressed as "a root of three and a root of fifteen". We write this as $\sqrt{3}+\sqrt{15}$, but as we explain below at 219.1, the operation of addition is not inherent in this composite expression. The current section covers addition of whole numbers, which always results in one expression.

Four different verbs are used in the book to mean "to add": *jamaʿa*, *ḥamala*, *zāda*, and *ḍāfa*. Lane's definitions of *jamaʿa* begin "*to collect; bring,* or *gather together*".[9] He starts off his definitions of *zāda* with "*to increase, or augment, or grow*", while the various meanings of *ḥamala* begin with "*to bear it, carry it, take it up and carry it, convey it, or carry it off or away*". Lane gives no definition of *ḍāfa* that relates to addition, but Wehr has "to be added, be annexed, be subjoined, be attached".[10] There are similar variations for words meaning "exceed"/"surpass" (*zāda*, *faḍala*, *ʿalā*) and "sum" (*majmū* and related forms, *jumla*).

Ibn al-Bannāʾ covers five types of addition in the *Condensed Book*:

1. Adding numbers with no known relation. He covers the basic process of adding numbers in Indian notation beginning at 74.9.
2. Adding sequences of numbers with a known disparity, at 76.7.
   2a. In one kind of disparity, the ratio between consecutive terms is constant.
   2b. In the second kind, the difference between consecutive terms is constant.
3. Adding consecutive numbers, their squares, and their cubes, at 79.13.
4. Adding consecutive odd numbers, their squares, and their cubes, at 80.5.
5. Adding consecutive even numbers, their squares, and their cubes, at 80.20.

The rules for summing finite series, extending from 76.7 to 82.4, are (mostly) originally Greek in origin, but were probably borrowed from some intermediate Arabic source.[11]

73.7, 73.17 Ibn al-Bannāʾ's distinction between a disparity in quantity (*kamm*), in which the difference between consecutive terms is constant, and a disparity in quality (*kayf*), in

[9] (Lane 1863–1893, 455). Lane gives definitions in third person singular perfect tense. Here and elsewhere I have changed them to the infinitive.
[10] (Lane 1863–1893, 1275, 646), (Wehr 1994, 640).
[11] (Saidan 1996, 341ff).

which the ratios of consecutive terms is constant, comes from Nicomachus.[12] The terms "quantity" and "quality" appear again in the passage at 92.17, but with different meanings.

The Arabic for "geometric progression" is *nisba handasiyya*, literally "geometric relation". Ibn al-Bannāʾ writes "known disparity" instead of "known relation" because the word *nisba* ("relation") might imply the geometric progression. See below at 193.1 for more on the word *nisba*.

74.17 Al-Hawārī gives two examples of the first type of addition, starting with the addition of 4043 to 2685. First, the two numbers are written on two lines, like this:

4	0	4	3
2	6	8	5

Then the units 3 and 5 are added, and the result is put above (we put changes from the previous figure in red):

			8
4	0	4	3
2	6	8	5

The tens are next. Because $4 + 8 = 12$ has two digits, 1 is added to the 6 in the hundreds place of the lower number, and a 2 is put above the tens place:

		2	8
4	0	4	3
2	7	8	5

There is nothing to add to the 7, so 7 is put above the hundreds place:

	7	2	8
4	0	4	3
2	7	8	5

Finally, the thousands place is 6, from the sum of 2 and 4. The answer is 6,728:

6	7	2	8
4	0	4	3
2	7	8	5

[12] In the *Arithmetical Introduction*, Book II, Chapters 20-23 (Nicomachus 1938, 263, 266-70).

Because this method requires erasing and replacing, it was intended to be worked out on a dust-board or wax tablet, and not with ink.

Operating on zero

The rules for operating on numbers expressed in Indian notation call for the addition, subtraction and multiplication of digits, and sometimes one or both of these digits is a zero. The zero signifies a place where there is no number at all, so we should ask what it meant to operate on it. For this we need to understand that the operations themselves were thought of in a more material sense than our binary operations on abstract sets. Even the notion of a set is a modern one – there was no word for "set" in Greek, Latin, Arabic, Sanskrit, or medieval Italian. Premodern mathematicians, Europeans and Indians included, had no concept of a set as an object.

Addition for al-Hawārī was not an operation on $\mathbf{R}^+$ satisfying the commutative and associative axioms. It was simply the appending of a number with another number, or the gathering of numbers together, which were all regarded as amounts of something (dirhams, men, hours, etc.). To add five to three was like combining the five silver dirhams in one purse with the three silver dirhams in another purse, or like extending a length of five *adhruʿ* by three more *adhruʿ*, or like adding three *mathāqīl* of grain to five *mathāqīl*. Not even Euclid found it necessary to provide a definition for addition conceived like this, though some Arabic arithmetic books, al-Hawārī's included (at 73.2), characterize the operation.

For the operation of addition, Ibn al-Bannāʾ provides a special instruction when there is a zero present (74.9): "Then you add each digit of one of the addends to its counterpart in the other. If there is no counterpart, then the answer is the addend, as if it had a counterpart". Al-Hawārī follows this rule in the present calculation: "Nothing corresponds to the seven in the upper line, so it is considered to be the sum of that rank and that of its counterpart as if it had something". Adding nothing to 7 to get 7 does not mean that 0 assumes the role of an operable quantity. Instead, no addition takes place at all. Think of it like combining the money in two purses: one with 7 dirhams and the other empty. There is no act of combining to perform. Subtraction works similarly. In the passage at 83.19 al-Hawārī is faced with the subtraction of 0 from 9: "So we subtract this nothing of the minuend from the nine of the subtrahend, leaving nine". No subtraction takes place when taking nothing away from an amount, so it leaves the amount unchanged.

Multiplication by zero is explained in the passage at 114.4: "multiplying the number by the zero or the zero by the number is identical. It comes from voiding the number or duplicating zero. Neither of these gives a number, so its sign is always a zero". The word behind "voiding" (*taṣfīr*) is related to the word for "zero" (*ṣifr*). The former could have been translated as "emptying" or "zeroing", and "zero" could be replaced with "nothing". This duplication conforms to the standard definition of multiplication, given by Ibn al-Bannāʾ at 95.2: "Multiplication consists of the duplication of one of two numbers by however many units are in the other". Duplicating nothing a number of times surely gives nothing, so the multiplication makes sense even if zero, being nothing, is not a number.

See the passages in the translation at 83.19, 84.13, 90.7, 90.18, 123.3, and 215.14 for other operations with zero and/or nothing.

75.9 Al-Hawārī's second example shows addition starting from the highest power term. He adds 978 to 456, first writing one above the other as before:

9	7	8
4	5	6

Working from the hundreds place, $9 + 4 = 13$, so 13 is placed above:

1	3		
	9	7	8
	4	5	6

Next, $7 + 5 = 12$, so a 2 is put above the 7, and 1 is added to the 3 next to it:

1	4	2	
	9	7	8
	4	5	6

Finally, $8 + 6 = 14$, so the 4 is put above the 8, and 1 is added to the 2 next to it to get the answer, 1,434:

1	4	3	4
	9	7	8
	4	5	6

76.7 For the second type of addition, Ibn al-Bannāʾ works with the famous chessboard problem. In some books, a grain of wheat is placed in the first square, two grains in the second, four in the third, etc. Ibn al-Bannāʾ simply places numbers in the squares, as did Abū Kāmil when he wrote about it in the late ninth century CE at the end of his *Book on Algebra*.[13] A 1 is placed in the first square, a 2 in the second, a 4 in the third, and continuing so that each square has double the previous square, like this:

128	64	32	16	8	4	2	1
	…	8192	4096	2048	1024	512	256

[13] (Abū Kāmil 1986, 218); (Abū Kāmil 2012, 725).

Ibn al-Bannāʾ gives the rule for finding the sum of the numbers from the first square up to the 2^nth square. Al-Hawārī gives the example for the 2^4th = 16th square. The following iteration is performed:

- Take the 1 in the first square. Add 1 to get 2.
- Square it to get 4. This is 1 more than what is in the first two squares $(1 + 2)$, and it is also what is in the third square (4).
- Then square the 4 to get 16. This is 1 more than what is in the first four squares $(1 + 2 + 4 + 8)$, and it is also what is in the fifth square (16).
- Square the 16 to get 256. This is one more than what is in the first 8 squares $(1 + 2 + 4 + 8 + 16 + 32 + 64 + 128)$, and it is also what is in the 9th square (256).
- Square the 256 to get 65,536. This is 1 more than what is in the first 16 squares, and it is also what is in the 17th square.

So, the sum of the numbers in the first 16 squares is 65,535.

The figure shown in the translation is the one found in the Medina, Tehran, and Tunis manuscripts. The Istanbul and Oxford manuscripts show this figure instead (only Istanbul has the 65536 written on the left):

65536

8	4	2	1
128	64	32	16
2048	1024	512	256
32768	16384	8192	4096

77.9 Ibn al-Bannāʾ then gives a variation in which the first square has a number other than 1, and the rule for filling out the remaining squares in the chessboard is the same: each square is double the one before it. For example, if the first square has a 3, then the succeeding squares are 6, 12, 24, etc. The rule to find the sum of the first 2^n squares is to follow the procedure as if a 1 were in the first square, then one multiplies the result by the number that is in the first square. Al-Hawārī gives the example of adding $4 + 8 + 16 + 32 + 64 + 128 + 256 + 512$. The sum of the first eight squares starting with 1 is $1 + 2 + 4 + 8 + 16 + 32 + 64 + 128 = 255$. Multiplying this by 4 gives 1020, which is the required sum.

78.1 Another variation is when the ratio of consecutive terms is some number other than $\frac{1}{2}$. The example given by al-Hawārī starts with 16, and each square is 2/3 of the succeeding square. He gives the first five numbers: 16, 24, 36, 54, 81. Putting Ibn al-Bannāʾ's rhetorical rule into modern form, the sum will be $\frac{16\cdot(81-16)}{24-16} + 81 = 211$. We leave the general rule as an exercise for the reader.

79.1 Ibn al-Bannāʾ then gives a rule for summing sequences of numbers in which the difference, rather than the ratio, of consecutive terms is constant. Al-Hawārī's example is to add the 6 numbers starting with 10, and with a difference of 3. If we were to write it all out the sum would be $10 + 13 + 16 + 19 + 22 + 25$, but we are working only with the known numbers 6, 10, and 3. The rule begins by finding the last number, which in this case is $3 \cdot (6-1)+10 = 25$. Then $(25+10)\cdot(\frac{1}{2}$ of 6$) = 105$ is the sum. Ibn al-Bannāʾ's rhetorical rule can be expressed in modern notation this way: if there are n numbers starting with a and with a difference of d, then the last number b is $d(n-1)+a$, and the sum is $(b+a)\cdot\frac{1}{2}n$.

79.13 The third type of addition covers consecutive numbers, their squares, and their cubes. It is true that the first of these is a special case of the second type of addition, but here the first and last numbers are both given, and the rule is then used to find the sums of the squares and the cubes.

To add the numbers from 1 to 10, multiply half of the 10 by one more than the 10: $5\times 11 = 55$. In modern notation, $\sum_{k=1}^{n} k = \frac{1}{2}k \cdot (k + 1)$.

To add the squares of these numbers, or $1 + 4 + 9 + 16 + \cdots + 100$, the rule is $(\frac{2}{3} \cdot 10 + \frac{1}{3}) \cdot 55 = 385$. In general, $\sum_{k=1}^{n} k^2 = \left(\frac{2}{3}n + \frac{1}{3}\right) \sum_{k=1}^{n} k$.

To add consecutive cubes, square the sum of the numbers. The example is $1 + 8 + 27 + \cdots + 1000 = 55^2 = 3025$. In general, $\sum_{k=1}^{n} k^3 = \left(\sum_{k=1}^{n} k\right)^2$.

Medieval Arabic mathematicians wrote "[a] square" of a number, with the implied indefinite article, rather than "the square" because their numbers admit multiplicity. See below at 163.2 for a more detailed explanation for the case of roots.

80.5 The fourth type of addition is to add the consecutive odd numbers, their squares, and their cubes. For the first of these, square half of one more than the last number. Al-Hawārī's example is to find $1+3+5+7+9$. The answer is $(\frac{1}{2}(9 + 1))^2 = 25$. In general, if the last number is n, then the sum $1 + 3 + 5 + \cdots + n = (\frac{1}{2}(n + 1))^2$.

The sum of the squares of consecutive odd numbers up to n is $\frac{1}{6}n(n + 1)(n + 2)$. In the example, al-Hawārī finds that $1^2 + 3^2 + \cdots + 9^2 = \frac{1}{6}9 \cdot 10 \cdot 11 = 165$.

If we let a be the sum of the odd numbers up to n, then the sum of the cubes of the odd numbers to n^3 is $a(2a-1)$. Al-Hawārī calculates that $1^3+3^3+5^3+7^3+9^3 = 25\cdot 49 = 1225$.

80.20 The fifth and last type of addition deals with consecutive even numbers, their squares, and their cubes. Al-Hawārī adds the even numbers from 2 to 10 by calculating $\frac{1}{2}(2 + 10) \cdot \frac{1}{2}10 = 30$.

For the squares $2^2 + 4^2 + \cdots + 10^2$ one adds $\frac{2}{3}$ of 10 to $\frac{2}{3}$ of 1, and the result is multiplied by the sum, 30: $6\frac{2}{3} + \frac{2}{3} = 7\frac{1}{3}$, and $7\frac{1}{3} \cdot 30 = 220$.

Al-Hawārī gives an alternative rule, using the example of adding the squares $2^2 + 4^2 + \cdots + 12^2$. Here one multiplies a sixth of the last number (12) by the product of the next two numbers (13 and 14). Taking $\frac{1}{6}$ of 12 gives 2, and 2 by 182 is 364, which is the required sum. This happens to be the same rule he gave for adding the odd squares.

The sum of the even cubes $2^3 + 4^3 + \cdots + 10^3$ "is given by multiplying the sum by its double". We already know that the sum $2 + 4 + \cdots + 10$ is 30, so we multiply 30 by 60 to get 1800, which is the answer.

83.1 **Section I.1.3. Subtraction.**

Just as with addition, several verbs are used for subtraction. The most common, and the one which appears in chapter titles and instructions, is *ṭaraḥa*. We translate it as "to subtract". The verb *ṭaraḥa* is also used for what we call "casting out", as in casting out nines to check the answer to a calculation. Two other common verbs for subtraction are *saqaṭa*, "to drop", and *naqaṣa*, which we also translate as "to subtract". Rarer are the verbs *nazala*, "to remove", and *dhahaba*, "to take away".

To announce the result of a subtraction the verb *baqiya* ("to remain, leave") is used. The word for "remainder" is *bāqī*. Often we translate a phrase whose literal meaning is "[there] remains" or "what remains" as "the remainder". The word for the "residue" after casting out nines or eights or sevens, covered starting at 87.15, is the related word *baqiya*, and we translate *bāqiya* as "residual". These two words appear one time each to mean "remaining" and "remainder", respectively, at 110.16 and 118.20.

83.2 Ibn al-Bannāʾ writes of two kinds of subtraction. The first is the subtraction of one number from another with Indian numerals, and the second is "casting out" to check the answer of a calculation. Between these two kinds al-Hawārī inserts a description of repeated subtractions, starting at 86.1, that he took from *Lifting the Veil*.

Two examples are given for the first kind. Al-Hawārī begins with the example $5035 - 4968$ and proceeds from the highest rank to the lowest. First, the greater number is written above the smaller:

5	0	3	5
4	9	6	8

The 4 is subtracted from the 5, and the result is written above:

1			
5	0	3	5
4	9	6	8

For the hundreds place, there is nothing (i.e., a 0) in the minuend, so one takes nothing away from the 9 in the subtrahend, leaving 9. This is subtracted from the 1 above the 5, which is really 10 since we are now working in the hundreds place. 9 from 10 leaves 1, so the first 1 is replaced with a 0, and this new 1 is placed above the hundreds place:

0	1		
5	0	3	5
4	9	6	8

In the tens place the 3 is smaller than the 6, so we subtract 3 from 6 to get 3, and this is subtracted from the 10 above, leaving 7. The figure then becomes:

0	0	7	
5	0	3	5
4	9	6	8

The situation is similar for the units place. Since 5 is less than 8, we subtract their difference, which is 3, from the 70 above. This leaves 67 as the answer:

0	0	6	7
5	0	3	5
4	9	6	8

84.13 Al-Hawārī gives a second example that starts with the units place. He begins with:

6	5	4	3
3	4	6	9

The 3 is less than 9, so we add 10 to the 3 and then subtract 9, leaving 4. This is placed above the 3. To compensate for the added 10, a 1 is added to the 6 next to the 9:

			4
6	5	4	3
3	4	7	9

For the tens place we have a similar situation: 4 is less than 7. So add 10 to the 4 and subtract, leaving 7. Then add one to the 4 in the bottom row to get:

		7	4
6	5	4	3
3	5	7	9

Next, taking 5 from 5 leaves nothing, so a 0 is placed above them:

	0	7	4
6	5	4	3
3	5	7	9

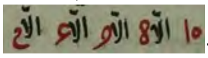
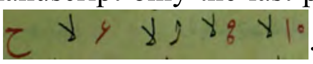
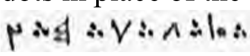
Taking 3 from 6 leaves 3, so the figure becomes:

3	0	7	4
6	5	4	3
3	5	7	9

The answer is the 3,074 on top.

85.16 *Rūmī* signs were used in a system of calculation practiced in Western North Africa and al-Andalus. See §3 in the Introduction.

86.1 The part on repeated subtractions is taken from Ibn al-Bannāʾ's commentary, and is not mentioned in the *Condensed Book*. It is not one of the two categories of subtraction he mentions at 83.2 at the start of the chapter. In modern notation the expression "ten less eight less seven less five less two" is $10 - (8 - (7 - (5 - 2)))$. This is explained in words from the inside out: "subtract two from five, and the remainder from seven, …" and in an English version of the Arabic notation it would be 10 ℓ 8 ℓ 7 ℓ 5 ℓ 2, where the ℓ stands for "less". We are keeping the direction of the Arabic figures, so we write it in the translation as 2 ʅ 5 ʅ 7 ʅ 8 ʅ 10. See the discussion at 219.4 below for an explanation of the word "less" (*illā*).

In two of the five manuscripts we consulted, Istanbul and Oxford, the numbers are separated by the word *illā*. Here it is in the Istanbul manuscript: . In the Medina manuscript only the last part of the word is drawn, so it looks like an upside-down "ℓ": . The two other manuscripts, Tunis and Tehran, put three dots in place of the word *illā* for this figure. Here is the figure from the Tunis manuscript: .[14] Both manuscripts, curiously, write the *illā* in other instances.

86.15 One could simply perform the operations as stated: subtract 2 from 5, then subtract the result from 7, etc. But Ibn al-Bannāʾ gives three other rules to work it out. For his first

[14] The first trio of dots on the right should not be there.

rule he distinguishes between the minuend, in this case 10, and the subtrahends, which here are 8, 7, 5, and 2. Add the even subtrahends (the 7 and 2) to the 10, and from this subtract the odd subtrahends, giving $19 - 13 = 6$.

86.18 Another way is to collapse the string of numbers in groups of three. We "subtract the middle from the sum of the extremes, leaving the remainder as one number". Taking the 10, 8, and 7, calculate $10 + 7 - 8 = 9$, and replace all three with the 9 to get 2 ⸜ 5 ⸜ 9. Repeating the process gives $9 + 2 - 5 = 6$. One does not need to start with the first three numbers. This works starting from any three consecutive numbers.

87.9 A third way is to perform alternating subtractions and additions, beginning with the 10. Subtract 8 from 10, then add 7 to the remainder, then subtract 5, and finally add 2.

87.11 See our extended discussions below, at 219.1 and 219.4, for an explanation of "appended" (*zāʾid*) and "deleted" (*nāqiṣ*). They do not mean "positive" and "negative". Because numbers were numbers of something counted or measured, negative numbers would have been meaningless to medieval arithmeticians.

87.15 The second kind of subtraction is what we call in English "casting out". In "casting out nines", which is still taught today, the remainder from division by 9 can be found by adding the digits and removing multiples of 9. Al-Hawārī gives the example of 6435. He adds the digits one by one, casting out nines as he goes. So $6 + 4 = 10$, and removing a 9 leaves 1. Then $1 + 3 = 4$, and $4 + 5 = 9$. "This is cast out entirely", meaning nothing remains. Nothing is literally no number at all, and not our modern number 0.

Al-Hawārī also gives examples of the rules for casting out eights and casting out sevens. Because multiples of 200 are divisible by 8, one only needs to deal with the first three places in casting out eights. Al-Hawārī's example is 5393. The 5000 is cast out entirely, as is the 200 from the 300. The remainder from the 100 is 4, then 2 is multiplied by the 9, and to these are added the 3, giving 25. Casting eights from this leaves 1. In general, the remainder of a number of the form $1ab$ (i.e., $100 + 10a + b$) is the same as the remainder of $4 + 2a + b$.

88.10 Casting out sevens is more complicated. No multiple of 10 is divisible by 7, so all digits must be taken into account. The remainders of each power of 10 are different for the first six powers, after which they repeat. These remainders must be memorized, and al-Hawārī illustrates Ibn al-Bannāʾ's rule of expressing this sequence of remainders in *abjad* form. The letters appearing there are:

The remainders of 1, 10, 100, 1000, 10000, and 100000 are 1, 3, 2, 6, 4, and 5, respectively. At 88.14 he gives a short poem designed to help the student memorize these letters.

88.17 Indian numerals were sometimes called *al-ghubār*, or "dust" numerals, after the dust-board on which they were commonly written.

89.4 The example al-Hawārī gives is to cast out sevens from 23,786,435. He writes the digits above their corresponding letters:

Arabic		Transliteration	Value
ا	*alif*	A	1
ب	*bā*	B	2
ج	*jīm*	J	3
د	*dāl*	D	4
ه	*hā*	H	5
و	*wāw*	W	6

2	3	7	8	6	4	3	5
J	A	H	D	W	B	J	A

For us, it is easier to use the Indian numerals:

2	3	7	8	6	4	3	5
3	1	5	4	6	2	3	1

Multiplying the digits in the corresponding places, and casting out sevens if necessary, gives:

6	3	0	4	1	1	2	5
2	3	7	8	6	4	3	5
3	1	5	4	6	2	3	1

For example, in the thousands place we multiply 6 by 6 to get 36, and casting out sevens leaves 1. The digits in the top row add up to 22, and casting out sevens again leaves 1.

90.3 Ibn al-Bannāʾ gives two variations for casting out sevens that do not require the memorization of the sequence of *abjad* numerals. The first is an iteration: first multiply the highest power digit by three (the residue of 10), cast out sevens if necessary, and then add the result to the previous digit. Al-Hawārī's example is 58,064. Starting with the highest power term, multiply 5 by 3 to get 15. Cast out sevens to get 1. Then add this 1 to the 8 to get 9. Now repeat: multiply 9 by 3 to get 27, leaving 6 after casting out. Add 6 to nothing (the 0) to get 6. Multiply by 3 to get 18, and cast out sevens, leaving 4. Add 4 to 6 to get 10. Thrice 10 is 30; cast out to get 2, and add it to the 4 to get 6. This is the answer.

90.15 The other variation takes into account two digits at a time. Casting out sevens from the 58 in 58,064 leaves 2. Although al-Hawārī does not write it, the residue of 58,064 is the same as the residue of 2,064. Next, the residue of 20 is 6, so we now consider 664. The residue of 66 is 3, so now the number is 34. Its residue is 6, which is the answer.

91.1 **Subsection on the way to test [calculations] by casting-out.**

Next, Ibn al-Bannāʾ turns to applications of these techniques of casting out to check the results of arithmetical operations. These include addition, subtraction, multiplication, and division/denomination. For the latter two it works even for fractions. Here are a couple of al-Hawārī's examples.

91.4 For addition, he works with the example $43 + 64 = 107$. He could cast out nines, eights, or sevens, and for this and subsequent examples he chooses sevens. The residue of 43 is 1, and the residue of 64 is also 1. Add them to get 2, which should be (and is) the residue of 107. If the residues added to 7 or more, one would cast out a 7 to make it less than 7.

92.3 In subtraction there is a problem if the residue of the minuend is smaller than the residue of the subtrahend. Take for example $29 - 13 = 16$. The residue of the 29 is 1, and the residue of the 13 is 6. We cannot subtract 6 from 1, so we add 7 to the 1 and then subtract: $8 - 6 = 2$, and this 2 is the residue of the remainder 16. For casting out nines, add nine to the residue of the minuend, and for casting out eights, add eight.

92.17 An example for the multiplication of fractions is $\frac{1}{3} \times 14\frac{1}{4} = 4\frac{3}{4}$. The residues of the multipliers are $\frac{1}{3}$ and $\frac{1}{4}$. Ibn al-Bannāʾ does not say so, but these are the residues of the numerators. If, for instance, the multiplier were $3\frac{1}{4}$, he would have found the residue to be $\frac{6}{4}$, since $3\frac{1}{4}$ as a single fraction is $\frac{13}{4}$, and the residue of 13 is 6.

The product of the $\frac{1}{3}$ by the $\frac{1}{4}$ is $\frac{1}{12}$, or as Ibn al-Bannāʾ puts it, "a third of a fourth". The numerator of this fraction is 1. The product $4\frac{3}{4}$ is $\frac{19}{4}$, and Ibn al-Bannāʾ takes the residue of the numerator, which is 5. But these are fourths, not twelfths, so he multiplies by 3 to get 15 (for $\frac{15}{12}$), whose residue is 1. This agrees with the residue of the multipliers.

The numerator 1 of "a third of a fourth" is equal to the numerator of the answer, so they are equal in quantity. The kinds of fractions they are, thirds of fourths, are the same, making them equal in quality. This terminology comes from Aristotle's *Categories,* probably via Ibn Sīnā.[15] See above at 73.7 for another use of the words "quantity" and "quality".

93.6 As Ibn al-Bannāʾ will explain in the section on division at 118.14, the term "division" is used for the division of a greater number by a smaller number, and "denomination" for the division of a smaller number by a greater number. For the division $a \div b = c$, where $a > b$, the a is the dividend and b the divisor. If $a < b$, then a is the denominated [number] and b is the denominating [number] or what we call the denominator. For both kinds, c is the quotient or result. So "the result and the divisor or denominating number" are c and b, and the "dividend or denominated number" is a.

The Arabic words for the "denominator" of a fraction (*imām*, and less frequently *maqām*) are unrelated to the verb "denominate" (*sammā*) and related nouns such as "denominating [number]" (*musammā minhu*).

[15] *Categories* 4b20-24; (Avicenna 2005, 10, 72).

93.10 For both division and denomination, al-Hawārī gives an example with whole numbers and an example with fractions.

The Arabic word for the "numerator" of a fraction is *basṭ*, or occasionally the related word *mabsūṭ*. These words are unrelated to the word for "number", which is *ʿadad*. We translate the related verb *basaṭa* as "to numerate". Its meaning is to find the numerator of a fraction that is "a combination of two or more names" (135.1). For example, to numerate the fraction "five sixths and three fourths" (at 93.15) means to express it as "thirty-eight fourths of a sixth" ($\frac{38}{4\cdot6}$). Finding the numerator is described at length in the first section on fractions, beginning at 135.8.

93.15 The notation for distinct fractions shows one next to the other, as explained later at 136.8 and 139.1. This problem is to divide $\frac{3}{4}\,\frac{5}{6}$ by $\frac{1}{2}$. We would write $\frac{3}{4}\,\frac{5}{6}$ as $\frac{5}{6}+\frac{3}{4}$, and the result of the division as $3\frac{1}{6}$. The residue of $3\frac{1}{6}=\frac{19}{6}$ is 5, or $\frac{5}{6}$. The residue of $\frac{1}{2}$ is 1, or $\frac{1}{2}$. Multiplying them, one gets $\frac{5}{12}$, or, as Ibn al-Bannāʾ says, "five halves of a sixth". The residue of the dividend, $\frac{3}{4}\,\frac{5}{6}=\frac{38}{24}$, is 3, with a denominator of 24. So the $\frac{5}{12}$ must be converted to 24ths, making it $\frac{10}{24}$, which is $\frac{3}{24}$ after casting out again. This matches the residue of the $\frac{38}{24}$.

94.1 Al-Hawārī gives an example of checking the result of the denomination of whole numbers even if it might be superfluous in practice. The example is to denominate 11 with 15, which for us gives the fraction $\frac{11}{15}$. The residue of the denominated number 11 is of course the same as the residue of the numerator 11 of the result, so there is nothing to check. The only aspect that makes this appear to be a problem is that al-Hawārī follows common Arabic practice by expressing the result not as $\frac{11}{15}$, but as "three fifths and two thirds of a fifth". We might write this as $\frac{3}{5}+\frac{2}{3}\frac{1}{5}$, but for al-Hawārī it would be shown as $\frac{2}{3}\frac{3}{5}$ (the notation for this fraction is explained at 123.18). The numerator and denominator of this fraction must be calculated in order to find its residue, but this brings us right back to the 11 and 15 we started with.

94.5 Checking the result of the denomination of fractions requires some work. Writing Ibn al-Bannāʾ's example in notation, it is to denominate $\frac{2}{3}\frac{2}{6}$ with $\frac{1}{3}\frac{5}{8}$, which results in $\frac{2}{3}$. After finding the numerators, the problem remains to denominate $\frac{8}{6\cdot3}$ with $\frac{16}{8\cdot3}$. The product of the residues of $\frac{2}{3}$ and $\frac{16}{8\cdot3}$ should equal the residue of $\frac{8}{6\cdot3}$, but we need to be sure the denominators are the same to make it work. The residue of the numerator of $\frac{2}{3}$ is 2. The residue of the numerator of $\frac{16}{8\cdot3}$ is also 2, and multiplying the 2 (as $\frac{2}{3}$) by the 2 (as $\frac{2}{8\cdot3}$) gives $\frac{4}{3\cdot3\cdot8}$. Because the 6 in the denominator of the $\frac{8}{6\cdot3}$ is lacking in the $\frac{4}{3\cdot3\cdot8}$, we multiply the numerator and denominator of the latter by 6 to get $\frac{24}{3\cdot3\cdot6\cdot8}$, and its residue is 3, which is the "answer". We now turn our attention to $\frac{8}{6\cdot3}$. The residue of the 8 is 1, and this must be multiplied by 3 and then by 8 to make the denominators match. This gives $\frac{24}{3\cdot3\cdot6\cdot8}$, and its residue is 3, which agrees with the answer.

95.1 **Section I.1.4. Multiplication.**

95.2 Arabic arithmetic books often define multiplication as "the duplication of one of two numbers by however many units are in the other", as Ibn al-Bannāʾ phrased it here in the *Condensed Book*.[16] This definition by "duplication" may come from Book VII of Euclid's *Elements*: "A number is said to multiply a number when that which is multiplied is added to itself as many times as there are units in the other, and thus some number is produced".[17] In fact, Abū l-Wafāʾ (tenth century) writes in his arithmetic book: "Euclid stated the meaning of multiplication in Book VII of his work *The Elements*, as did Nicomachus of Gerasa in the *Arithmetic*. They said that multiplication is the duplication of one of two numbers by the amount of what is in the other in units".[18]

It would not matter so much that Nicomachus, in fact, gives no definition of multiplication in his *Arithmetical Introduction* if it were not for what Abū l-Wafāʾ says about division: "Division according to the example (*qiyās*) of Euclid and Nicomachus is the partitioning (*tafrīq*) of one of two numbers by the amount of units in the other".[19] Neither Euclid nor Nicomachus defines division in their books, and if this statement is not intended to make such a claim, it at least implies that this definition, too, originates with the Greeks.

Contrary to what Abū l-Wafāʾ's testimony might imply, there is evidence that the "duplication" definition of multiplication was already circulating among Arabic arithmeticians before they began reading Euclid. The definition appears in al-Khwārazmī's algebra book, written sometime between 813 and 833 CE. This book presents the practical algebra connected with finger-reckoning, and the only hint of Greek influence is the presence of letters to label points in some diagrams.[20] Later, al-Khwārazmī copied the definition into his book on calculating with Indian numerals. This way of characterizing multiplication is rather intuitive, and it is entirely possible that arithmeticians developed or learned it independent of Euclid.

It may have been his desire to tie practical Arabic arithmetic with the Greek tradition that led Abū l-Wafāʾ to imply a false origin for both definitions. We have found another instance of this kind of "false history" in the tenth century philosopher al-Fārābī. In the third part of his *Enumeration of the Sciences* he classifies the various branches of mathematics. He describes algebra as being "concerned with the ways of figuring out how to find numbers that apply Euclid's principles on the rational and irrational in the tenth book of the *Elements*…"[21] Yet not one of the extant books on algebra written before him, and to our knowledge none written after him, show any sign of applying principles from Book X to algebra.[22] Al-Fārābī's wishful connection of algebra to Euclid's *Elements* seems to have been a way to legitimize the prevalence of irrational numbers in algebra and to suggest a Greek inspiration for the art.

Whatever its source, the "duplication" definition of multiplication was devised with whole numbers in mind. Abū l-Wafāʾ criticizes it on this point, and shows through an example of

[16] Most arithmetic books give this definition. We found four arithmetic books that give no definition at all: those of al-Uqlīdisī, Kūshyār ibn Labbān, al-Baghdādī, and al-Qurashī. (Titles of their books are given in Appendix C.) The word for "duplication" is almost always *ḍaʿafa*.
[17] (Euclid 1956, vol. 2, 278).
[18] (Saidan 1971, 124.16).
[19] (Saidan 1971, 126.10).
[20] (al-Khwārizmī 2009, 123.6).
[21] (al-Fārābī 1953, 74.10).
[22] There were some books that apply algebra to solve problems relating to Book X, but not the other way

the price of cloth that it also works for fractions. He justifies this extension by appealing to propositions about lines, reinterpreted numerically, from Book II of the *Elements*.[23] Ibn al-Bannāʾ gives a different definition for the multiplication of fractions, below at 149.2. Neither he nor al-Hawārī defines the multiplication of irrational numbers, even if these multiplications are performed in the book (183.2). Definitions in practical books like Ibn al-Bannāʾ's should be read more as intuitive characterizations that introduce students to the concepts rather than precise definitions of the kind that Aristotle or David Hilbert would have approved.

95.3 Ibn al-Bannāʾ distinguishes between two types of multiplication. In the first type, one puts a copy of the multiplicand in place of each unit of the multiplier. The example is "three men: each of them has five dirhams. You multiply five by three, which gives fifteen dirhams". Each of the three men is substituted with five dirhams, to get a total of fifteen dirhams. There is a change in meaning (*maʿnan*) because the three (men) became fifteen (dirhams). Also, the units shift from (three) men to (fifteen) dirhams, which is a change in terms (*lafẓ*).

The example for the second type is "five dirhams, how many thirds does it contain?". This type is called "conversion" because the units for the amount "five dirhams" is converted from dirhams to thirds of a dirham. There are 5 of the former and 15 of the latter. The total amount "five dirhams" remains the same, so there is no change in meaning. But the shift in units, from dirhams to thirds of a dirham, is a change in terms. "All of what is in the multiplier in units", or three (thirds of a dirham), "is equal to the one [dirham] of the multiplicand". Thus the "[number of] units in the multiplier", or three, "is the number of what is in one of the multiplicand in parts", or, more loosely translated, "is the number in each part [i.e., each dirham] of the multiplicand". Conversion of fractions is covered later, at 157.9, and the two corresponding types of division are explained beginning at 117.9.

Ibn al-Bannāʾ's description of the first type, at 95.3, is problematic. It reads "in putting down the multiplier, each one of them is equal to the one of the multiplicand". Reading the example for this type at 95.10, it could be translated as: "in putting down the multiplier, each one of them stands for (*mithl*) the single entity (*wāḥid*) of the multiplicand". One should substitute for each unit in the multiplier the entire multiplicand, not just its unit ("the one"). But this translation is at odds with the description of the second type of multiplication, which begins: "all of what is in the multiplier in units is equal to the one of the multiplicand". The same phrase "the one of the multiplicand" in this case must mean its unit, and not the multiplicand as a single entity. Perhaps the description of the first type is misstated.

In *Lifting the Veil*, Ibn al-Bannāʾ completes his descriptions and examples by writing "The first type is combining (*tarkīb*) and the second is dissecting (*tafṣīl*)", though al-Hawārī does not copy it. In the example of the first type the three copies of five dirhams are combined, and in the example of the second type each of the five dirhams is dissected into three pieces.

Modern mathematicians work with only one kind of multiplication because our numbers all belong to the same abstract set. Instead, in Arabic arithmetic a number is a number

around.

[23] (Saidan 1971, 124).

of something, whether it be men, dirhams, mithqals, or abstract units. This is why Ibn al-Bannāʾ distinguishes between a change in terms and a change in meaning. (See also our comments on division, below at 117.9.) This idea of different kinds of number is also behind the names of the powers of ten (hundreds, thousands, etc.), the names of fractions (thirds, fourths, etc.), and the names of the powers of the unknown in algebra (numbers, things, *māl*s, cubes, etc.). So "three men" was a particular kind of 3, like "three boats", "three thousands", "three fifths", and "three things".

Arabic authors often designated numbers by their kind, and several examples are found in al-Hawārī's book. Just above, at 95.7, we read "the units in the multiplier"; at 114.12, we find "the ranks of the result"; at 208.1; "if we make the starlings twenty-four"; and at 215.15, "we square half of the things". These nouns mean, respectively, "the number of units", "the number of ranks", "the number of starlings", and "the number of things". If one has 24 starlings, for example, then that collection of birds is an instance of the number 24, and one can indicate their number by saying merely "the starlings".

This idea that a number has two aspects, its meaning (value) and its term (the kind of number), breaks down with irrational roots, though our authors do not discuss it. This is because it makes no sense to have an irrational number of anything, like $\sqrt{5}$ bricks, for example. This becomes problematic in algebra, where the "coefficient" (the "number", or meaning) of a term had to remain rational even when multiplying by an irrational number. See the discussion below in the last paragraph of our commentary to 219.1.

95.15 Rules for multiplying numbers, Ibn al-Bannāʾ tells us, fall into one of three categories: those with shifting, those with half-shifting, and those without any shifting. Methods involving shifting are designed for the dust-board or wax tablet, where it is easy to erase and rewrite the digits. Those without shifting were intended for the *lawḥa* with ink pen, and will also work on paper. He also describes some techniques of mental arithmetic at the end of the chapter, beginning at 108.9, that he took mainly from Ibn al-Yāsamīn's *Grafting of Opinions*.

95.17 The first multiplication rule with shifting is called "sleeper multiplication", perhaps because the numbers are placed horizontally as if they are sleeping on a bed. There is another method of "sleeper multiplication", without shifting, that features numbers written horizontally. It is described at 103.14.

96.3 Al-Hawārī gives the example of multiplying 43 by 54. They are arranged on the dust-board like this:

	4	3
5	4	

The overall scheme will be to multiply the 4 of the 43 by the 54, then the 54 will be shifted to the right, and then the 3 from the 43 will be multiplied by the 54. First the 4 from the 43 is multiplied by the 5, and the 20 is placed above:

2	0	4	3
	5	4	

The 4 from the 43 is then multiplied by the 4 below, giving 16. The 1 from the 16 is added to the 0 we just wrote, and the 6 of the 16 replaces the 4:

2	1	6	3
	5	4	

The 54 is then shifted to the right one unit, giving this figure:

2	1	6	3
		5	4

Now we multiply the 3 by the 5 below. The resulting 15 is added to the 16 above to get 31:

2	3	1	3
		5	4

Now the 3 by 4 gives 12. The 1 is added to the 1 on top, and the 2 replaces the 3:

2	3	2	2
		5	4

The result of the multiplication of 43 by 54 is 2,322.

It is important to know when to add a digit and when to replace a digit. One adds to digits that have been calculated, while one replaces digits from the original problem. This way, the digits of the multiplicand, 43, are replaced with the digits of the evolving calculated number.

97.1 The other method of multiplication with shifting is called vertical multiplication. It works the same way as the sleeper multiplication just described. Al-Hawārī's example is to multiply 42 by 37. The digits are arranged vertically, with units on top:

2	
4	7
	3

Like before, we first multiply the 4 of the 42 by the 3 of the 37 to get 12. This is placed next to the 3, with the 2 above the 1:

2	
4	7
2	3
1	

The 4 is then multiplied by the 7, giving 28. This will be placed to the left, too. As before, we replace the digit 4 of the multiplicand with the 8, and we add the 2 to the calculated 2 that lies below it:

2	
8	7
4	3
1	

Now the multiplier is shifted up one unit:

2	7
8	3
4	
1	

The 2 on the left will now be multiplied by the 37. First, $2 \times 3 = 6$, so we add the 6 to the calculated 8, giving 14. Because of the extra digit, the 1 is added to the 4 below:

2	7
4	3
5	
1	

Finally, $2 \times 7 = 14$. The 4 replaces the 2, and the 1 is added to the 4 below it:

4	7
5	3
5	
1	

The result of multiplying 42 by 37 is 1,554.

98.4 The method by half-shifting is a technique for squaring numbers. Al-Hawārī's example is to multiply 463 by itself. First, the number is written with dots between the digits:

4 • 6 • 3

The 4 is squared, and the resulting 16 is written above:

```
1 6
  4 • 6 • 3
```

Next, double the 4 to get 8, and put this in place of the first dot:

```
1 6
  4 8 6 • 3
```

This 8 is regarded as being shifted, and it is multiplied by the 6 to its right to get 48. This result is added above the 8, treating it like 160 + 48:

```
2 0 8
  4 8 6 • 3
```

Next the 6 is squared, giving 36. This is added to the 2080 above:

```
2 1 1 6
  4 8 6 • 3
```

Now the 6 is doubled to get 12, and this is shifted one place to the right so that it replaces the "6 •". One must also shift the 8, treated as 80 since it is one place to the left, that was doubled before. This is added to the 12, making 92:

2	1	1	6		
	4	8	9	2	3

Now the last 3 must be multiplied by this shifted 92. The 3 by the 9 gives 27, which is added to the 16 above the 89, making 43:

2	1	4	3		
	4	8	9	2	3

Next 3 by 2 is 6, and this is added above the 2:

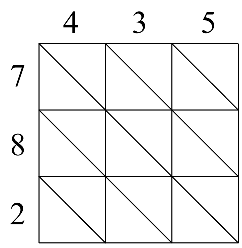
2	1	4	3	6	
	4	8	9	2	3

Last, we square the 3 to get 9, and add this above the 3:

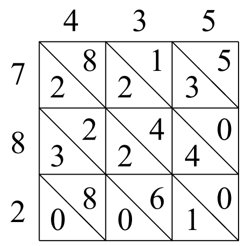
2	1	4	3	6	9
	4	8	9	2	3

The result of multiplying 463 by itself is 214,369.

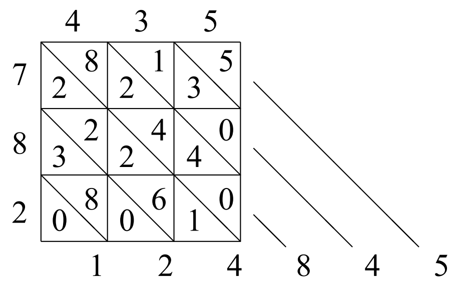
99.14 There are several different ways to perform the third kind of multiplication, without any shifting. Ibn al-Bannāʾ first explains "table multiplication", which we call lattice multiplication. Al-Hawārī's example is to multiply 435 by 287. These are drawn around a grid with diagonals:



Ibn al-Bannāʾ mentions that the 287 can be placed on the left or the right. Most manuscripts, and our translation, show it on the right, but putting it on the left makes the final addition easier. Next, in each of the nine squares we put the product of the column digit by the row digit. For example, for the upper left square $7 \times 4 = 28$, so a 2 goes under the diagonal and an 8 above the diagonal. The spaces are then filled out like this:



We then add the numbers between the diagonal lines. The upper right shows a 5, which is the units. Between the first and second diagonal lines are a 1, 3, and 0. These add to 4, which is the tens place of the answer. Adding the numbers between the next two diagonals gives $8 + 2 + 4 + 4 = 18$, so the hundreds place is an 8, and the 1 is carried to the next sum. Continue like this to get the answer 124,845.



101.16 For "vertical multiplication" al-Hawārī multiplies 183 by 347. These are arranged vertically, separated by some space:

7			3
4			8
3			1

The units place of the answer will be in the top row between the numbers, the tens place in the row below that, etc. First, we multiply the 3 from the 183 one by one by the digits of 347. The 21 from 3×7 is put on the right like this:

7		1	3
4		2	8
3			1

Next, $3 \times 4 = 12$. The 2 is added to the 2 in the tens place to become 4, and the 1 is placed below:

7		1	3
4	4	~~2~~	8
3		1	1

So far we have 141. As in the Oxford manuscript, we will cross out discarded digits. Next, the 3 is multiplied by the 3 to get 9, which is added to the 1 on the bottom to make 10:

7		1	3
4	4	~~2~~	8
3	0	~~1~~	1
		1	

Each digit on the left is then multiplied by the 8, bringing the diagram to this state:

7				1	3
4		0	~~4~~	~~2~~	8
3	8	~~6~~	~~0~~	~~1~~	1
		8	~~4~~	~~1~~	
				2	

Last, the 1 is multiplied by the 7, 4, and 3:

7					1	3
4			0	~~4~~	~~2~~	8
3	5	~~8~~	~~6~~	~~0~~	~~1~~	1
	3	~~9~~	~~8~~	~~4~~	~~1~~	
			6	~~3~~	~~2~~	

Thus $183 \times 347 = 63{,}501$.

103.14 The horizontal version of "vertical multiplication" is called "sleeper multiplication", like the method described above at 95.17. Al-Hawārī multiplies 253 by 987, first writing them on two lines like this:

2	5	3
9	8	7

First, the 3 is multiplied by the digits in 987, beginning with the 7. The 21 is written above:

	2	1
2	5	3
9	8	7

Next is $3 \times 8 = 24$. The 4 is added to the 2 of the 21, making 6, which is placed above, and the 2 is put to the left:

	6	
2	~~2~~	1
2	5	3
9	8	7

So far the result is 261. Again, we cross out defunct digits as we go, but most manuscripts just leave them alone. Next, $3 \times 9 = 27$, and this is added the same way:

	9	6	
2	~~2~~	~~2~~	1
	2	5	3
	9	8	7

Now the 5 is multiplied by the 7, 8, and 9, giving this figure:

	2			
	~~7~~	3	1	
	~~3~~	~~9~~	~~6~~	
5	~~2~~	~~2~~	~~2~~	1
		2	5	3
		9	8	7

Finally, after multiplying the digits by 2, the final figure is:

		9			
		~~3~~			
		~~2~~	7		
	4	~~7~~	~~3~~	1	
	~~6~~	~~3~~	~~9~~	~~6~~	
2	~~5~~	~~2~~	~~2~~	~~2~~	1
			2	5	3
			9	8	7

The result of multiplying 253 by 987 is 249,711.

104.1 For *rūmī* calculation see 85.16 above and §3 in the Introduction.

106.11 This next method, called "repetition", is just a curiosity of calculation. It works when the digits in each number are all the same and they both have the same number of digits, like 7777 by 9999, or, in al-Hawārī's example, 444 by 333. These are written on two lines like this:

		4	4	4
3	3	3		

Under this we put a 1, 2, 3, etc., until we get to the end of one number and the beginning of the next. Then the numbers descend back to 1. These numbers will serves as multipliers:

		4	4	4
3	3	3		
1	2	3	2	1

Multiply the 3 of 333 by the 4 of 444 to get 12. Multiplying the 12 by the 1 on the left gives 12, which is put above like this:

1	2				
			4	4	4
	3	3	3		
	1	2	3	2	1

Next, the 12 is multiplied by the 2 to get 24, and this is added above, one place to the right. The 2 is added to the 2 of the last 12 to make 4, so we replace it as we did in the last scheme:

	4				
1	~~2~~	4			
			4	4	4
	3	3	3		
	1	2	3	2	1

The 12 is then multiplied by the 3, giving 36, and this is added above the same way:

	4	7			
1	~~2~~	~~4~~	6		
			4	4	4
	3	3	3		
	1	2	3	2	1

Continuing, we multiply the 12 by the 2, and then finally the 12 by the 1. The final figure is:

	4	7	8	5	
1	~~2~~	~~4~~	~~6~~	~~4~~	2
			4	4	4
	3	3	3		
	1	2	3	2	1

So the product of 333 by 444 is 147,852.

108.9 The techniques of multiplication presented so far are meant to be worked out in Indian notation on a board, and can be applied to any two (positive) integers. The remaining techniques are either shortcuts originating in finger-reckoning, or they are board techniques that work for specific kinds of numbers. Many of these rules were copied by Ibn al-Bannāʾ from Ibn al-Yāsamīn's *Grafting of Opinions*, a book that is a kind of hybrid between Indian calculation and finger reckoning. Even though that book is expressly devoted to calculation with Indian numerals, it is organized like a book on finger reckoning in that it covers multiplication and division before passing to addition and subtraction; also, it still retains the rules of mental calculation that Ibn al-Bannāʾ copied into his book. In the following breakdown of Ibn al-Bannāʾ's remaining rules, we indicate which come from finger-reckoning (FR) and which were taken from Ibn al-Yāsamīn (Y):

	108.9	109.7	109.19	111.7	112.5	112.16	113.6	113.16	114.4
FR	×	×	×			×	×	×	×
Y		×	×	×	×	×		×	

In addition, at 109.16 al-Hawārī gives a variation on a finger-reckoning rule that he attributes to al-Yāsamīn.

Multiplication "by excess" is a trick for doing the calculation mentally when at least one of the numbers is between 10 and 19. To multiply 12 by 15, think of the 12 as 2 more than 10. Divide the 2 by the 10, which is $\frac{1}{5}$, and multiply this by the 15 to get 3. Then add this 3 back to the 15 to get 18, and multiply by 10 to get the answer, 180. Al-Hawārī then does it again, switching the roles of the 12 and 15. In general, to multiply a number of the form $10 + a$ (a is a digit) by another number b, one calculates $(\frac{a}{10}b + b) \cdot 10$. Al-Hawārī gives a second example, 13×17, that gives fractions. Try it in your head! $\frac{3}{10}$ of 17 is $\frac{51}{10}$, or $5\frac{1}{10}$. Add this to the 17 to get $22\frac{1}{10}$. Finally, multiply by 10 to get the product, which is 221.

109.7 Another trick for mental calculation is called "denomination". Al-Hawārī's example is to multiply 6 by 12. First denominate (we would say "divide") one of the numbers with their sum. Dividing 6 by 18 gives $\frac{1}{3}$. Then multiply this by the other number, 12, to get 4. Last, multiply this 4 by the sum, 18, giving 72.

Written algebraically, to multiply a by b one performs these operations: $\frac{a}{a+b} \cdot b \cdot (a + b)$. This method can only be useful if the two numbers have a common divisor. Otherwise, the multiplication of $\frac{a}{a+b}$ by b will require one to find $a \cdot b$, which is the original problem.

109.16 A variation on this rule can be written in modern notation as $a \cdot b = (a - \frac{a}{a+b}a)(a+b)$. Al-Hawārī gives Ibn al-Yāsamīn as his source, though this rule is not in that author's *Grafting of Opinions*. We do not know its origin.

109.19 Another denomination method is to divide one of the two numbers by a power of ten, and then multiply the result by the other number. This is then multiplied by the power of ten to get the product. Al-Hawārī first works out the example 24×8. He divides the 8 by 10 to get $\frac{4}{5}$. Then $\frac{4}{5} \times 24 = 19\frac{1}{5}$. Multiplying this by 10 gives the product 192. The advantage here is that the numbers are kept small until the end when the power of ten is multiplied back.

110.5 The Arabic word we translate as "power of ten" is *ʿaqd*. The use of this word in arithmetic originated in finger-reckoning. Historian A. S. Saidan wrote: "In Arabic *ʿuqda* is the name of the finger-joint, and *ʿaqd* in this sense should mean 'to bend the finger-joint'".[24] He later described the word in more detail:

> This placement [of the fingers] is called *ʿaqd*, plural *ʿuqūd*. Thus the finger-reckoner understood numbers as formed of places, namely units, tens, hundreds, etc., each place having one or the other of the nine *ʿuqūd*: one, two, … nine. With this understanding the word *ʿuqūd* came to mean what we may now call digits. But in usage *ʿaqd* and place were not always clearly distinguished.[25]

The first occurrence of the word *ʿaqd* in al-Hawārī's book is in a rule of Ibn al-Bannāʾ given above at 87.17. There it takes the meaning of "power of ten" or "place". The other seven instances of the word are in the present chapter covering finger reckoning rules, from 109.19 to 111.3. When reading "power of ten" here one should keep in mind its association with the positioning of the fingers.

The definition at 110.5 is not accurate. The *only* non-zero rank should be a ten or a hundred or the like, not just the first (*awwal*). The way the condition is stated, a number like 310 would be a "simple power of ten".

110.12 Al-Hawārī then works out 12×15, which he solves by working through $15 \div 10 = 1\frac{1}{2}$, $1\frac{1}{2} \times 12 = 18$, $18 \times 10 = 180$. He then works it out again, this time subtracting 5 from the 15 so that the first operation gives a whole number. He takes $15 - 5 = 10$, then $10 \div 10 = 1$, $1 \times 12 = 12$, and $12 \times 10 = 120$. To compensate for the subtracted 5, he calculates $5 \times 12 = 60$, and this is added to the 120 to get 180, which is the product.

111.1 Next al-Hawārī finds 3×15 similarly, but by adding rather than subtracting. Adding 2 to the 3 gives 5, and dividing that by 10 gives $\frac{1}{2}$. This $\frac{1}{2}$ may not be a whole number, but it is easier to work with than $\frac{3}{10}$. Then $\frac{1}{2} \times 15 = 7\frac{1}{2}$, and "we raise each [digit] by ten" to get 75. From this we must subtract 2×15, or 30, to get 45, which is the answer. With practice, methods like these prove to be quite useful.

[24] (Saidan 1968, 707).
[25] (al-Uqlīdisī 1978, 10).

111.7 Recall the "repetition" method at 106.11, which was covered just before the techniques of mental multiplication. That method requires that the number of digits in the multipliers be the same, and that all the digits in each multiplier be equal, as in al-Hawārī's example 333×444. This next method, called "nines", works when one of the two multipliers consists of all 9s. Al-Hawārī works through the example 444×999. We put them on two lines, and above them we put a row of dots equal to the sum of the number of places of the two numbers, in this case six:

•	•	•	•	•	•
			4	4	4
			9	9	9

First, $4 \times 9 = 36$. The 6 replaces the right-hand dot, and the 3 replaces the middle dot of the remaining dots:

•	•	3	•	•	6
			4	4	4
			9	9	9

Next, the difference $9 - 4 = 5$ replaces the dots between the 3 and 6:

•	•	3	5	5	6
			4	4	4
			9	9	9

Last, the remaining dots are replaced with 4s:

4	4	3	5	5	6
			4	4	4
			9	9	9

Then $444 \times 999 = 443{,}556$.

To see why this works, note that multiplying a number of the form aaa (i.e., $100a+10a+a$) by 999 is the same as aaa by $1000 - 1$, which is of the form $aaa000 - aaa$. The digits of the answer must then be a, a, $a - 1$, $9 - a$, $9 - a$, and $10 - a$.

112.5 Next is another method of multiplying by a number expressed with all 9s. This one has no restriction on the other number. Al-Hawārī's example is $999 \times 9{,}354$. Add to the 9,354 as many 0s as there are 9s in 999 to get 9,354,000. Then subtract the 9,354 to get 9,344,646, which is the desired product.

112.16 The method called "squaring" derives from the fact that $[\frac{1}{2}(b+a)]^2 - [\frac{1}{2}(b-a)]^2 = ab$. The method is much simpler to apply mentally than the modern formula suggests. To multiply 17 by 19, al-Hawārī squares half their sum, 18, to get 324. From this he subtracts a square of half the difference between them, which is 1, to get 323. This is the desired product.

Incidentally, in his *Lifting the Veil* Ibn al-Bannāʾ appropriates this rule as the foundation for arithmetical proofs for the rules for solving the three composite algebraic equations. The same rules are stated and illustrated in the present work starting at 214.7, but without proofs.[26]

113.6 Another "squaring" method entails squaring one of the two numbers and multiplying or dividing the result by their ratio. To multiply 25×15, al-Hawārī squares the 25 to get 625. Because 25 is the greater of the two numbers, its square is multiplied by the ratio of 15 to 25, or $\frac{3}{5}$, to get 375, which is the answer. He then works it out by squaring the 15 to get 225. Because 15 is the smaller number, its square is divided by the ratio $\frac{3}{5}$, again giving 375. Algebraically, $a \cdot b = a^2 \cdot \frac{b}{a}$.

113.16 In this next method, Ibn al-Bannāʾ makes use of rules that can be written in modern notation as $ab = b^2 - (b-a)b$ and $ab = a^2 + (b-a)a$, for $a < b$. Al-Hawārī multiplies 36 by 14 using both rules. For the first rule, the difference 22 is multiplied by 36 to get 792. This is taken from 1296, a square of 36, leaving the answer 504. For the second rule, the difference 22 is multiplied by 14 to get 308. This is added to 196, a square of 14, to get the same answer, 504.

Al-Hawārī did not pick a good example to illustrate the utility of this trick. In one step in the first rule he has to multiply 22 by 36, which itself is no easier than finding 36 by 14 directly. The rules have an advantage if the difference between a and b is a nice number, like in the example of 16 by 26. The difference is 10, which multiplied by 26 easily gives 260. Subtract this from 676, a square of 26, to get 416. Or multiply the 10 by the 16 to get 160, and add this to 256, a square of 16, to again get 416.

114.4 The last trick deals with multiplying multiples of powers of 10. To multiply 30 by 140 al-Hawārī first multiplies 3 by 14 to get 42, and to this he adds back the zeros to get the answer, which is 4,200.

115.1 Students should memorize the multiplications of the whole numbers from 1 to 10.

117.1 **Section I.1.5. Division.**

117.2 Ibn al-Bannāʾ may not have taken into account non-integers in his definition of multiplication at 95.2 above, but he did so for division. He copied two definitions from Ibn al-Yāsamīn, one at 117.2, which calls for the decomposition of the dividend into equal parts, and the other at 117.5, based instead on ratio. Al-Hawārī copies Ibn al-Bannāʾ's

[26] For arithmetical proofs in Arabic algebra, see (Oaks 2018a).

explanations from *Lifting the Veil* that the first "applies to discrete quantities", that is, to whole numbers, while the second "concerns continuous quantities", which in his case are the numbers of the arithmeticians that include fractions and irrationals. This second definition is echoed in Ibn al-Bannāʾ's definition of a fraction, below at 133.1, but there the numbers in the ratio are assumed to be whole numbers. Neither definition is known from Greek sources (but see above at 95.2 for a discussion of Abū l-Wafāʾ's attempt to link the first definition of division to Euclid and Nicomachus). At 117.7 Ibn al-Bannāʾ then gives the definition of division that is observed by "most people", which instead counts how many divisors are in the dividend.

117.9 The two meanings of division that he then offers stem from the two definitions at 117.2 and 117.7. The first is "the division of a type by another type, like dirhams by men", and the second is "the division of a type by the same type". This distinction corresponds to the notion of a change in terms for the two kinds of multiplication, above at 95.3.

117.16 The examples of the two meanings, here and at 118.1, mirror the examples for multiplication at 95.10 and 95.12. For the first meaning, Ibn al-Bannāʾ divides 15 dirhams equally among five men, and for the second, he divides a piece of wood of fifteen spans into pieces of wood of three spans. The explanations given here correspond to those for multiplication at 95.3 and 95.6: decomposing (the 15 dirhams) and uniting (the 15 spans into groups of three) are the opposites of the combining (5 dirhams of 3 men) and dissecting (5 dirhams into thirds of a dirham) that we saw for multiplication.

118.14 A distinction was often made in medieval Arabic arithmetic between dividing a greater number by a smaller number and dividing a smaller number by a greater number. Books written in the finger-reckoning tradition typically "divide" (*qasama*) the greater by the smaller, and "relate" (*nasaba*) the smaller to the greater.[27] *Nasaba* is the verb associated with *nisba*, the word for "ratio". Ibn al-Bannāʾ and al-Hawārī use *qasama* similarly, but they follow al-Ḥaṣṣār and Ibn al-Yāsamīn by "denominating" (from *sammā*) the smaller with the greater. To denominate 3 with 7 means to give 3 the denomination, or name, "sevenths". Saying the result "three sevenths" is like saying "three cents", but with a different denomination. The prepositions are different between division and denomination, too. One divides a number *by* (*ʿalā*) a smaller number, while one denominates a number *with* (*min*) a greater number.

118.17 Al-Hawārī illustrates the method of dividing a greater number by a smaller number with the example 245 ÷ 12. First, the numbers are put on two lines like this, with the highest power terms lined up:

2	4	5
1	2	

The 12 goes into the 24 of the 245 two times, so we write a 2 under the 12. Because the 2 and 4 of 245 are exhausted, they are replaced with zeros:

[27] The earliest extant book is Abū l-Wafāʾ's arithmetic book (Saidan 1971, 113).

$$\begin{array}{ccc} 0 & 0 & 5 \\ 1 & 2 & \\ \hline & 2 & \end{array}$$

Next, shift the 12 one place to the right:

$$\begin{array}{ccc} 0 & 0 & 5 \\ & 1 & 2 \\ \hline & 2 & \end{array}$$

The 12 does not go into the 5 at all, so a zero is put below and the remainder is 5:

$$\begin{array}{ccc} 0 & 0 & 5 \\ & 1 & 2 \\ \hline & 2 & 0 \end{array}$$

The remaining 5/12 is "two sixths and half a sixth", which is written as $\frac{1\ 2}{2\ 6}$. This kind of fraction is explained at 138.1. The final answer is $\frac{1\ 2}{2\ 6}$ 20, read right-to-left.

119.18 The rules Ibn al-Bannāʾ gives from here up to and including 124.8 are from finger-reckoning. The first technique is to partition the dividend and then add the respective quotients. Al-Hawārī's example is to divide 44 by 11. He breaks up the 44 into 22 and 22, then he divides 22 by 11 to get 2 for each of the 22s. The quotient is then $2 + 2 = 4$. This technique might seem more useful with a problem like dividing 399 by 19. Thinking of 399 as $380 + 19$, we divide each number by 19 to get $20 + 1 = 21$, which is the answer. In general, $(a + b) \div c = (a \div c) + (b \div c)$.

The second technique is to factor the divisor and divide by each of them in turn. To divide 96 by 12, al-Hawārī decomposes the 12 into $2 \cdot 6$. He finds that $96 \div 2 = 48$, and this he divides by the 6 to get 8, which is the answer. In general, $a \div (b \cdot c) = (a \div b) \div c$.

The third technique is to "reconcile" (*wafiqa*) the dividend and the divisor, which means to cancel common divisors. To divide 35 by 15, al-Hawārī takes a fifth of the dividend and the divisor to change the problem into $7 \div 3$, which gives "two and a third". In general, $ab \div ac = b \div c$.

120.20 In "apportionment" (*muḥāṣṣa*), a certain quantity is deducted proportionally from two or more amounts. In al-Hawārī's example three people want to give a total of ten dinars to a bankrupt friend, and they do it in such a way that each of them gives an amount proportional to how much each of them has, and they have 4, 5, and 6 dinars, respectively. In modern notation, we want to find three numbers a, b, and c such that $a + b + c = 10$ and $\frac{a}{4} = \frac{b}{5} = \frac{c}{6}$.

Al-Hawārī begins by adding up the wealth of the three people: $4 + 5 + 6 = 15$. These "surpass them"; in other words, they surpass the three "apportioned parts" that will be given, and which will add up to ten dinars. For the first friend's gift al-Hawārī multiplies the 10 by the 4 dinars to get 40 dinars, and he divides the result by 15 to get his share, or "apportioned part", which is $2\frac{2}{3}$ dinars. The same procedure gives the other shares. For the second friend this is $5 \cdot 10 \div 15 = 3\frac{1}{3}$ dinars, and for the third it is $6 \cdot 10 \div 15 = 4$ dinars. This works because the first must give 4/15 of the total amount of 10 dinars, the second 5/15, and the third 6/15, and the proportions add to 1.

121.14 Al-Hawārī mentions four variations on this method, each one being a reordering of the operations. For example, where the share of the first friend was calculated above by 10×4, then $\div 15$, the first variation is to calculate $4 \div 15$, then $\times 10$. The second is to calculate $10 \div 15$, then $\times 4$. In the third, we find $15 \div 4$, then we divide 10 by the result. In the fourth we find $15 \div 10$, then we divide 4 by the result. These variations were probably devised for cases in which one or another would be easier. For example, if instead of 4, 15, and 10, we had 8, 26, and 22, it would be easier to apply the first variation, by dividing 8 by 16 to get a half, then multiplying this by the 22 to get 11. The original way has us multiplying 8 by 22 and then dividing by 16, which is more difficult.

122.2 In case there are fractions, multiply the numbers corresponding to the wealth of the friends by the least common multiple of their fractional parts. And if the fractions have common divisors, divide by those factors. The amount of money given to the bankrupt friend remains the same.

By "they are all different (*mutabāyna*)" in the passage at 122.10 Ibn al-Bannāʾ means they are relatively prime. Also, we translate *ishtirāk*, a word whose ordinary meaning is "common", as "common divisor".

Al-Hawārī considers the situation in which a total of 12 dinars is given by three friends who have $4\frac{1}{3}$, $5\frac{1}{4}$, and $6\frac{1}{6}$ dinars, respectively. He first calculates the least common multiple of the denominators by collecting factors: a 3 from the third, two 2s from the 4, and nothing from the 6 because we already have a 2 and a 3. So the least common multiple is 12. The $4\frac{1}{3}$, $5\frac{1}{4}$, and $6\frac{1}{6}$ are then multiplied by 12 to get 52, 63, and 74. Next he checks to see if there is a common factor among these numbers that can be cancelled out. In this case, the numbers are relatively prime. He then finishes the problem, following the rule in the first example.

123.18 Next, Ibn al-Bannāʾ gives the rule for expressing the result of a denomination as a related fraction (see below at 135.10 and 138.1 for this type). We saw one example of this notation already at 119.7, where "two sixths and half a sixth" was written as $\frac{1\ 2}{2\ 6}$.

123.22 The example is to denominate 11 with 15. Decompose the 15 into 5 and 3 and put them under a line like this:

$$\frac{\quad\quad}{3\ \ 5}$$

Then divide the 11 by the 3 and put the remainder, 2, above it:

$$\begin{array}{cc} 2 & \\ \hline 3 & 5 \end{array}$$

Then divide the quotient 3 from $11 \div 3$ by the 5. Since 3 is less than 5, 3 is the remainder, and it too is written above:

$$\begin{array}{cc} 2 & 3 \\ \hline 3 & 5 \end{array}$$

This is spoken as "three fifths and two thirds of a fifth", and we might write it in our notation like this: $\frac{3}{5} + \frac{2}{3 \cdot 5}$.

This process was followed in situations where the denominating number is composite and greater than ten. It was convention that the denominators descend. One said "three fifths and two thirds of a fifth", $\frac{2\ 3}{3\ 5}$, rather than "two thirds and a fifth of a third" $\frac{1\ 2}{5\ 3}$. Also, it was preferred to start with the largest possible denominator, so we read "half a sixth" more often than "a third of a fourth".

124.8 There are three "lesser known" ways to denominate. Al-Hawārī's example of the first is to denominate 4 with 12. He switches the numbers and divides 12 by 4, then denominates 1 with the result. Here $12 \div 4 = 3$, and the denomination gives $\frac{1}{3}$. This works best in simple cases where there is a common factor. It would not work so well with a problem like $11 \div 15$, since $15 \div 11 = 1\frac{4}{11}$, and it would be too difficult to denominate 1 by this number. This method uses the property that $\frac{a}{b} = 1/\frac{b}{a}$.

His example for the second way is to denominate 9 with 15. Denominating 1 with the 15 gives "a third of a fifth", or $\frac{1\ 0}{3\ 5}$. Multiplying this by 9 gives "three fifths", or $\frac{3}{5}$. Here, to find $a \div b$, one first finds $1 \div b$, then the result is multiplied by a.

In the third way, al-Hawārī denominates 10 with 16. The idea is to turn denomination into division by multiplying the 10 by some number to make it greater than 16. He chooses 8, to get $10 \times 8 = 80$. Dividing this by 16 gives 5, and to compensate for the multiplied 8 he denominates the result with 8 to get $\frac{5}{8}$. This works best when the new numerator (here 80) is a multiple of the denominator, and the method is convenient only when there is a common factor (in this case 2). In modern notation, the motive becomes somewhat lost: if $a < b$ then $\frac{a}{b} = \frac{ac}{b}/c$.

124.20 Ibn al-Bannāʾ now turns to the decomposition of numbers into their prime factors. This will be useful later in the manipulations of the denominators of fractions. He gives several rules, some of which warrant explanation because of the way they are worded.

The first case is a number that "does not begin with units". Recall that the units are 1, 2, …, 9. The 0, signifying nothing, stands for an empty place. So a number that "does not begin with units" has a 0 in the units place, and is divisible by 10, 5, and 2. This is what he means when he says that it "has a tenth and a fifth and a half".

125.5 One way of finding factors is to consider the residue after casting out nines or eights or sevens. Ibn al-Bannāʾ first covers various cases for a number whose units digit is even. The result of casting out nines is also the remainder after dividing by 9, so from this we can tell if the number is divisible by 3 or 9. If it is cast out entirely by nines – in other words, if nothing is left – then the number "has a ninth and a sixth and a third", or, in our language, it is divisible by 9, 6, and 3.

Al-Hawārī gives the example of 36. He does not mention that it is also divisible by 18. There is no need to say it is divisible by 2 since the number is already known to be even. If the residue is 3 or 6 then the number is divisible by 3, and because we already knew it was even, it is also divisible by 6. Al-Hawārī gives the examples 66 and 42.

If the residue is some other number (1, 2, 4, 5, 7, or 8), then cast out eights. If it is cast out entirely then the number is divisible by 8 and 4, and if the residue is 4, then it is divisible by 4. If the residue is some other number, then cast out sevens. Because 7 is prime, the only rule is that if it is cast out entirely then the number is divisible by 7.

Similar rules are then given for odd numbers.

126.14 **Subsection on finding deaf parts.**

If all these rules have been applied and the original number is still not decomposed completely, "then look for deaf parts by dividing by them". (We explain the term "deaf part", meaning "prime", below at 134.2.)

127.9 To find deaf parts, Ibn al-Bannāʾ explains how to draw a table of odd numbers to make the sieve. This is the famous "sieve of Eratosthenes", after the third century BC Greek mathematician. Ibn al-Bannāʾ's description ultimately derives from Nicomachus's *Arithmetical Introduction*.[28] Al-Hawārī draws the table to find the prime numbers less than 145. The first number in the table is 3, which is prime. Putting a bar over every third number after that gives:

[28] (Nicomachus 1959, 31); (Nicomachus 1938, 203ff).

19	17	$\overline{15}$	13	11	$\overline{9}$	7	5	3
37	35	$\overline{33}$	31	29	$\overline{27}$	25	23	$\overline{21}$
55	53	$\overline{51}$	49	47	$\overline{45}$	43	41	$\overline{39}$
73	71	$\overline{69}$	67	65	$\overline{63}$	61	59	$\overline{57}$
91	89	$\overline{87}$	85	83	$\overline{81}$	79	77	$\overline{75}$
109	107	$\overline{105}$	103	101	$\overline{99}$	97	95	$\overline{93}$
127	125	$\overline{123}$	121	119	$\overline{117}$	115	113	$\overline{111}$
145	143	$\overline{141}$	139	137	$\overline{135}$	133	131	$\overline{129}$

The next number in the table without a bar is the 5, so it is prime. Put a bar likewise above every fifth number after it, starting with 15, 25, etc. Then do the same for the next number without a bar, which is 7, then for 11. There is no need to do this for 13 or any greater prime, since their squares are greater than 145. The table should now look like this:

19	17	$\overline{15}$	13	11	$\overline{9}$	7	5	3
37	$\overline{35}$	$\overline{33}$	31	29	$\overline{27}$	$\overline{25}$	23	$\overline{21}$
$\overline{55}$	53	$\overline{51}$	$\overline{49}$	47	$\overline{45}$	43	41	$\overline{39}$
73	71	$\overline{69}$	67	$\overline{65}$	$\overline{63}$	61	59	$\overline{57}$
$\overline{91}$	89	$\overline{87}$	$\overline{85}$	83	$\overline{81}$	79	$\overline{77}$	$\overline{75}$
109	107	$\overline{105}$	103	101	$\overline{99}$	97	$\overline{95}$	$\overline{93}$
127	$\overline{125}$	$\overline{123}$	$\overline{121}$	$\overline{119}$	$\overline{117}$	$\overline{115}$	113	$\overline{111}$
$\overline{145}$	$\overline{143}$	$\overline{141}$	139	137	$\overline{135}$	$\overline{133}$	131	$\overline{129}$

Numbers with a bar are composite, and the remaining numbers are prime. Al-Hawārī writes that “these deaf parts are counted only by one”, or as we would say, prime numbers are divisible only by 1. One can say, for example, that 5 *counts* 40 because one can count to 40 by fives. But one cannot count to a prime number by anything but ones.

128.1 Al-Hawārī writes that he is filling the table to 145, but below that he writes that we know, presumably from the table, that 151 is prime. The Oxford and Medina manuscripts show the table to 145, while the Tunis and Tehran manuscripts show it to 199. The table in the Istanbul manuscript is drawn with 15 columns and 10 rows. The copyist must have been confused. He began by filling in the odd numbers, from 3 to 37, and then continued by writing in only prime numbers from 83 to 157. The rest of the squares are blank, and no marks are drawn above any number.

129.1 **Section I.1.6. Restoration and reduction.**

In problems involving proportion it is often necessary to restore or reduce a number to another number. "Restoration" (*al-jabr*) is what we do when we want to increase a number to a greater number, and "reduction" (*al-ḥaṭṭ*) is for reducing a number to a smaller number. These operations are applied, for instance, to set the "coefficient" of the highest power in an algebraic equation to 1 (at 217.1). *Al-jabr* is also the word used in the simplification of subtractions and equations in algebra (in the text at 211.2, 220.5, 223.1, and in our commentary at 223.1). Restoration and reduction of fractions is covered at 154.1.

131.1 Chapter I.2. Fractions.

Fractions in Arabic arithmetic are numbers that result from partitioning the unit. Ibn al-Bannāʾ, for example, writes in the passage at 137.1 the fraction "fifteen…parts of twenty-four parts of the unit" ($\frac{15}{24}$). The unit is partitioned into 24 parts, and the fraction is 15 of those parts. But to define fractions, Ibn al-Bannāʾ feels obliged to respect the Greek notion that the unit is indivisible. Because of this he identifies fractions with ratios of whole numbers, so the fraction just mentioned is considered to be the ratio of 15 to 24. A ratio according to Euclid is "a sort of relation in respect of size between two magnitudes of the same kind".[29] Ratios are not mathematical objects in themselves, but are only relations between such objects. This way, the unit maintains its integrity. Ibn al-Bannāʾ's definition only serves to provide Arabic fractions with a semblance of a Greek foundation. It had no impact on actual calculations, and in Arabic arithmetic books, al-Hawārī's included, fractions were understood to be fractional portions of the unit.[30]

Euclid defines "part" and "parts" at the beginning of *Elements*, Book VII, the first of his three books on number theory: "A number is a *part* of a number, the less of the greater, when it measures the greater…" For example, four is "a part" of twelve because twelve can be broken into three equal parts, each one of them four. He continues, "…but *parts* when it does not measure it".[31] Eight, for example, is "parts" of fourteen because, if we consider the number two as one part, then eight is four of the seven parts making up fourteen, or, as we might say, eight is four-sevenths of fourteen. Similarly, five is "parts" of twelve where one part is the unit.

The Greek term meaning "part" is translated into Arabic as *juzʾ*. This is the same word meaning "part" used in the statements of simple fractions in Arabic with (usually prime) denominators greater than ten, like Ibn al-Bannāʾ's $\frac{15}{24}$ quoted above. The big difference is that in Greek "a part" is always a positive integer that is part of a greater integer, while in Arabic fractions it is the unit itself that is partitioned.

We describe the different ways Arabic arithmeticians expressed fractions below at 134.2. It is there that we explain Ibn al-Bannāʾ's phrase "the part and its name".

133.3 Al-Hawārī goes further than his teacher in addressing the ontology of fractions. Had he worked with the common notion of fractions as parts of a divisible unit, he could

[29] Book V, Definition 3 of the *Elements* (Euclid 1956, vol. 2, 114). The word "size" here is translated from *pēlikotēs*, which was translated into Arabic as *qadr*. The meaning of *pēlikotēs* in the Book V definition is not yet that of *numerical* size.
[30] The desire to validate fractions as ratios goes back at least to the tenth century, when Abū l-Wafāʾ also defined fractions as ratios of whole numbers (Saidan 1971, 71).
[31] (Euclid 1956, vol. 2, 277).

have written that a fraction like "three sixths" is named in terms of sixths, much like "three cats" is named in terms of cats. In both cases the "three" modifies the name. But al-Hawārī follows Ibn al-Bannāʾ in formally identifying fractions with ratios. He argues that the ratio of three to six is not named in terms of either number, nor in terms of the two together.[32]

Al-Hawārī's distinction between "sensible" and "intelligible" ultimately comes from Aristotle. For Aristotle, mathematical objects are sensible objects. We experience lines, squares, spheres, and numbers through our senses as attributes of the physical things we see and touch. But a ratio, being a relation between two objects, cannot be apprehended through the senses. It is an intelligible object that can only be imagined in the mind.[33] Al-Hawārī's statement that ratios are intelligible objects is the only philosophical observation he makes in the book. We suspected that he may have copied it from somewhere, so we checked the obvious places: Aristotle, Ibn Sīnā, and Ibn al-Bannāʾ. Aristotle makes no comment on the ontology of ratios in his extant works, nor did we find anything in Ibn al-Bannāʾ's writings. But we did find this remark of Ibn Sīnā on relations in general: "As for predicating [the quiddity of the relative] with respect to another, this occurs only in the mind".[34]

We set this aside for a moment and continue with the rest of the passage. Al-Hawārī next paraphrases Ibn al-Bannāʾ's *Lifting the Veil*[35] when he explains the word "fraction" by comparing a fractional number with fractured land. Perhaps he was thinking of the incremental increase of fractions with the same denominator, like $\frac{1}{17}$, $\frac{2}{17}$, $\frac{3}{17}$, etc., that model the incremental strata of some rocky landscape. He also compares fractions with geometric magnitudes, probably because lines, surfaces, and bodies can be partitioned into arbitrarily many parts. Al-Hawārī contrasts these continuous magnitudes with "discrete quantities", echoing the notion in Aristotle and Euclid that numbers do not admit fractions due to the indivisibility of the unit.

We contend that Ibn al-Bannāʾ related nearly the entire passage at 133.3, from "For example…" to "…similar abstractions", verbally to al-Hawārī. Ibn al-Bannāʾ was familiar with Ibn Sīnā's work, and al-Hawārī paraphrased rather than quoted Ibn al-Bannāʾ's comparison of fractions with terrain from *Lifting the Veil*.

134.1 Section I.2.1. The names of fractions and numerating them.

134.2 Fractions in most Arabic books on Indian arithmetic, al-Hawārī's included, are borrowed from finger-reckoning. This particular system has its origin in the ancient Egyptian practice of expressing fractions as sums of unit fractions, like writing $\frac{2}{5}$ as the sum

[32] The "separation" he speaks of may be the preposition "to" (*ilā*) in the phrase "the ratio of three to six". Or it may be the mark one makes when writing ratios in notation. Al-Mawāḥidī, another commentator of the *Condensed Book*, uses the generic " ∴ " which also serves to separate numbers in other contexts. The proportion $3 : 4 :: 6 : 8$ is written as "8 ∴ 6 ∴ 4 ∴ 3" in (al-Mawāḥidī manuscript, fol. 78a).

[33] We also find intelligible mathematical objects in al-Khayyām (Omar Khayyam), who followed Aristotle closely. He wrote that while one-, two-, and three-dimensional algebraic powers correspond to geometric magnitudes that exist in sensible things, the zero degree term and powers higher than three are not associated with any sensible objects. These objects are instead "imagined in the continuous magnitudes", or are "taken as abstracted in the intellect from material things". See (Oaks 2011a); (Rashed and Vahabzadeh 2000, 171).

[34] (Avicenna 2005, 122).

[35] (Ibn al-Bannāʾ 1994, 270.3-4).

of $\frac{1}{3}$ and $\frac{1}{15}$. It was later modified, probably by Greek calculators, with approximation techniques based in sexagesimal arithmetic.[36]

At some point before the ninth century CE the restriction to unit fractions was dropped. One could now say fractions like "five sevenths" and "two ninths". But in Arabic fractions could only be formed from nine names, or "heads" (*ruʾūs*): "a half", "a third", "a fourth", up to "a tenth". There were no words in Arabic for "eleventh", "twelfth", "thirteenth", etc., so fractions with denominators greater than ten were expressed when possible by combinations of the heads. For instance, al-Khwārazmī wrote "a fifth and four fifths of a fifth" for $\frac{9}{25}$ and "a ninth and a tenth" for $\frac{19}{90}$.[37] The various ways of combining the heads already present in ninth century texts remained common throughout the medieval period, and are explained by al-Hawārī in the present section.

Numbers in medieval Arabic that could be expressed in words were called "expressible" (*munṭaq*).[38] These included integers and fractions reducible to combinations of the heads. Numbers inexpressible in words, for which only approximations could be found, were often called "deaf" (*aṣamm*). These included irrational roots, and also fractions irreducible to combinations of the heads, that is, fractions whose denominators contain prime factors greater than ten. The problem of deaf fractions was overcome some time before the early ninth century with the introduction of "parts". For example, al-Khwārazmī wrote $\frac{4}{13}$ as "four parts of thirteen parts of a dirham".[39] With the dirham divided into thirteen equal parts, the fraction is four of those parts. So now, in addition to being able to say "four fifths" and "four sevenths", one could say "four parts". Thus, "a part" became the tenth name for the naming of fractions.

This accounts for the origin of the term "deaf parts" for prime numbers, above at 126.14. To express the result of a denomination as a related fraction, explained at 123.18 above, one needs to find the prime factorization of the denominated number. If this number contains a factor not divisible by 2, 3, 5, or 7, then one needs to look for "deaf parts", that is, to find the greater prime denominators required for the fraction. The example at 151.14 results in "fifty parts of one hundred thirty-seven parts and five ninths of a part of one hundred thirty-seven parts". The deaf parts here are the 137 parts. Prime numbers do not quite coincide with the denominators of deaf fractions, since the former include 2, 3, 5, and 7, but the association was close enough.

In arithmetic, the naming of fractions with parts typically terminates with "of a unit/one" or "of a dirham" (units were often counted in dirhams, a silver coin, in many calculations). In one problem al-Karajī writes "thirty-five parts of eighty-three parts of a unit" and then, just after, "thirty-four parts of eighty-three parts of a dirham".[40] Often, as in al-Hawārī, the designation is left off altogether. So his "a part of eleven" is short for "a part of eleven parts of one/a unit/a dirham". Also, it is not just abstract units that are partitioned in Arabic fractions. In the inheritance problems solved by al-Khwārazmī it is sometimes a share of the estate, as in "twenty-three parts of fifty-nine parts of a share",[41] and in the context of

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36 (al-Uqlīdisī 1978, 11).
37 (al-Khwārizmī 2009, 147.15, 253.19).
38 Our authors use this word to mean "irrational". See below at 163.2.
39 (al-Khwārizmī 2009, 169.14).
40 (Saidan 1986, 181.12,14).
41 (al-Khwārizmī 2009, 265.7).

algebra Abū Kāmil writes “fifteen parts of thirty-nine parts of a thing”, where the “thing” is the name of the first degree unknown.[42]

Grammatically, the ten names function like other Arabic nouns. Saying “three fourths” or “three parts” is like saying “three apples”. The “fourths”, “parts”, and “apples” are the names, or kinds of object counted, while the “three” tells how many there are (the same applies also to “shares” and “things”). Our own words “numerator” and “denominator” reflect this idea. The word “denominator” derives from the Latin *dēnōmināre*, “to call, to name”, and our word “name” comes from the related Latin word *nōmen*. Our “numerator” is the “number” of this name.

There was no rule that the language of parts could only be used for deaf fractions. Al-Khwārazmī and Abū Kāmil routinely use it to name fractions like $\frac{25}{36}$, $\frac{14}{15}$, $\frac{5}{12}$, and $\frac{4}{25}$. And when convenient for the calculations the fraction can even be improper, like al-Khwārazmī’s “twenty-eight parts of thirteen of a dirham”.[43] Al-Hawārī, though, works with combinations of the heads whenever possible, and his numerator is always less than his denominator.

Although schoolbooks today often still explain fractions in terms of parts, mathematicians define them in terms of division. The value of al-Khwārazmī’s “four parts of thirteen parts of a dirham” may be equal to the result of dividing 4 by 13, but the two ways of regarding fractions are not equivalent. With quotients in mind, we allow numerators and denominators to be irrational, like $\frac{1}{\sqrt{2}}$ and $\frac{\sqrt{2}}{2}$. But these numbers cannot be fractions in medieval Arabic. The first makes no sense because one cannot partition the unit (or anything else, for that matter) into an irrational number of parts. And the second does not work because the number of parts making up a fraction cannot be irrational. To have $\sqrt{2}$ parts is just as meaningless as having $\sqrt{2}$ loaves of bread. One can perform the corresponding divisions, however, to get the acceptable “root of a half” ($\sqrt{\frac{1}{2}}$). We explain this below at 188.1.

135.1 When the denominator is a composite number greater than ten it was common to express it with “two or more names”. Al-Hawārī’s first example is “two eighths and a seventh of an eighth”, which in notation looks like this: $\frac{1\ 2}{7\ 8}$. We can write this as a modern fraction by working it out to get $\frac{15}{56}$. Although it may sound overly complicated to us, saying “two eighths and a seventh of an eighth” gives a good idea of the magnitude of the fraction. We know that “two eighths” is a fourth, and we are adding to that the small amount “a seventh of an eighth”. The fraction is expressly presented as being just a little over $\frac{2}{8}$, while with our $\frac{15}{56}$ this is not immediately clear. One of us even found himself thinking in terms of these fractions while measuring wood with a ruler scaled in inches. The fractional part was clearly three fourths and a fourth of a fourth, and it would have been much less transparent as well as superfluous to convert $\frac{1\ 3}{4\ 4}$ to $\frac{13}{16}$. We can all grasp the magnitudes of the fractions $\frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{10}$, and combinations of these are often easier to understand than fractions with large composite denominators.

[42] (Abū Kāmil 2012, 471.5).
[43] (al-Khwārizmī 2009, 169.13).

An example in which the related fraction is *not* easier to grasp is Ibn al-Bannāʾ's artificial example $\frac{2\ 4\ 5}{3\ 5\ 6}$ (at 135.10 below), which is equal to $\frac{89}{90}$. And just as our fractions do not admit of a unique representation ($\frac{1}{2} = \frac{2}{4} = \frac{3}{6}$ etc.), related fractions can be written in many ways. One example is al-Hawārī's $\frac{3\ 4}{6\ 9}$ in the example at 171.6, which is equal to $\frac{1}{2}$.

Al-Hawārī uses the language of parts when the denominator is not prime in only two instances, at 155.2 and 156.7. There the denominators are 87 and 93.[44]

135.8 To expand on Ibn al-Bannāʾs remark, al-Hawārī copies *Lifting the Veil* from 135.10 to 137.10 to explain how to find the numerators of three kinds of fractions: related, distinct, and portioned. These types are covered again immediately after by al-Hawārī himself, along with other types. Finding the numerator is necessary to perform every kind of operation: addition, subtraction (including casting out, above at 93.9), multiplication, division/denomination, conversion, and finding square roots.

135.10 Ibn al-Bannāʾ works with the example $\frac{2\ 4\ 5}{3\ 5\ 6}$ to show how to find the numerator and denominator of a *related* fraction. These fractions are expressed with more than one name, or denominator, so that each part is related to, or is a fraction of, what precedes it. In this example, the 5 in the top is multiplied by the second denominator, also a 5, to get 25, and this is added to the 4 to get 29. So the first two terms of the fraction are equivalent to $\frac{29}{5\cdot 6}$, or $\frac{29}{30}$. Then this 29 is multiplied by the 3 to get 87, which is added to the 2 above, resulting in the numerator 89. The fraction is the same as our $\frac{89}{90}$.

136.8 Sometimes two fractions are gathered with the conjunction "and" (*wa*), like the example "five sixths and four fifths". In notation, one is written next to the other like this: $\frac{4}{5}\,\frac{5}{6}$. This is called a *distinct* fraction, and its value is the sum of the individual fractions. To find the numerator, Ibn al-Bannāʾ multiplies the $\frac{5}{6}$ by the denominator 5 of the other fraction to get $\frac{25}{30}$ (though he does not name this fraction). He then multiplies the $\frac{4}{5}$ by the 6 to get $\frac{24}{30}$. Together they are $\frac{49}{30}$, or forty-nine "parts of thirty" (or "fifths of sixths" or "sixths of fifths"), so the numerator is 49.

137.1 A *portioned* fraction is one like "three fourths of five sixths", written as $\frac{5\ \bullet\ 3}{6\ \bullet\ 4}$. This may be equivalent to the product of the fractions, but it was conceived and stated as a fraction of a fraction. In modern notation the example becomes $\frac{15}{24}$.

137.11 Al-Hawārī now returns to commenting on the *Condensed Book*. Ibn al-Bannāʾ names five different kinds of fraction, and al-Hawārī shows how to find the numerator of each.

137.13 *Simple fractions*. The numerator of a simple fraction, like "a seventh", written $\frac{1}{7}$, is the number above the line. From the example at 139.2, we know that for Ibn al-Bannāʾ simple fractions are not restricted to those with a numerator of 1. So a fraction like "four sixths" ($\frac{4}{6}$) is also simple, and its numerator is the 4 above the line.

[44] Ibn al-Bannāʾ's example $\frac{15}{24}$ at 137.1 is first stated in terms of "fourths of a sixth", and the version in terms

Combined fractions. This may not have been considered as a separate type since neither al-Hawārī nor Ibn al-Bannāʾ describe it. A combined fraction is a fraction whose representation has all 0s in the numerator except on the extreme left, like Ibn al-Yāsamīn's $\frac{21\ 0\ 0\ 0}{2\ 6\ 13\ 7}$. In modern notation, the fraction is $\frac{21}{2\cdot 6\cdot 13\cdot 7}$, or $\frac{21}{1092}$, and the numerator is 21. (Ibn al-Yāsamīn preferred to write his fractions in reverse order, so he shows it like this: $\frac{0\ 0\ 0\ 21}{7\ 13\ 6\ 2}$.[45])

138.1 *Related fractions*. A related fraction is the common type with "two or more names", which we have seen several times already. Al-Hawārī's example is $\frac{2\ 3\ 4\ 5}{3\ 5\ 7\ 8}$. Ibn al-Bannāʾ gave two rules, the first being the same as the one from *Lifting the Veil* cited above at 135.10. The fraction can be written in modern form as $\frac{596}{3\cdot 5\cdot 7\cdot 8} = \frac{596}{840}$. The second rule is explained as clearly as the first.

139.1 *Distinct fractions*. Distinct fractions are those that are added, or gathered together. They have already been described above at 136.8. Al-Hawārī finds the numerator for "five sevenths and half a seventh and four sixths". In notation this is $\frac{4}{6}\ \frac{1\ 5}{2\ 7}$, which we would write as $\frac{11}{14} + \frac{4}{6}$. Combining them is equivalent to our method of cross multiplication, so the numerator is $11 \cdot 6 + 4 \cdot 14 = 122$.

139.10 *Portioned fractions*. The numerator of fractions of fractions is the product of the individual numerators. These were described above at 137.1.

140.1 *Excluded fractions of disconnected type*. These are called "excluded" because they are expressed as a fraction removed or excluded from a greater fraction. For an explanation of the terms "less", "diminished", and "excluded", see our remarks at 219.4 below. They are "disconnected" because the second fraction is not taken of the first fraction, but is independent of it. The example here is "six eighths less a ninth", written as $\frac{1}{9}$ لا $\frac{6}{8}$. This fraction is drawn in the Medina manuscript as . (See our commentary above at 86.1 for an explanation of the sign for "less".) In modern notation we write it as $\frac{6}{8} - \frac{1}{9}$. By "is not taken from what precedes it", al-Hawārī means that this is not $\frac{6}{8} - \frac{1}{9}\left(\frac{6}{8}\right)$. The procedure is like that for distinct fractions, but involves taking the difference rather than the sum of the products. Here $6 \times 9 = 54$, and $1 \times 8 = 8$. The numerator is $54 - 8 = 46$.

140.14 *Excluded fractions of connected type*. Al-Hawārī converts the example "a half less its third" to the disconnected version "a half less a sixth". He then finds the numerator of "six sevenths and half a seventh less its third". By "its third" he means that you take away a third of the six sevenths and half a seventh. In notation it is written just like an excluded fraction of disconnected type, $\frac{1}{3}$ لا $\frac{1\ 6}{2\ 7}$, though the meaning is different. In modern notation we would write $\frac{1\ 6}{2\ 7}$ as $\frac{13}{14}$, and the whole fraction is equivalent to our $\frac{13}{14} - \frac{1}{3}\left(\frac{13}{14}\right)$. Take the numerator of the greater part, which is $\frac{1\ 6}{2\ 7}$, to get 13, and multiply it by the

of "twenty-fourths" is only added to show the denominator.

[45] (Zemouli 1993, 171). On Ibn al-Yāsamīn's way of writing numbers, see (Abdeljaouad 2005b).

denominator of the smaller part, which is 3, to get 39. Then multiply the two numerators: $1 \times 13 = 13$. The numerator is then $39 - 13 = 26$, and the fraction is equal to $\frac{26}{42}$.

The word "connected" for excluded fractions takes the same meaning as the word "related" for related fractions, in that the fraction that follows is taken as a portion of the preceding fraction. So it is natural that in cases where the diminished amount is itself a related fraction the notation was sometimes written with a single bar. For this example, if the "less" were instead an "and", it would have been "six sevenths and half a seventh and a third of a half of a seventh", or $\frac{1\ 1\ 6}{3\ 2\ 7}$. When the last fraction is taken away instead of added, as in "six sevenths and half a seventh less its third", it might be written as $\frac{1\ \searrow\ 1\ 6}{3\ \ \ 2\ 7}$ instead of the ambiguous $\frac{1}{3} \searrow \frac{1\ 6}{2\ 7}$. Four of the five manuscripts we consulted show the latter, but the Tehran manuscript shows the fraction with the bar extending all the way across:  (this manuscript shows the Eastern forms of the numerals).

It was not a problem to have one notation with two possible meanings. While working through a problem, one knows whether the fraction is connected or disconnected. Notation was used by the individual to perform calculations, not to communicate the work to others or to preserve a record for future consultation. A rhetorical version served those purposes, and there the ambiguity is erased by the language.

141.7 Al-Hawārī then gives some "additional remarks" of Ibn al-Bannāʾ's. Recall from 86.1 above that an expression like "ten less eight less seven less five less two" is $10-(8-(7-(5-2)))$. Here Ibn al-Bannāʾ explains that when the word "and" (*wa*) appears before each "less", then the numbers are all taken away from the first term. Al-Hawārī's example is "five and a third less its fourth and less its seventh and less its fifth", and is written in notation as $\frac{1}{5} \searrow \frac{1}{7} \searrow \frac{1}{4} \searrow \frac{1}{3}\,5$. We might write it as $5\,\frac{1}{3} - \frac{1}{4}\left(5\,\frac{1}{3}\right) - \frac{1}{7}\left(5\,\frac{1}{3}\right) - \frac{1}{5}\left(5\,\frac{1}{3}\right)$. The figure can be reduced to $\frac{1}{5}\,\frac{1}{7}\,\frac{1}{4} \searrow \frac{1}{3}\,5$, with a single "less".

The "particle of exclusion" is most often the word "less" (*illā*), as it is here. But sometimes in Arabic mathematics other words, meaning "except" or "other than" (*ghayr* or *siwa*), are used. The Arabic word for "detached" (*munfaṣil*) is related to the word we later translate as "apotome" (*munfaṣila*) in the chapter on roots. See our remarks below at 173.10.

142.6 Here al-Hawārī gives a disconnected example. If we said "five and a third less a fourth of one and less a seventh of one and less a fifth of one" it would be equivalent to the modern $5\frac{1}{3} - \frac{1}{4} - \frac{1}{7} - \frac{1}{5}$, or $5\frac{1}{3} - \frac{83}{140} = \frac{1991}{420}$. He does not show the notation for this.

142.10 Again quoting Ibn al-Bannāʾ, al-Hawārī explains the meaning of the repeated "less" and cites the same rule given at 86.9 for whole numbers. He does this for the connected type (like "…less its fifth") and the disconnected type (like "…less a fifth of one" or simply "…less a fifth"). He does not mention that the work starts from the last term.

143.1 The "three of its fourths" must become disconnected from the "five sixths". So the fraction is rephrased as an excluded fraction of disconnected type.

143.5 Al-Hawārī finds the numerator for the mixed fraction $\frac{3\ 5}{4\ 6}5$, or $5\,\frac{23}{24}$ in modern notation. He first multiplies the whole number 5 by the two denominators to get 120. He then adds this to the numerator of the fraction, which is 23, to get 143. We can write the number as the fraction $\frac{143}{24}$.

143.13 If the whole number comes after the fraction (that is, it is placed on the left), then the number is the fraction *of* the whole number. Al-Hawārī's example is $10\,\frac{6}{8}\,\frac{4}{7}$, which is "four sevenths and six eighths of ten" which we might write as $\left(\frac{4}{7}+\frac{6}{8}\right)\cdot 10$. The numerator of the fractions is 74, since together they are $\frac{74}{56}$. Multiply this 74 by the 10 to get the numerator 740.

144.4 If the whole number is written between two fractions, then it might be attached to the first fraction or to the second. If it is attached to the first, or "to what precedes it", then one proceeds as in al-Hawārī's example $\frac{3}{6}\,5\,\frac{4}{9}$, which is "four ninths of five, and three sixths". That is, take $\frac{4}{9}$ of 5, then add $\frac{3}{6}$. Al-Hawārī multiplies the 4 by the 5 to get 20, the numerator of the first part. We can now look at it as the addition of $\frac{3}{6}$ to $\frac{20}{9}$. He effectively cross multiplies, adding 20×6 to 3×9 to get the numerator 147.

145.1 The whole number might be attached to the second fraction, or "what is after it". Al-Hawārī's example is "two thirds of seven and four sevenths", or $\frac{4}{7}\,7\,\frac{2}{3}$. In modern notation this is equivalent to $\frac{2}{3}\cdot\left(7\,\frac{4}{7}\right)$. First we find the numerator of the mixed fraction $7\,\frac{4}{7}$ to get 53. The problem is now $\frac{2}{3}\cdot\frac{53}{7}$. Then the 53 is multiplied by the 2 to get 106, which is the numerator.

As we explained above at 140.14, the ambiguity of the notation is not problematic for the person working through the calculations.

146.3 Al-Hawārī often does not cancel common factors in his fractions. The most glaring examples could have been simplified easily, like "four sixths" at 139.2, "six eighths" at 140.8, and "three sixths" at 144.10. But here Ibn al-Bannāʾ states that one must decompose the numerator and denominator into their prime factors and cancel the common factors, and al-Hawārī explains it for the case of portioned fractions.

147.1 Section I.2.2. Adding and subtracting fractions.

147.2 Now we turn to operating on fractions, beginning with addition. To add two fractions Ibn al-Bannāʾ gives the rule that one multiplies the numerator of each fraction by the denominator of the other, and then the sum of the results is divided by the product of the denominators. Al-Hawārī's example is to add $\frac{6}{8}\,\frac{4}{5}\,3$ to $\frac{1\ 3\ \ 4}{2\ 8\ 10}$. He gives instructions to put the addend over the augend. The operations are clearer if we write the first fraction as $\frac{182}{40}$ and the second as $\frac{71}{160}$. He multiplies 182 by 160 to get 29,120. Next he multiplies 71 by 40 to get 2,840. Adding 29,120 to 2,840 gives 31,960, which is the numerator of the sum. The denominator is the product of the denominators of the two numbers: $40\times 160 = 6{,}400$. So the sum is $\frac{31{,}960}{6{,}400}$, which he prefers to express as "four and nine tenths and seven eighths of a tenth and half an eighth of a tenth", and to write as $\frac{1\ 7\ \ 9}{2\ 8\ 10}4$.

Al-Hawārī then remarks "And its answer is given by (*bi-*) five". This might be read as an approximation, since the answer, as we would write it, is $4\frac{159}{160}$. But phrases like this follow the sample calculations for the other operations on fractions (subtraction, multiplication, division, denomination), and the "answer" in these cases is often not close to the number given. At 149.8, he writes "and its answer is given by one" for the number that as a decimal is approximately .3246. At 150.2, he writes "And its answer is given by two" for the number $1\frac{5}{6}$, and at 151.4 he has "And its answer is four" for $3\frac{1}{63}$. All we can say is that the stated "answer" is the smallest whole number greater than the calculated value. Then in two cases, at 147.15 and 151.14, he writes "And its answer is given by removal/subtraction (*ṭarḥ*)". In these two questions one of the numbers is diminished. We have not been able to decipher the meanings of these phrases, which in any case are given *after* the exact answer to the question is found.

147.15 Al-Hawārī's next example is to subtract $\frac{1}{3}$ ℓ $2\,\frac{7}{10}$ from $\frac{5\bullet 3}{6\bullet 4}4$. In our notation this is $\frac{111}{24}-\frac{32}{30}$. Al-Hawārī rhetorically works through $(111\cdot 30)-(32\cdot 24) = 2,562$. This is the numerator of the difference. The denominators of the two fractions are already given as 6, 4, 3, and 10. Al-Hawārī silently cancels a 6, and instead of working with "thirds of fourths of tenths" he switches to the more customary "halves of sixths of tenths". The result is stated as "three and five tenths and three sixths of a tenth and half a sixth of a tenth", or $\frac{1\ 3\ \ 5}{2\ 6\ 10}3$.

149.1 **Section I.2.3. Multiplying fractions.**

149.2 Ibn al-Bannāʾ characterizes, or defines, the multiplication of fractions as "the portioning of one of the two fractions by the amount of the other". Said a little differently, one takes the portion of one number according to the fraction of the other, which is of course also the meaning of portioned fractions described above at 137.1. So to multiply three fourths by two thirds, for example, one takes two thirds of the three fourths to get one half. As noted in the text, this definition is different from the definition of multiplication for whole numbers given at 95.2.

Both definitions work for the multiplication of a whole number by a fraction. The first is to find "the whole number by the amount of the fraction". To multiply 10 by $\frac{3}{5}$, for example, take $\frac{3}{5}$ of 10 to get 6. By the second definition, one duplicates the fraction as many times as there are units in the whole number. Duplicating $\frac{3}{5}$ ten times again gives 6.

149.6 The rule for multiplying fractions is just what we know it should be: multiply the numerators and divide the result by the product of the denominators. Al-Hawārī sets up his examples the same way he did for adding and subtracting fractions, with one number placed above the other. To multiply $\frac{1\ 3}{3\ 4}$ by $\frac{1\ 4\ 3}{5\ 6\ 9}$ he multiplies the numerators 13 and 111 to get 1443, which he divides by the denominators 3, 4, 5, 6, and 9. This gives "four ninths and a fourth of a fifth of a sixth of a ninth", or $\frac{1\ 0\ 0\ 4}{4\ 5\ 6\ 9}$. He would have arrived at this form by canceling the common 3 to get $\frac{481}{4\cdot 5\cdot 6\cdot 9}$, and then by following the rule for denomination given above at 123.18.

150.2 In the next example the numbers are set up similarly. The notational versions are written one above the other, the numerators are found, and the multiplication follows. Here the numerators of $\frac{1}{8}\,4\,\frac{1}{3}$ and $10\,\frac{2\bullet 1}{3\bullet 5}$ are 33 and 20, respectively, so their product is 660. The denominators are given as 8, 3, 3, and 5. Everything but a leftover 2 and a 3 cancel, so the answer is "one and five sixths".

151.1 **Section I.2.4. Division and denomination.**

151.2 This rule is also what we would expect. To divide or denominate $\frac{a}{b}$ by or with $\frac{c}{d}$, divide/denominate ad by/with bc. Al-Hawārī's example for division is to "divide six and a third by four fifths of seven eighths of three":

$$\frac{1}{3}\,6$$

$$3\,\frac{7\bullet 4}{8\bullet 5}$$

The numerator of the top fraction is 19, and this is multiplied by the denominator of the bottom number, which is 40, to get 760. The other numerator and denominator are 84 and 3, and their product is 252. The answer is what we get from dividing 760 by 252, which is "three and a seventh of a ninth", or $\frac{1\ 0}{7\ 9}\,3$. Here the denominators 7 and 9 do not come from the denominators in the given numbers. To get them, note that 252 goes into 760 three times with a remainder of 4. The fraction 4/252 is the same as 1/63, and 63 is 7×9. Thus the remainder is "a seventh of a ninth".

151.14 The example for denomination is to "denominate three and a fourth less two ninths of it with six and two eighths and three fifths". The calculations are easier to follow if we rewrite the problem as $\frac{91}{36}\div\frac{274}{40}$. After cancelling eights, the denominator of the quotient is 1233, which is $9\cdot 137$.

152.9 There is no need to do all this work if the denominators of the two fractions are the same. In this case, just divide the numerator of the top number by the numerator of the bottom number. Al-Hawārī again gives examples for both division and denomination, and his calculations are clear.

153.7 If the numerators are equal, then for division one divides the denominator of the divisor by the denominator of the dividend, and similarly for the case of denomination. The example for division is to "divide five by five sixths". Al-Hawārī notes that the denominator of a whole number is 1, so he divides 6 by 1 to get 6. Similarly, denominating $\frac{5}{6}$ with 5 gives "a sixth".

154.1 **Section I.2.5. Restoration and reduction.**

This chapter is the version for fractions of Section I.1.6 (at 129.1 above). Ibn al-Bannāʾ breaks up his explanations of restoration into six "problems": (1) restoring a fraction to a fraction, (2) restoring a fraction to a whole number and a fraction, (3) restoring a fraction to a whole number, (4) restoring a whole number to a whole number and a fraction, (5) restoring a whole number and a fraction to a whole number, and (6) restoring a whole number and a fraction to a whole number and a fraction. He omits the two cases of (a) a whole number to a fraction, and (b) a whole number and a fraction to a fraction, since in both cases the first number is necessarily greater than the second, so the problem belongs to reduction. There are also six types of reduction, this time omitting the two that necessarily belong to restoration.

The rule is the same as that given above in Section I.1.6. Reviewing one example each for restoration and reduction should suffice. For the third type of restoration, al-Hawārī restores "two thirds of five sevenths so that it gives ten". The answer is found by dividing 10 by $\frac{5 \bullet 2}{7 \bullet 3}$, which gives 21. This means that to restore $\frac{5 \bullet 2}{7 \bullet 3}$ to 10, one multiplies it by 21. The example for the first type of reduction is to "reduce seven tenths so that it becomes a third". Denominate $\frac{1}{3}$ with $\frac{7}{10}$ to get $\frac{10}{21}$, which is "three sevenths and a third of a seventh", or $\frac{1\ \ 3}{3\ \ 7}$. So, to reduce $\frac{7}{10}$ to $\frac{1}{3}$ one multiplies the $\frac{7}{10}$ by $\frac{1\ \ 3}{3\ \ 7}$.

157.1 **Section I.2.5. Converting.**

Sometimes one wants to change a fraction from one name to another, like from thirds to fifths. Two thirds, for example, is the same as three and a third fifths. We might awkwardly write this as $\frac{3\frac{1}{3}}{5}$, but it would have been expressed by al-Hawārī as "three fifths and a third of a fifth", which in notation is $\frac{1\ \ 3}{3\ \ 5}$. Al-Hawārī begins the section on converting (*taṣrīf*) by quoting from *Lifting the Veil*.

157.2 Ibn al-Bannāʾ speaks about two kinds of conversion. The first type "concerns only the name". Ibn al-Bannāʾ's example is to convert "five sixths and three fourths" ($\frac{3\ \ 5}{4\ \ 6}$) to tenths. "Naming the fraction in tenths" results in "one and five tenths and five sixths of a tenth", or $\frac{5\ \ \ 5}{6\ \ 10}1$. Al-Hawārī will illustrate Ibn al-Bannāʾ's rule below at 158.11.

157.9 The second type "concerns how many of that name, taken as units, are in the whole [fraction]". The example asks how many tenths are in $\frac{3\ \ 5}{4\ \ 6}$. Ibn al-Bannāʾ relates this to the problem of converting whole numbers in the section on multiplication above at 95.6. The answer is found by multiplying the fraction by 10, so it "does not require division by the denominator of the converted number" (158.1).

158.7 Al-Hawārī then returns to give the rule in the *Condensed Book*. His example is to convert six eighths and four tenths to ninths. The numerator and denominator of the fraction "to be converted" are 92 and 80. The procedure is to multiply the 92 by the 9, giving 828, and then this is divided by the 80. We would write the quotient as $10\frac{28}{80}$, but in Arabic it is $\frac{4\ \ \ 3}{8\ \ 10}10$. Al-Hawārī writes "four eighths" instead of "a half" because he wants to keep the same denominators as the original fractions. Dividing this by 9 is easy: $10 \div 9$ is $\frac{1}{9}1$, and to divide $\frac{4\ \ \ 3}{8\ \ 10}$ by 9 one just puts a $\frac{0}{9}$ on the right: $\frac{4\ \ \ 3\ \ 0}{8\ \ 10\ \ 9}$. Gathering them together gives the answer: $\frac{4\ \ \ 3\ \ 1}{8\ \ 10\ \ 9}1$. For this problem, al-Hawārī did not bother to rearrange the denominators in descending order.

161.1 **Chapter I.3. Roots.**

163.1 **Section I.3.1. Taking a root of a whole number and a root of a fraction.**

163.2 The term for "square root" in Arabic was simply "root" (*jadhr* or *jidhr*). A root of a number might be rational, like $\sqrt{25} = 5$ and $\sqrt{\frac{4}{9}} = \frac{2}{3}$, or it might be irrational, like $\sqrt{3}$ and $\sqrt{\frac{17}{25}1}$. In Ibn al-Bannāʾ and al-Hawārī, as in many Arabic authors, the word for "rational" is *munṭaq*, which literally means "expressible". One can *say* $\sqrt{\frac{4}{9}}$ as "two thirds", while one cannot say the result of taking a square root of 3 other than as "a root of three". Our authors write *ghayr munṭaq* ("inexpressible") to mean "irrational". Since irrational numbers in medieval mathematics are all surds, we translate this term by "surd" (recognition of transcendental numbers came only in the nineteenth century). Some other Arabic texts write *aṣamm* ("deaf") for "irrational", but our authors use this word to mean "prime" (see our comments at 66.7 and 134.2 above). The corresponding words in other languages have the same association with speech. In Greek the word for "rational" was *rhêtos*, meaning "something that may be said or spoken", while "irrational" was either *alogos* ("deprived of speech") or *arrhêtos* ("unsayable"). In medieval Latin the word *surdus*, meaning "deaf", was used for "irrational", and Italian texts show the related word *sordo*. The Latin *surdus* is the origin of our word "surd".

Roots are expressed in medieval Arabic in a slightly different way than they are in English today. Where we speak of "*the* root of ten", they said "*a* root of ten". In modern arithmetic numbers are unique. There is only one 3, one $\sqrt{10}$, etc. But just as in Greek arithmetic,[46] in medieval arithmetic numbers admitted multiplicity. One can have twelve roots of ten, for example. So when an Arabic arithmetician takes the square root of a number, the result is expressed with the implied indefinite article. Read "a root of ten" to mean a single $\sqrt{10}$. This may seem to be just a linguistic curiosity, but it becomes relevant in the duplication of roots and for understanding their preference for single roots as opposed to multiple roots, discussed below at 179.1. Al-Hawārī will, however, sometimes write "*the* root" when he is pointing to a specific quantity in a calculation. This is done with the same intention as when he writes, for example, "we add *the* four in the hundreds rank…" (at 75.11). Other words, like "difference", "square", and "ratio", will lack the definite article, too. Sometimes we insert a "the" where there is none in the Arabic to make the reading easier. We did this in the passage at 174.1, for example, where neither "the" in our translation "the difference between the squares" is in the original (with the exception of "the result", the other instances of "the" in that passage are present in the Arabic).

163.4 Ibn al-Bannāʾ differentiates between irrational roots that can be expressed with the word "root" once, and those that require the word more than once. A number in the first category, like $\sqrt{3\frac{9}{10}}$, is said to be "rational in square" because its square is rational, while "a root of a root of ten" is an example of a "medial" root. In our notation this number is $\sqrt{\sqrt{10}}$, which we usually write as $\sqrt[4]{10}$. Rational numbers are also rational in square,

[46] (Mueller 1981, 59).

so when we want to speak of an irrational number that is rational in square we say it is "rational in square only".

The phrase "rational in square" is more literally translated as "expressible in power (*fī l-quwwa*)". The phrase *fī l-quwwa* is a translation of the Greek term *dynámei*, which ordinarily means "in power" or "in value", but was applied in mathematics to mean the size of the square on a line or of a number.

The word "medial" (*muwassaṭ*) is translated from the Greek *mésos*. Reinterpreting Euclid's geometric definition of *mésos* in *Elements* Proposition X.21 in arithmetical terms, medial numbers are those that take the form $\sqrt{\sqrt{p}}$, where p is a non-square rational. These "roots of roots" are called "medial" because they are the mean proportion between 1 and a number rational in square only. For example, the medial $\sqrt{\sqrt{10}}$ satisfies $1 : \sqrt{\sqrt{10}} :: \sqrt{\sqrt{10}} : \sqrt{10}$. Ibn al-Bannāʾ's characterization, if taken to the letter, diverges from the Greek. Other numbers, like $\sqrt{\sqrt{\sqrt{3}}}$ and $\sqrt{2+\sqrt{8}}$ are also expressed with the word "root" more than once, though it is unlikely that he considered them to be medial, too.

163.11 Vowels are generally not indicated when writing Arabic, so the word for "root" shows only the three consonants *j-dh-r*. Some pronounce this as *jidhr* and others *jadhr*. Ibn al-Bannāʾ preferred the latter.

163.14 Ranks, as well as numbers, are said to either have a root or not have a root. Every other rank, starting with the units, has a root (the units, hundreds, ten thousands, etc.), while the remaining ranks have no root (the tens, thousands, hundred thousands, etc.). As Ibn al-Bannāʾ explains, a rank has a root "if there is a number in it that has a root". For example, the number 400 is in the hundreds rank, and it has a root, but none of the numbers 1,000, 2,000, up to 9,000 have a root.

164.3 – 165.16 Here al-Hawārī copies Ibn al-Bannāʾ's rules for determining when a whole number might have a (rational) square root. For example, if the units digit (the first digit) is a 5, and the tens digit is not a 2, then the number does not have a root.

Squaring numbers of the form $9n + r$ for $r = 1, 2, \ldots, 8$ and then casting out nines shows that the remainder, when it is not cast out entirely, must be a 1, 4, or 7. Determining the remainders of the squares after casting out eights and sevens works similarly.

166.1 The examples that al-Hawārī and Ibn al-Bannāʾ give for extracting roots are too small to show fully how the rule works, and no figures are shown in the book. Fortunately, the same method is explained by Ibn al-Yāsamīn with greater numbers and with some figures. We give one of his examples here.[47]

[47] (Zemouli 1993, 250).

To find a root of 876,096 we begin by writing the number on a line, and we mark the first place with an "r" for "root", the second place with an "n" for "no root", alternating "r"s and "n"s to the last place of the number:

n	r	n	r	n	r
8	7	6	0	9	6

We start with the last "r" on the left. The number below it is 87, and the greatest square that can be subtracted from it is 81. So we put its root, 9, under it, and we subtract 81 from 87 and replace the 87 with the remainder:

n	r	n	r	n	r
	6	6	0	9	6
	9				

The 9 below is then doubled and shifted one place to the right:

n	r	n	r	n	r
	6	6	0	9	6
	1	8			

Now we look for the greatest digit n such that when we multiply it by the number of the form "18n" (i.e., $180 + n$) it will cancel as much of the 660 above it as possible. This digit is 3. We put the 3 below the 0, and then start the multiplication. Instead of multiplying the 3 by the 183 at once, Ibn al-Yāsamīn, Ibn al-Bannāʾ, and al-Hawārī perform the multiplication one digit at a time. First we multiply the 3 by the 1 to get 3, and this is subtracted from the 6 above:

n	r	n	r	n	r
	3	6	0	9	6
	1	8	3		

Next, we multiply the 3 by the 8 to get 24, and this is subtracted from the 36 above:

n	r	n	r	n	r
	1	2	0	9	6
	1	8	3		

Then the 3 is multiplied by itself, and the 9 is subtracted from the 120 above:

n	r	n	r	n	r
	1	1	1	9	6
	1	8	3		

Now the 3 is doubled, and the 186 is shifted one place to the right:

n	r	n	r	n	r
	1	1	1	9	6
		1	8	6	

Next, we want the greatest digit n such that when it is multiplied by "186n" it cancels as much of the 11196 above as possible. This digit is a 6. We put it under the 6, and we multiply it by 1866, again one digit at a time. Multiplying it by 1 gives 6, and we take 6 away from the 11 above it:

n	r	n	r	n	r
		5	1	9	6
		1	8	6	6

Then we multiply the 6 by the 8, and we take 48 away from the 51 above it:

n	r	n	r	n	r
			3	9	6
		1	8	6	6

The 6 is then multiplied by the 6 next to it to get 36, and this is subtracted from the 39 above:

n	r	n	r	n	r
				3	6
		1	8	6	6

And finally, the 6 is multiplied by itself to get 36, and this cancels the 36 above. The answer is found by adding the 6 to half of the 1860 that follows it, giving 936. So $\sqrt{876{,}096} = 936$.

166.9 To find a fractional approximation to a root of a non-square number, Ibn al-Bannāʾ gives different rules depending on whether or not the remainder is greater than the integer part of the root. The rule for remainders less than or equal to the root is the well-known approximation $n + \frac{r}{2n}$, where n is the integer part of the root and r is the remainder. The rule for greater remainders is the same, but based at the next root up. In modern notation, the value obtained by this rule is $n+\frac{r+1}{2n+2}$, and we can calculate that it is equal to $(n+1)+\frac{r'}{2(n+1)}$ where the remainder r' is now negative, counting back from $(n+1)^2$ instead of forward from n^2.

Ibn al-Bannāʾ gives proofs/derivations for this method in *Lifting the Veil*. These are not copied by al-Hawārī, so we translate part of one proof here to show the reasoning behind it. This part covers the case where the approximation to the root is smaller than the true root. Ibn al-Bannāʾ's arguments are arithmetical, and to make the reading easier we give modern algebraic equivalents in brackets. In it, he approximates the fraction f that is added to the known root n of a square n^2 so that together they equal the unknown root m of a greater square m^2. The "surplus" $m^2 - n^2$ is our remainder r.

> And its cause is that a root of the number is divided into two parts, a root of the smaller and a fraction [$m = n + f$]. Multiplying that by itself is like multiplying each one by itself and one of them by double the other [$(n+f)^2 = n^2 + f^2 + f \cdot 2n$]. So the surplus between[48] the number and the square of the smaller is equal to a square of the fraction and a product of the fraction by double a root of the smaller [$m^2 - n^2 = f^2 + f \cdot 2n$]. You are allowed to drop a square of the fraction, and you make the surplus equal to a product of the fraction by double a root of the smaller [f^2 will be small, so we can suppose that $m^2 - n^2 = f \cdot 2n$]. So divide the surplus by double a root of the smaller, resulting in the fraction by approximation [$f = \frac{m^2-n^2}{2n}$].[49]

He continues: "And it is clear that the resulting fraction is greater than the true fraction, so the approximation is always in excess of the required root of the number".[50] This is true for both rules, whether the remainder exceeds the root or not. Reinterpreting the rules in terms of modern functions, the approximations for the first rule lie on the line tangent to $f(x) = \sqrt{x}$ at $x = n^2$, where n is the integer part of the root. The approximations from the second rule lie on the line tangent to $f(x) = \sqrt{x}$ at $x = (n+1)^2$. *We* know that the approximations will always be greater than the actual values because the graph of $f(x) = \sqrt{x}$ is concave down. Ibn al-Bannāʾ's reasoning is simpler.

Examples are calculated at 167.8, 167,14, and 168.1. Because n can be any approximation to the root, not just a whole number, Ibn al-Bannāʾ understood that the rule can be iterated. This is done beginning at 168.8.

166.13 Al-Hawārī's first example of calculating a square root is to find $\sqrt{625}$. He sets it up and begins the process according to Ibn al-Bannāʾ's instructions:

r	n	r		r	n	r		r	n	r
6	2	5	⟶	2	2	5	⟶	2	2	5
				2					4	

[48] Here "surplus (*faḍla*) between" could be translated as "difference between", but the notion of a surplus or excess would be lost.
[49] (Ibn al-Bannāʾ 1994, 285.10).
[50] (Ibn al-Bannāʾ 1994, 285.15).

The next digit to be found is the units digit. Because al-Hawārī presumes that 625 is a perfect square, he knows that it must be a 5: no other unit would produce the 5 in 625.

167.3 He then multiplies the 5 by 4 to get 20, which leaves a 2 when confronted with the 22 above it. Then the 5 is squared, which cancels the 25 left above:

$$\begin{array}{ccc} r & n & r \\ 2 & 2 & 5 \\ & 4 & 5 \end{array} \longrightarrow \begin{array}{ccc} r & n & r \\ & 2 & 5 \\ & 4 & 5 \end{array}$$

By "the five and the doubled two after it" he means the 5 in the units place on the lower line, and half of the 4 (really 40) to its left. The root is then 25.

167.8 Al-Hawārī's next example is not a perfect square. He finds an approximation to a root of 20 using the rule from 166.9:

$$\begin{array}{cc} n & r \\ 2 & 0 \\ & \end{array} \longrightarrow \begin{array}{cc} n & r \\ & 4 \\ & 4 \end{array}$$

Since the number above is not exhausted and is not greater than the root, he adds to the root, which is the 4 below, the remainder above divided by twice the root, or $4 \div (2 \cdot 4)$, to get $4\frac{1}{2}$. A square of $4\frac{1}{2}$ is $20\frac{1}{4}$, which is close to 20.

167.14 The same rule is applied to approximate a root of 54, since the remainder, 5, is less than the root, 7.

168.1 Al-Hawārī's next example is to find a root of 92. On the dust-board, it would have progressed like this:

$$\begin{array}{cc} n & r \\ 9 & 2 \\ & \end{array} \longrightarrow \begin{array}{cc} n & r \\ 1 & 1 \\ & 9 \end{array}$$

This time, the remainder 11 is greater than the root 9: so he adds 1 to the 11 to get 12, and he adds 2 to double the root to get 20. Then the fractional part is 12/20, or 3/5. The approximation is then $\frac{3}{5}\,9$. As al-Hawārī notes, its square is $\frac{4\ 0}{5\ 5}\,92$. We would write it as $92\frac{4}{25}$ or 92.16.

168.8 In the approximation method just described, one finds the integer part of the root first, and the approximation is calculated as a fraction to be added on. This approximation is greater than the desired root, so Ibn al-Bannāʾ's rule for obtaining an even closer approximation is calculated by subtracting off another fraction.

168.10 In the example of approximating $\sqrt{92}$, the "close smaller square" is a square of the integer part of the first approximation $9\frac{3}{5}$, or 9, which is 81, and which is less than 92. The "close greater square" is a square of $9\frac{3}{5}$, or $92\frac{4}{25}$, which is greater than 92.

168.16 To obtain a better approximation with this greater square, he denominates the $\frac{4}{25}$ with double the root $9\frac{3}{5}$ to get $\frac{1}{120}$, or "half a sixth of a tenth". He subtracts this from the root $9\frac{3}{5}$ to get $\frac{1\ \ 5\ \ 5}{2\ \ 6\ \ 10}9$, or, in our terms, $9\frac{71}{120}$. This approximation is more accurate: its square as a decimal is 92.0000694…

169.1 Another way to get a better approximation is to multiply the original number by some large square number, take its root, and then divide by a root of that large square. The example is to approximate a root of 12 by first multiplying the 12 by 16, taking its root, and then dividing the result by 4. The approximation reached this way is $\frac{1\ \ 3}{4\ \ 7}3$, whose square as a decimal begins 12.00127… If we were to approximate $\sqrt{12}$ by the rule at 166.9 we would get $\frac{1}{2}3$, whose square is 12.25.

169.8 Ibn al-Bannāʾ extends the technique for finding roots of whole numbers by allowing one to put down numbers with fractional parts, and he gives two examples in *Lifting the Veil* that are copied by al-Hawārī at 169.10 and 169.17. The statement that "the remainder will be smaller than the remainder with whole numbers" is best illustrated in the example at 169.17. The remainder after the first step is $\frac{3}{4}$, which is smaller than the 3 that would have been the remainder had he worked with whole numbers. When working with fractions, there is no smallest possible (positive) remainder because one can always find a closer fraction to the root.

169.10 In *Lifting the Veil*, Ibn al-Bannāʾ shows how to use the standard rule from 166.1 with fractions to find $\sqrt{729}$ and $\sqrt{625}$. Al-Hawārī copies them in reverse order, doing $\sqrt{625}$ first. Instead of choosing the greatest whole number whose square does not exceed the number above, now a fractional part can be added to that number. The work proceeds like this:

$$\begin{array}{ccc} r & n & r \\ 6 & 2 & 5 \\ & & \end{array} \longrightarrow \begin{array}{ccc} r & n & r \\ 0 & 0 & 0 \\ 2\frac{1}{2} & & \end{array} \longrightarrow \begin{array}{ccc} r & n & r \\ 0 & 0 & 0 \\ & 5 & \end{array} \longrightarrow \begin{array}{ccc} r & n & r \\ 0 & 0 & 0 \\ & 5 & 0 \end{array}$$

Ibn al-Bannāʾ finds the answer by taking half of the 50 in the last figure.

169.17 The next example is to find a root of 729. The work goes like this:

$$\begin{array}{ccc} r & n & r \\ 7 & 2 & 9 \\ & & \end{array} \longrightarrow \begin{array}{ccc} r & n & r \\ \frac{3}{4} & 2 & 9 \\ 2\frac{1}{2} & & \end{array} \longrightarrow \begin{array}{ccc} r & n & r \\ 1 & 0 & 4 \\ 2\frac{1}{2} & & \end{array} \longrightarrow \begin{array}{ccc} r & n & r \\ 1 & 0 & 4 \\ & 5 & \end{array} \longrightarrow \begin{array}{ccc} r & n & r \\ 1 & 0 & 4 \\ & 5 & 2 \end{array}$$

From the second to the third figure he calculates that $\frac{3}{4}$ of 100 is 75, so he adds this to the 29 to get 104. In the last step $2 \cdot 50 + 2^2 = 104$, so the number on top is exhausted. The answer is then half of the 50 together with the 2, which is 27.

170.6 The example of calculating $\sqrt{100}$ by adding half the zeros back to $\sqrt{1}$ is clear.

170.10 Al-Hawārī gives examples of taking roots of fractions. There are two methods to do this. The first is that "you multiply the numerator by the denominator and you divide a root of the result by the denominator". This method is preferred when the denominator is not a perfect square, because you avoid dividing by an ugly fractional approximation of a root. The second way is to "divide a root of the numerator by a root of the denominator". This way is easier if the denominator is a perfect square. In modern notation, in the first method the square root of the fraction $\frac{a}{b}$ is found by calculating $\sqrt{ab} \div b$, and in the second by $\sqrt{a} \div \sqrt{b}$.

170.14 – 173.3 Ibn al-Bannāʾ classifies four kinds of fraction, depending on whether the numerator and/or the denominator is a square.

As a single fraction, the example at 170.16 is $\frac{25}{36}$, and at 171.1 it is $\frac{49}{4}$. The fraction at 171.6 is $\frac{27}{54}$, which is the same as $\frac{1}{2}$. The approximation for its root is equal to $\frac{1451}{2052}$, which is correct to four decimal places. Applying the standard method to $\frac{1}{2}$ gives an approximation of $\frac{3}{4}$, which is much less accurate.

The fraction at 171.15 is $\frac{175}{16}$. For the first method, al-Hawārī multiplies the denominator by the numerator to get 2800, and he takes a root of the product. The result, $\frac{1}{2}\frac{48}{53}52$, is then divided by 16. For the second method, he calculates a root of 175 as $\frac{3}{13}13$, and he divides this by 4 to get $\frac{4}{13}3$. He is wrong when he writes "This method is closer than the first [method]". The first method is correct to five places, while the second is correct only to three.

The fraction at 172.12 is $\frac{9}{14}$.

173.4 Binomials and apotomes.

The only mention of binomials and apotomes in Ibn al-Bannāʾ's *Condensed Book* is his rule for finding their roots at 173.4. He does not explain what binomials and apotomes are, and he omits their classification into the six types and how to find examples of them. All that should precede the calculations. He may have inserted this rule as an afterthought, since the heading for the current section, given above at 163.1, does not cover roots of irrational numbers: "on taking a root of a whole number and a root of a fraction".

173.10 Ibn al-Bannāʾ provided the missing material in his *Lifting the Veil*, and al-Hawārī copied it from there into his book. This includes the definitions of binomials and apotomes (173.12-18), the classification of the six types (174.1-13), and the finding of the six binomials and six apotomes (174.14-175.10). Al-Hawārī then finds a root of a sample binomial and its apotome by Ibn al-Bannāʾ's rule (175.11-21).

Then, at 176.1, al-Hawarī gives a variation on Ibn al-Bannāʾ's rule and applies it to find roots of examples of each of the six types of binomial and apotome (176.6-177.15). After calculating each root, al-Hawārī adds a brief description of it. These descriptions are translated from Euclid's *Elements*, and they are found with this same particular wording in a number of Arabic geometry and arithmetic books written before al-Hawārī's time. In the translation, we put the descriptions in quotation marks to indicate the borrowing. We do not know al-Hawārī's immediate source for these descriptions, but they may originate from al-Ḥajjāj's translation of the *Elements*.[51] We single them out in part because the meanings of the terms "medial" and "bimedial" become altered in the arithmetical setting of al-Hawārī's book. In Euclid's geometrical setting medial/bimedial *areas* are intended, meaning that it is their sides (square roots) that are medial. Al-Hawārī's $\sqrt{20}$ (at 177.5), for example, corresponds to a medial area because its "side", $\sqrt{\sqrt{20}}$, is medial. Reading the Arabic literally, it seems that the number $\sqrt{20}$ itself is being called medial.

We should give two related definitions before proceeding. Two (positive) numbers are said to be *commensurable* if their ratio is rational, like $\sqrt{12}$ and $\sqrt{27}$, or any two rational numbers. Two numbers are *commensurable in square* if their squares are commensurable. Examples include $\sqrt{\sqrt{12}}$ and $\sqrt{\sqrt{27}}$, 5 and $\sqrt{13}$, and any two commensurable numbers. Our authors write "numerical ratio" (*nisba ʿadadiya*, at 174.17) or "in ratio" (*min nisba*, at 180.8) to mean "commensurable", and a word whose ordinary meaning is "different" (*mutabāyna*, at 180.11 and 182.5) for "incommensurable". There may not have been standard terms for these meanings.

The nomenclature and theory of binomials and apotomes that was part of Arabic arithmetic derive from a numerical reading of Book X of Euclid's *Elements*.[52] Our word "binomial", like the original Greek *(ek) duo onomatōn*, means "two names". In Arabic, the term for "binomial" is *dhū l-ismīn*, meaning "two unified names". A binomial is a number that can only be expressed in the form "x and y", where the incommensurable numbers x and y are either rational or rational in square.[53] Examples are "eight and a root of sixty" and "a root of five and a root of three", which we would write as $8+\sqrt{60}$ and $\sqrt{5}+\sqrt{3}$, respectively. Our $8+\sqrt{60}$ contains the arithmetical operation of addition, but as we explain at 219.1

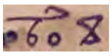
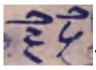
[51] We have found the descriptions in three commentaries on Euclid's Book X: Anonymous (attributed to al-Māhānī, #82 [M2], ninth century), Sulaymān ibn ʿIṣma al-Samarqandī (#181 [M1], ca. 900), and al-Ahwāzī (#193 [M1], tenth century); in Ibn Sīnā's epitome of Euclid's *Elements* ([M1], ca. 1000); and in two works dealing with algebra: al-Karajī's *Marvelous [Book] of Arithmetic* (*al-Badīʿ*) ([M3], early eleventh century. He gives a minor variation in the descriptions of the second and third apotomes) and Ibn al-Bannāʾ's *Book on the Fundamentals and Preliminaries in Algebra* ([M6], before 1300. Only for binomials). Gregg De Young writes that Ibn Sīnā's book "was heavily influenced by, if not based completely upon, the al-Ḥajjāj translation tradition" (De Young 1991, 665). His evidence comes partly from Ibn Sīnā's proofs to propositions X.64-66 (De Young 1991, 661). From this, and the fact that their wording is different in the Isḥāq-Thābit translation, we cautiously attribute the wording of these phrases to the translation of al-Ḥajjāj.

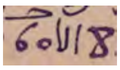
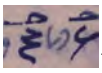
[52] Definitions are given in Propositions X.36 and X.73, and the classifications are given in two sets of definitions presented after Propositions X.47 and X.84.

[53] By contrast, commensurable roots can be added to make a single root, like the example at 179.16: $\sqrt{2}$ and $\sqrt{8}$ together are $\sqrt{18}$.

below, the premodern “eight and a root of sixty” consists instead of two numbers gathered together.

Ibn al-Bannāʾ’s verb for forming a binomial is *waṣala*, “to join” two numbers together (pp. 174-5). Sometimes the related noun *muttaṣil*, “joining”, takes the place of the usual term for “binomial”. At 188.19, Ibn al-Bannāʾ spells it out as “[a] joining of two names”, and in three instances just after that we find it as merely “joining”. On that page we translate these variants as “binomial” to make the reading clearer, while at 177.9 and 177.15 the word is better rendered as “joining”.

Al-Hawārī does not show notation for roots, but other authors do. In his 1370 commentary on Ibn al-Bannāʾ’s *Condensed Book*, Ibn Qunfudh writes these two binomials in notation as  and .[54] Reversing the order of the terms, we can transcribe them as “$8\sqrt{60}$” and “$\sqrt{5}\,\sqrt{3}$”, respectively. The letter *jīm* (ج) without the dot, for *jadhr* (“root”), is placed above the 60, 5, and 3 to indicate square root. We have already seen this way of writing gathered numbers – one after the other – several times for distinct fractions, first at 136.8.

Our word “apotome” comes from the Greek word *apotomē*, meaning “something cut off”, and the Arabic translation is *munfaṣil*, meaning an amount from which something has been “detached”. Examples are “eight less a root of sixty” and “a root of five less a root of three”. We write these in modern notation as $8-\sqrt{60}$ and $\sqrt{5}-\sqrt{3}$, but again, our operation of subtraction distorts the premodern idea of a diminished quantity. An apotome was regarded as a single quantity from which something has been removed. The “eight less a root of sixty”, for example, should be thought of as a deficient eight, and not as the subtraction of something from eight. We explain this more thoroughly below at 219.4. Ibn Qunfudh shows these two examples in notation with the word *illā* (“less”) between the numbers:  and .[55] We transcribe them as $8\ \ell\ \sqrt{60}$ and $\sqrt{5}\ \ell\ \sqrt{3}$. This notation is the same as we have seen for excluded fractions of disconnected type, first at 140.8, and with an image from the Medina manuscript in our commentary at 140.1. Further, Ibn al-Bannāʾ and al-Hawārī use the term *munfaṣilat* with the meaning of “detached” for the excluded parts of these fractions in the passages at 141.7, 142.6, and 142.10.

Binomials and apotomes are not restricted to the arithmetic of fractions and roots. In algebra, too, the same concepts and notation are behind polynomials. At 183.15, al-Hawārī makes a direct comparison between the quadratic irrationals of this section and algebraic expressions: “The principle behind multiplying appended and deleted terms will be covered in [the chapter on] algebra”.

Euclid’s theory of binomials and apotomes is expressed in a geometric context, so he necessarily takes into account the dimensions of his magnitudes. For example, instead of the multiplication of two numbers, he writes of the formation of a rectangle from two sides. He defines a binomial as a line, and lines by their nature cannot have “square roots”. So, Euclid defines its square root as the side of a square equal in area to a rectangle having the binomial as one of its sides and a rational line (his version of a unit) for the other side. This theory was transferred to the setting of numbers in Arabic arithmetic. Numbers are

[54] (Ibn Qunfudh manuscript, 197.24, 198.12).
[55] (Ibn Qunfudh manuscript, 198.12-13).

homogeneous and thus dimensionless, so some of Euclid's distinctions become superfluous.

Because we will be interspersing the notation for binomials and apotomes with calculations in modern notation, we will express the binomials and apotomes with "+" and "−" from now until the end of the section. To transcribe them in notation that better reflects the notation in the texts, omit the "+" and replace the "−" with an "ℓ". We write them this way, right-to-left, in the conspectus in Appendix A.

174.1 In *Lifting the Veil*, Ibn al-Bannāʾ explains the classification of the six types of binomial and apotome, and al-Hawārī gives examples of each. There are three basic types. Writing the greater term first, in order they are $p \pm \sqrt{q}$, $\sqrt{p} \pm q$, and $\sqrt{p} \pm \sqrt{q}$ where both p and q are rational, and where one of these values appears under a root it is not a perfect square. Each of the basic types is divided into two subtypes stemming from a characteristic of its root. Writing the general binomial in modern notation as $\sqrt{P} + \sqrt{Q}$, in which P and Q are both rational, $P > Q$, and at most one is a square, al-Hawārī's rule for calculating $\sqrt{\sqrt{P} + \sqrt{Q}}$ at 176.1 can be written as

$$\sqrt{\tfrac{1}{2}(\sqrt{P} + \sqrt{P - Q})} + \sqrt{\tfrac{1}{2}(\sqrt{P} - \sqrt{P - Q})}.$$

A root of the associated apotome, $\sqrt{\sqrt{P} - \sqrt{Q}}$, is

$$\sqrt{\tfrac{1}{2}(\sqrt{P} + \sqrt{P - Q})} - \sqrt{\tfrac{1}{2}(\sqrt{P} - \sqrt{P - Q})}.$$

In one of the two subtypes $\sqrt{P}$ is commensurable with $\sqrt{P - Q}$, so the two terms can be combined into a single root. This holds for the first three of the six types of binomial/apotome. In the other subtype the two are incommensurable, and this holds for the last three of the six types.

Ibn al-Bannāʾ's classification follows that of Euclid, who gives constructions for each of the six types of binomial in Propositions X.48-53, and for the six types of apotome in Propositions X.85-90.

Just as the digits in a number come in ranks, so do numbers expressed with roots. Two numbers are of the same rank if they are the same "distance" from being rational. Thus $\sqrt{5}$ and $\sqrt{21}$ are of the same rank, as are $\sqrt{\sqrt{5}}$ and $\sqrt{\sqrt{21}}$ and the pair $\sqrt{\sqrt{\sqrt{5}}}$ and $\sqrt{\sqrt{\sqrt{21}}}$.

174.14 Euclid then gives constructions for roots of the six binomials in Propositions X.54-59, and for roots of the six apotomes in Propositions X.91-96. Ibn al-Bannāʾ gives numerical rules in place of Euclid's constructions. We rewrite them below in modern notation. Al-Hawārī does not give examples here, so we provide our own. In these descriptions it is assumed that a, b, and c are positive rational numbers, and numbers under roots are positive non-squares.

A number is a *first binomial* if it can be written in the form $a + \sqrt{a^2 - b^2}$, and a *first apotome* if it can be written as $a - \sqrt{a^2 - b^2}$. If we let $a = 5$ and $b = 2$, then the binomial is $5 + \sqrt{21}$ and the corresponding apotome is $5 - \sqrt{21}$.

A number is a *fourth binomial* if it can be written in the form $a + \sqrt{a^2 - b}$, where b is not a perfect square. If $a = 5$ and $b = 8$, then the fourth binomial is $5 + \sqrt{17}$ and the associated apotome is $5 - \sqrt{17}$.

A number is a *second binomial* if it can be written in the form $\sqrt{a^2(a^2 - b^2)} + (a^2 - b^2)$. A *second apotome* can be written in the form $\sqrt{a^2(a^2 - b^2)} - (a^2 - b^2)$. For example, if we let $a = 3$ and $b = 2$, the binomial is $\sqrt{45} + 5$ and the associated apotome is $\sqrt{45} - 5$. Medieval mathematicians would say "a root of forty-five" instead of "three roots of five" because the latter is a collection of *three* numbers. They preferred the single number $\sqrt{45}$. See below at 179.1 for more on this. Ibn al-Bannāʾ made an error in defining this type by writing "a root of their difference" instead of simply "their difference".

Ibn al-Bannāʾ also made an error in defining the fifth binomial. In our notation his description translates into $\sqrt{a^2 + b^2} + a$. In fact, it should be of the form $\sqrt{a^2 + b} + a$, where $\sqrt{a^2 + b}$ is incommensurable with $\sqrt{b}$. If we make $a = 5$ and $b = 3$, the fifth binomial is $\sqrt{28} + 5$ and the apotome is $\sqrt{28} - 5$. Al-Hawārī's example of the fifth binomial at 177.5 below is correct.

A number is a *third binomial* if it can be written in the form $\sqrt{ca^2} + \sqrt{c(a^2 - b^2)}$. If we let $a = 5$, $b = 2$, and $c = 3$, then the third binomial is $\sqrt{75} + \sqrt{63}$ and the associated apotome is $\sqrt{75} - \sqrt{63}$.

A number is a *sixth binomial* if it can be written in the form $\sqrt{a^2 + b} + \sqrt{b}$, and if the two roots are incommensurable. If $a = 6$ and $b = 7$, then the sixth binomial is $\sqrt{43} + \sqrt{7}$ and the associated apotome is $\sqrt{43} - \sqrt{7}$.

175.11 Al-Hawārī finds roots of the first binomial $8 + \sqrt{60}$ and its associated apotome $8 - \sqrt{60}$ (from $a = 8$ and $b = 2$) by the method given by Ibn al-Bannāʾ at 173.4. We can rewrite this as the modern formula:

$$\sqrt{\sqrt{P} \pm \sqrt{Q}} = \sqrt{\tfrac{1}{2}P + \sqrt{\tfrac{1}{4}P^2 - \tfrac{1}{4}Q^2}} \pm \sqrt{\tfrac{1}{2}P - \sqrt{\tfrac{1}{4}P^2 - \tfrac{1}{4}Q^2}}.$$

For the binomial, he subtracts $\frac{1}{4}$ of a square of the smaller ($\frac{1}{4}$ of $\sqrt{60}^2$ is 15) from $\frac{1}{4}$ of a square of the greater ($\frac{1}{4}$ of 8^2 is 16), leaving 1. Its root is 1. This 1 is added to half of the greater term ($\frac{1}{2}$ of 8 is 4) to get 5, and it is also subtracted from the 4 to get 3. Then $\sqrt{5}+\sqrt{3}$ is a root of $8+\sqrt{60}$. He then finds a root of the associated apotome $8-\sqrt{60}$ by the same procedure to get $\sqrt{5}-\sqrt{3}$.

176.1 – 177.15 Al-Hawārī gives the variation on the rule for finding roots of binomials and apotomes that we described above at 174.1, and he applies it to his examples for each of the six types.

176.1 To find a root of the first binomial $8+\sqrt{55}$ (with $a = 8$ and $b = 3$), take the difference of their squares $8^2-(\sqrt{55})^2$ to get 9, and take its root to get 3. Now figure $\frac{1}{2}(8+3)$ and $\frac{1}{2}(8-3)$, and take their roots. A root of the binomial is then $\sqrt{5\frac{1}{2}}+\sqrt{2\frac{1}{2}}$, and a root of the apotome $8-\sqrt{55}$ is $\sqrt{5\frac{1}{2}}-\sqrt{2\frac{1}{2}}$. Euclid (Propositions X.54, 91) and the Arabic translation call the root of the binomial "a binomial" and the root of the apotome "an apotome", which is indeed what they are. Al-Hawārī expands these phrases to "one of the binomials" and "one of the six apotomes".

176.10 Next, al-Hawārī finds a root of $\sqrt{112}+7$, which is an example of the second binomial ($a = 4$, $b = 3$). Following the same procedure, $(\sqrt{112})^2-7^2 = 63$. Then $\frac{1}{2}(\sqrt{112}+\sqrt{63})$ and $\frac{1}{2}(\sqrt{112}-\sqrt{63})$ are $\sqrt{85\frac{3}{4}}$ and $\sqrt{1\frac{3}{4}}$. (For these steps, al-Hawārī calculated $\sqrt{112}+\sqrt{63} = \sqrt{343}$ and $\sqrt{112}-\sqrt{63} = \sqrt{7}$. Instructions on how to add and subtract roots are given below, starting at 179.1.) So a root of the binomial $\sqrt{112}+7$ is $\sqrt{\sqrt{85\frac{3}{4}}}+\sqrt{\sqrt{1\frac{3}{4}}}$.

Euclid (Proposition X.55) and the Arabic translation call a root of the second binomial "the first bimedial". Euclid defines this term in Proposition X.37: "If two medial straight lines commensurable in square only and containing a rational rectangle be added together, the whole is irrational; and let it be called a first bimedial straight line". Reinterpreting this in our arithmetical context, the root is called a bimedial because it is composed of the two medial numbers $\sqrt{\sqrt{85\frac{3}{4}}}$ and $\sqrt{\sqrt{1\frac{3}{4}}}$. They are commensurable in square only because the ratio of their squares, $\sqrt{85\frac{3}{4}}/\sqrt{1\frac{3}{4}}$, is 7, but their ratio $\sqrt{\sqrt{85\frac{3}{4}}}/\sqrt{\sqrt{1\frac{3}{4}}}$ is $\sqrt{7}$, which is irrational. The two medial numbers contain a rational rectangle because their product, $\sqrt{\sqrt{85\frac{3}{4}}}\cdot\sqrt{\sqrt{1\frac{3}{4}}} = 3\frac{1}{2}$, is rational.

A root of the second apotome $\sqrt{112}-7$ is $\sqrt{\sqrt{85\frac{3}{4}}}-\sqrt{\sqrt{1\frac{3}{4}}}$. Euclid (Proposition X.92) and the Arabic translation call this "a first apotome of a medial [straight line]".[56] Because the root was considered to be a diminished $\sqrt{\sqrt{85\frac{3}{4}}}$, it does not consist of "two names". Thus it is medial and not bimedial.

[56] We enclose in square brackets English words implied but not present in the Greek, and were thus not

176.15

Al-Hawārī's example of the third binomial is $\sqrt{32}+\sqrt{14}$ ($a = 4$, $b = 3$, and $c = 2$). Its root is $\sqrt{\sqrt{24\frac{1}{2}}}+\sqrt{\sqrt{\frac{1}{2}}}$. This root is called a "second bimedial" in Euclid (Proposition X.56) and "the binomial of the second bimedial" in the Arabic translation. The second bimedial is defined in *Elements*, Proposition X.38: "If two medial straight lines commensurable in square only and containing a medial rectangle be added together, the whole is irrational; and let it be called a second bimedial straight line".[57] The bimedial numbers $\sqrt{\sqrt{24\frac{1}{2}}}$ and $\sqrt{\sqrt{\frac{1}{2}}}$ are commensurable in square only because the ratio of their squares $\sqrt{24\frac{1}{2}}/\sqrt{\frac{1}{2}}$ is 7, while their ratio is the irrational number $\sqrt{7}$. Their product $\sqrt{\sqrt{24\frac{1}{2}}}\cdot\sqrt{\sqrt{\frac{1}{2}}}$ is $\sqrt{3\frac{1}{2}}$, which is irrational. A "medial area" (i.e., a "medial rectangle") "is the area which is equal to the square on a medial straight line".[58] In arithmetical terms, the $\sqrt{3\frac{1}{2}}$ is a medial area because its square root $\sqrt{\sqrt{3\frac{1}{2}}}$ is medial. In arithmetic there is no use for such a designation, because, unlike geometric magnitudes, numbers do not possess dimension. A "medial area" translated to arithmetic becomes simply a number rational in square only, or, in other terms, a number of the form $\sqrt{p}$ where p is a non-square rational.

A root of the corresponding apotome is $\sqrt{\sqrt{24\frac{1}{2}}}-\sqrt{\sqrt{\frac{1}{2}}}$. Euclid (Proposition X.93) and the Arabic translation call this "a second apotome of a medial [straight line]".

176.20 Next, al-Hawārī finds a root of $7+\sqrt{30}$, which is an example of the fourth binomial ($a = 7$, $b = 19$). Following the same procedure he gets $\sqrt{3\frac{1}{2}+\sqrt{4\frac{3}{4}}}+\sqrt{3\frac{1}{2}-\sqrt{4\frac{3}{4}}}$. Again, following Euclid (Propositions X.57, X.94) the Arabic translation calls this "the major" and the corresponding apotome "the minor". A major is defined in *Elements* Proposition X.39: "If two straight lines incommensurable in square which make the sum of the squares on them rational, but the rectangle contained by them medial, be added together, the whole straight line is irrational: and let it be called *major*".[59] In al-Hawārī's example, $\sqrt{3\frac{1}{2}+\sqrt{4\frac{3}{4}}}$ is incommensurable in square with $\sqrt{3\frac{1}{2}-\sqrt{4\frac{3}{4}}}$ because the ratio of their squares, $(3\frac{1}{2}+\sqrt{4\frac{3}{4}})/(3\frac{1}{2}-\sqrt{4\frac{3}{4}})$, is $2\frac{4}{15}+\sqrt{4\frac{31}{225}}$, which is irrational. The sum of their squares is 7, which is rational. The rectangle contained by them, or their product, is $\sqrt{7\frac{1}{2}}$, which is a medial area when interpreted in terms of geometry. The minor is defined similarly for the apotome in Proposition X.76.

translated into Arabic.

[57] (Euclid 1956, vol. 3, 85).

[58] This is Heath's explanation in (Euclid 1956, vol. 3, 55).

[59] (Euclid 1956, vol. 3, 87).

177.5 The example for the fifth binomial is to find a root of $3+\sqrt{20}$ ($a = 3$, $b = 11$). Its root by the same procedure is $\sqrt{\sqrt{5}+\sqrt{2\frac{3}{4}}}+\sqrt{\sqrt{5}-\sqrt{2\frac{3}{4}}}$. Euclid (Proposition X.58) called this "[the line whose] power is a rational and a medial [area]".[60] Here the rational area is the 3, and the medial area is the $\sqrt{20}$. The Arabic translation is quite literal: "[the number whose] power is a rational and a medial", with the result that term "medial" would be misinterpreted in the arithmetical setting. The number $\sqrt{20}$ is rational in square, not medial like its square root.

Euclid (Proposition X.95) calls the corresponding apotome $\sqrt{20}-3$ "[a straight line] which produces with a rational [area] a medial whole".[61] In our numbers, the root $\sqrt{\sqrt{5}+\sqrt{2\frac{3}{4}}}-\sqrt{\sqrt{5}-\sqrt{2\frac{3}{4}}}$ produces by squaring it the quantity $\sqrt{20}-3$ which, if one adds back the rational area 3, gives the medial area $\sqrt{20}$. The Arabic translation describes it as "the joining with a rational to become a whole medial".

The Greek and Arabic words behind "power" in the descriptions "[the line/number whose] power is a rational and a medial" warrant some explanation. The Greek term is *dynaménē*, deriving from *dynamis*, a word meaning "power" or "value", and in a mathematical setting, "square". In geometry, a *dynamis* is a characteristic of a line. One did not make reference to a *dynamis* directly by naming its opposite vertexes, but rather to a side in respect of *dynamis*. As Jens Høyrup proposed, a *dynamis* is "a square identified with its side" or "a line seen under the aspect of a square" (Høyrup 1990, 210). The word often occurs in the dative form, *dynámei* ("in power", or "in square"), as we described above at 163.4 for the phrase "rational in square". Thus, our added words "[the line/number whose]" in the description of the fifth binomial are implied in Euclid's description.

The forms of the word *dynamis* in Euclid were translated into Arabic by corresponding forms of *quwwa*, a word which also means "power". This word, too, is a characteristic of a side or a number. Gustav Junge and William Thomson, in an appendix to their edition and translation of the medieval Arabic translation of Pappus of Alexandria's *Commentary on Book X of Euclid's* Elements, wrote this of *quwwa*:

> *The Dictionary of Technical Terms* (Calcutta, A. Sprenger, Vol. II, p. 1230, top.) defines it as "*Murabbaʿu-l-Khaṭṭi*", i.e., "the square of the line", "the square which can be constructed upon the line", and goes on to say that the mathematicians treat the square of a line as a *power* of the line, as if it were potential in that line as a special attribute. Al-Ṭūsī (Book X, Introd., p. 225, l. 9.) says: —"The line is a length actually (reading "*bi-l-fiʿli*" for "*bi-l-ʿalqi*")[62] and a square (*murabbaʿun*) potentially (*bi-l-quwwati*) i.e., it is possible for a square to be described upon it. (Pappus 1930, 181)

So here, too, an Arabic reader familiar with Euclid would have understood the description as meaning "[the number whose] power is a rational and a medial". But for a young arithmetic student studying al-Hawārī's book, we guess that the meaning of this description,

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[60] Our translation. Heath translates it as "the side of a rational plus a medial area" (Euclid 1956, vol. 3, 129).
[61] (Euclid 1956, vol. 3, 206).
[62] We have corrected misplaced quotation marks here.

and of the others, too, would have been obscure, and would have been read as being mere labels.

As a last remark on this passage, al-Hawārī writes the fraction "three fourths" in his calculations of the fourth and sixth binomials and apotomes, but here he writes it in the colloquial way reminiscent of the unit fractions from finger-reckoning as "a half and a fourth".

177.11 The example for the sixth binomial is $\sqrt{10}+\sqrt{11}$ ($a = 1$, $b = 10$), and its root is $\sqrt{\frac{1}{2}+\sqrt{2\frac{3}{4}}}+\sqrt{\sqrt{2\frac{3}{4}}-\frac{1}{2}}$. It is called by Euclid (Proposition X.59) "[the line whose] power [consists of] two medial [areas]",[63] where in this case the medial areas are the $\sqrt{11}$ and the $\sqrt{10}$. The Arabic translation is again literal: "[the number whose] power is a bimedial". And as before, the word "bimedial" takes a different meaning in an arithmetical context.

Euclid (Proposition X.96) calls the root of the corresponding apotome, here $\sqrt{\frac{1}{2}+\sqrt{2\frac{3}{4}}}-\sqrt{\sqrt{2\frac{3}{4}}-\frac{1}{2}}$, "[a straight line] which produces with a medial [area] a medial whole".[64] The root produces by squaring it the apotome $\sqrt{11}-\sqrt{10}$. This is a deficient, or incomplete, $\sqrt{11}$, so to make it whole we add to it the medial area $\sqrt{10}$ that it lacks to produce the whole medial area $\sqrt{11}$. The Arabic translation has "the joining with a medial to become a whole medial".

179.1 Section I.3.2. Adding and subtracting roots.

Before plunging into the rules and examples for operating on roots, we must clarify some distinctions made in medieval arithmetic that are lost in modern notation. Consider the following:

A. Duplicate a root of five three times. (explained at 186.1)

B. Three roots of five.

C. Multiply three by a root of five. (explained at 183.1)

D. A root of forty-five.

While we might translate each of these into $3\sqrt{5}$, all four were conceived of differently in their medieval forms. Both (A) and (C) are operations to be performed, while (B) and (D) are quantities. To "duplicate a root of five three times" in (A) means to collect together three copies of $\sqrt{5}$. This wording is well-suited for integer multiples, but it cannot work for irrationals. One cannot duplicate a number $\sqrt{23}$ times, for instance. For fractional multiples, al-Hawārī's wording is different. For them, he "partitions" the root (at 186.8).

[63] Our translation. Heath translates it as "the side of the sum of two medial areas" (Euclid 1956, vol. 3, 130).
[64] (Euclid 1956, vol. 3, 209).

The immediate and intermediate result of (A), duplicating a root of five three times, is (B), three roots of five. This quantity was not regarded as a single number like our $3\sqrt{5}$. It is a collection of three numbers, all of them $\sqrt{5}$. Just as the binomial "eight and a root of fifty-five" is the pair of numbers 8 and $\sqrt{55}$ gathered together, "three roots of five" is the gathering of $\sqrt{5}$, $\sqrt{5}$, and $\sqrt{5}$. Likewise, below at 179.20 we will see that "half a root of twenty" was regarded as being "less than one root", while "two roots of five" are "more than one root". The "half" and the "two" tell us *how many* roots there are. They are not the coefficients or scalar multiples that we work with in $\frac{1}{2}\sqrt{20}$ and $2\sqrt{5}$.

Multiples of a root and fractions of a root are routinely converted to whole, single roots before operating on them. To convert "three roots of five" to a single number the 3 is multiplied by the $\sqrt{5}$. To do this the 3 is squared, and the result is multiplied by 5 to get $\sqrt{45}$. In other words, one performs (C), and the result is (D). The multiplication in (C) is a different operation from (A). The most evident difference is that (C) allows any number as a multiplier. One can multiply $\sqrt{7}$ by $\sqrt{5}$, for instance, to get "a root of thirty-five". The result of the multiplication must be a single root. It cannot be anything like our $\sqrt{7}\sqrt{5}$ since this symbolic amount, like our $3\sqrt{5}$, has no equivalent in medieval arithmetic.

179.2 We noted above at 176.10 that $\sqrt{112} + \sqrt{63} = \sqrt{343}$. This is because, as we would work it out, $\sqrt{112}+\sqrt{63} = \sqrt{16\cdot 7}+\sqrt{9\cdot 7} = 4\sqrt{7}+3\sqrt{7} = 7\sqrt{7} = \sqrt{343}$. Ibn al-Bannāʾ gives the rule that two roots can be added or subtracted only if their product is a perfect square. In this example $112 \cdot 63 = 7{,}056$, which is a square of 84. On the other hand, $\sqrt{5}$ cannot be added to $\sqrt{12}$ to form a single root because $5 \cdot 12 = 60$ is not a square number.

179.7 Once it is known that two roots can be added, Ibn al-Bannāʾ gives a rule for finding their sum. Al-Hawārī first applies it to the example of adding $\sqrt{3}$ to $\sqrt{27}$. Here $3 \cdot 27 = 81$ is a square, so the roots can be added. Two roots of 81 are 18, and this is added to $3 + 27$ to get 48. Then $\sqrt{3} + \sqrt{27} = \sqrt{48}$. In modern notation, $\sqrt{a} + \sqrt{b} = \sqrt{a + b + 2\sqrt{ab}}$.

This rule is found in many arithmetic and algebra books, extending back to the *Book on Algebra* of Abū Kāmil (late ninth century). It is a practical rule for converting two roots into a single root. It is not covered in Euclid's *Elements*, and it is not connected with any theory of quadratic irrationals.

179.11 Al-Hawārī gives a second method of finding the sum. Using the same example he calculates $\frac{\sqrt{27}}{\sqrt{3}} = 3$ (division of roots is explained below, at 187.2), and he adds 1 to get 4. This 4 is then multiplied by the $\sqrt{3}$ to get $\sqrt{48}$. The same procedure works if we reverse the roles of 3 and 27. In modern notation, $\sqrt{a} + \sqrt{b} = \left(\sqrt{\frac{a}{b}} + 1\right) \cdot \sqrt{b}$.

179.16 Another example: add $\sqrt{2}$ to $\sqrt{8}$ to get $\sqrt{18}$.

[65] (Saidan 1986, 530.24).

179.20 Al-Hawārī borrows his problem “add half a root of twenty to two roots of five” from Ibn al-Bannāʾ’s algebra book, where it is solved the same way.[65] In his explanation al-Hawārī notes that “half a root of twenty is less than one root”, so he transforms it to one root, which is $\sqrt{5}$. (The verb behind “transform” here, and elsewhere in our translation, is *radda*, which ordinarily means “to return”, or “to send back”.) At this point, al-Hawārī could add this to the two roots of five to get three roots of five, which would lead to $\sqrt{45}$. But his goal is to illustrate the technique of forming single roots from fractions and multiples of roots, so he continues: “And two roots of five are more than one root, so we transform them to one root”, which is $\sqrt{20}$. Then the problem is the same as adding $\sqrt{5}$ to $\sqrt{20}$. This is worked out by the rule at 179.7 to get $\sqrt{45}$.

180.6 Al-Hawārī, or more likely Ibn al-Bannāʾ, speaks here about adding roots of different ranks, like adding a root to a root of a root (i.e., to a fourth root). The square root must be made into a root of a root before adding. But there is no nice rule to tell if two fourth roots can be added to get a single fourth root. Al-Hawārī seems to think that the rule for square roots works here too, for he writes below, in the passage at 180.15, that “their surface is likewise not a square, so we add them with the coordinating conjunction”, where this conjunction is the word “and”.

180.10 If the roots cannot be added, then one puts the word “and” between them. The example is to add $\sqrt{3}$ to $\sqrt{15}$. Since $3 \cdot 15 = 45$ is not a perfect square, we cannot write them as a single root. Their sum is thus expressed as the binomial “a root of three and a root of fifteen”. The “and” (*wa*) does not mean “plus”. See the discussion at 219.1 below. Also, “their surface” is their product. See our comments at 66.17 above.

180.15 Here al-Hawārī asks for the sum of $\frac{1}{2}$ of $\sqrt{\sqrt{80}}$ and $\frac{1 \bullet 1}{4 \bullet 3}$ of $\sqrt{684}$, where the $\frac{1 \bullet 1}{4 \bullet 3}$ is “a third of a fourth”. First, the $\frac{1}{2}$ of $\sqrt{\sqrt{80}}$ is “a root of a root of five” ($\sqrt{\sqrt{5}}$), and the $\frac{1 \bullet 1}{4 \bullet 3}$ of $\sqrt{684}$ is “a root of four and three fourths” ($\sqrt{\frac{3}{4}4}$). To add them we make them the same rank, so we write $\sqrt{\frac{3}{4}4}$ as $\sqrt{\sqrt{\frac{1\ 4}{2\ 8}22}}$ by squaring the $\frac{3}{4}4$. As we just mentioned, the rule al-Hawārī seems to apply to determine if the two can be added does not work. The sum of the roots is expressed as “a root of a root of five and a root of a root of twenty-two and four eighths and half an eighth” ($\sqrt{\sqrt{5}}\ \sqrt{\sqrt{\frac{1\ 4}{2\ 8}22}}$).

181.4 The first rule for subtracting roots is similar to the first rule for adding roots. Al-Hawārī subtracts $\sqrt{8}$ from $\sqrt{32}$ by first multiplying $8 \cdot 32 = 256$, taking its root to get 16, and doubling it to get 32. This is subtracted from the sum of the two numbers $8 + 32 = 40$, leaving 8. The difference of the roots is then $\sqrt{8}$. In modern notation, $\sqrt{a} - \sqrt{b} = \sqrt{a + b - 2\sqrt{ab}}$.

181.10 The second rule for subtracting roots is like the second rule for adding them. Al-Hawārī’s example is to subtract $\sqrt{12}$ from $\sqrt{27}$. He divides $\sqrt{27}$ by $\sqrt{12}$ to get $\frac{1}{2}1$

(division of roots is explained below, at 187.2). Subtracting 1 leaves $\frac{1}{2}$. Multiplying this by $\sqrt{12}$, the divisor, gives $\sqrt{3}$, which is the answer. In modern notation, $\sqrt{a} - \sqrt{b} = \left(\sqrt{\frac{a}{b}} - 1\right) \cdot \sqrt{b}$.

182.1 Ibn al-Bannāʾ again mentions that if the roots are less than one root or more than one root, they must be returned to one root, like in the addition problem above at 179.20. And if their ranks are different they must be made the same, as above at 180.15.

182.4 Here al-Hawārī gives examples in which the roots cannot be subtracted. For example, subtracting $\sqrt{8}$ from $\sqrt{10}$ can only be expressed as "a root of ten less a root of eight". The "less" (*illā*) does not mean "minus". See below at 219.4.

183.1 **Section I.3.2. Multiplying roots.**

183.4 Although we have not found any Arabic book that characterizes the multiplication of irrational roots in a way like Ibn al-Bannāʾ did for the multiplication of whole numbers (95.2) and fractions (149.2), the basic rule was well known. In the latter ninth century Abū Kāmil stated the rule and gave two proofs of it, while other authors, like al-Baghdādī, merely state the rule and give examples.[66] Al-Hawārī's first example is to multiply $\sqrt{8}$ by $\sqrt{9}$. The answer is a root of the product of the numbers, $\sqrt{72}$.

183.7 Next, al-Hawārī multiplies $\sqrt{\sqrt{5}}$ by $\sqrt{\sqrt{7}}$. Since the roots are of the same rank (both are roots of roots), again he multiplies the numbers and applies the roots. The answer is $\sqrt{\sqrt{35}}$. The next example, at 183.11, is $\sqrt{\sqrt{\sqrt{3}}} \cdot \sqrt{\sqrt{\sqrt{8}}} = \sqrt{\sqrt{\sqrt{24}}}$.

183.15 Here al-Hawārī multiplies 3 by the apotome $\sqrt{7} \,\ell\, 2$. He distributes just the way we would, first multiplying the 3 by $\sqrt{7}$ to get $\sqrt{63}$, then the 3 by 2 to get 6. The answer is $\sqrt{63} \,\ell\, 6$.

183.20 Ibn al-Bannāʾ gives the rule for multiplying a rational number by a root that al-Hawārī has already applied above in the calculations at 179.20, 180.15, 181.12, and 183.15. The first example is to multiply 3 by $\sqrt{7}$, which he had just done as part of the previous problem. Al-Hawārī calculates $3^2 = 9$, and $9 \cdot 7 = 63$, so the answer is $\sqrt{63}$. He then continues through 185.12 with examples of multiplying numbers by multiples and fractions of roots. After that, starting at 186.1, he will duplicate and partition roots.

184.4 Next, to find $2 \cdot \sqrt{\sqrt{3}}$, he finds $2^2 = 4$, $4^2 = 16$, and $16 \cdot 3 = 48$, so the answer is $\sqrt{\sqrt{48}}$.

66 For Abū Kāmil, see (Oaks 2011b); for al-Baghdādī, see (al-Baghdādī 1985, 200.5).

184.11 To multiply 2 by two roots of 7 is like multiplying the 2 by a pair of $\sqrt{7}$s. The two $\sqrt{7}$s must be turned into a single root first, and together they are $\sqrt{28}$. Now the 2 can be squared and multiplied by 28 to get the answer, which is $\sqrt{112}$.

184.18 Similarly, to multiply 5 by three roots of a root of 2 is like multiplying 5 by three $\sqrt{\sqrt{2}}$s. The three roots together are $\sqrt{\sqrt{162}}$, and multiplying the 162 by a square of a square of 5, which is 625, gives the answer $\sqrt{\sqrt{101{,}250}}$.

185.6 Al-Hawārī multiplies $\frac{2}{3}$ by half of $\sqrt{20}$. The latter is the same as $\sqrt{5}$, so the $\frac{2}{3}$ is squared to give $\frac{4}{9}$, and this is multiplied by the 5, resulting in $\sqrt{\frac{2}{9}2}$.

185.12 To multiply $\sqrt{5}$ by half of $\sqrt{\sqrt{40}}$ one first turns the half into a whole by squaring the $\frac{1}{2}$ to get $\frac{1}{4}$, and then squaring this to get $\frac{1}{16}$, and then multiplying it by the 40 to get $\sqrt{\sqrt{\frac{1}{2}2}}$. Also, the $\sqrt{5}$ must be converted into a root of a root, and this is $\sqrt{\sqrt{25}}$. Multiplying $\frac{1}{2}2$ by 25 gives $\frac{1}{2}62$, so the answer is $\sqrt{\sqrt{\frac{1}{2}62}}$.

186.1 To duplicate a root means to collect together copies of it. In the second of two examples, to duplicate $\sqrt{7}$ five times means to gather five $\sqrt{7}$s into a single root. This is done by multiplying 5 by $\sqrt{7}$. The answer is $\sqrt{175}$.

186.8 The same works for partitioning a root, which means to take a fraction of a root. Half a root of ten is found by multiplying $\sqrt{10}$ by $\frac{1}{2}$. Likewise, a third of four eighths of $\sqrt{\sqrt{60}}$ is found by multiplying the fraction by the root.

187.1 **Section I.3.2. Dividing and denominating roots.**

187.2 Dividing or denominating a root by a root follows the same general rules as multiplying roots. Al-Hawārī begins by dividing $\sqrt{20}$ by $\sqrt{3}$. First he finds $20 \div 3 = 6\frac{2}{3}$, then he takes its root to get $\sqrt{\frac{2}{3}6}$. The next example works the same way. Al-Hawārī wrote at 187.7 and at 187.14 “divide” where he should have written “denominate”.

187.10 To divide $\sqrt{\sqrt{6}}$ by $\sqrt{\sqrt{2}}$, divide the 6 by 2 to get 3, then apply the roots to get $\sqrt{\sqrt{3}}$. The next example, dividing $\sqrt{\sqrt{18}}$ by $\sqrt{\sqrt{32}}$, works the same way.

188.1 One should only perform a division with single roots, so multiples and fractions of a root must first be converted to single roots. Also, the ranks of the roots of the dividend and divisor must be the same. We mentioned above at 134.2 that there are no medieval counterparts to fractions with irrational numbers, like our $\frac{1}{\sqrt{2}}$ and $\frac{\sqrt{2}}{2}$, but one can perform the divisions. To divide 1 by $\sqrt{2}$ the ranks should be made equal, so 1 is converted to $\sqrt{1}$. Then 1 is divided by 2 to get $\frac{1}{2}$, and finally a root is applied to get $\sqrt{\frac{1}{2}}$. Here, $\frac{1}{2}$ is valid as a fraction, and a root can be taken of any number. To divide $\sqrt{2}$ by 2 one follows the same rule by first rewriting the 2 as $\sqrt{4}$, to again get $\sqrt{\frac{1}{2}}$.

188.6 To divide $\sqrt{\sqrt{14}}$ by $\sqrt{2}$, the $\sqrt{2}$ has to be written as $\sqrt{\sqrt{4}}$. Then $14 \div 4 = 3\frac{1}{2}$, so the answer is $\sqrt{\sqrt{\frac{1}{2}3}}$.

188.11 Dividing two $\sqrt{15}$s by 2 should be trivial – one simply gets a single $\sqrt{15}$. But al-Hawārī illustrates the general rule by converting the two roots into a single root and then dividing the result by 2. The two roots together are $\sqrt{60}$, and the 2 as a root is $\sqrt{4}$. Then $60 \div 4 = 15$, so the answer is $\sqrt{15}$. The next example, dividing half of $\sqrt{24}$ by $\sqrt{2}$, works similarly.

188.18 To divide a quantity by a binomial or an apotome one multiplies both amounts by the conjugate to eliminate the square root. Al-Hawārī's first example is to divide 12 by the binomial $5\ \sqrt{3}$. He multiplies the 12 by the apotome $5\ \ell\ \sqrt{3}$ to get $60\ \ell\ \sqrt{432}$. Then he multiplies $5\ \sqrt{3}$ by $5\ \ell\ \sqrt{3}$ to get 22. Dividing the $60\ \ell\ \sqrt{432}$ by 22 results in $\frac{8}{11}2\ \ell\ \sqrt{\frac{9}{11}\frac{9}{11}}$. We would write this as $2\frac{8}{11} - \sqrt{\frac{108}{121}}$, or $2\frac{8}{11} - \frac{6}{11}\sqrt{3}$, or even $\frac{30-6\sqrt{3}}{11}$. Restating the latter two rhetorically would result in improperly expressed amounts. These only became possible with Descartes.[67]

Dividing 10 by $3\ \ell\ \sqrt{7}$ is done the same way. The numerator and denominator are multiplied by the binomial $3\ \sqrt{7}$, which leads to the answer $15\ \sqrt{175}$.

191.1 **Part II. On the rules by which one arrives at knowledge of the required unknown from the posited known.**

Ibn al-Bannāʾ covers three problem-solving methods in Part II: the rule of three, double false position, and algebra. Ibn al-Bannāʾ and other Arabic authors call the first of these "the four proportional numbers". It is often applied directly to solve a problem, and it is also the core of the method we call single false position. In that method, a convenient value is chosen for the answer (the false posited value, or position), and the true value is found via proportion by a calculation on the posited value and its error. Ibn al-Bannāʾ does not cover single false position in his *Condensed Book*.

Double false position was called the method of "the two errors" (*al-khaṭaʾān*). Like other arithmeticians in the Maghreb, Ibn al-Bannāʾ calls it the method of "scales" (*kiffāt*) because of the shape of the diagram drawn to work out the problems. In this method, two

[67] See (Oaks 2017, 153).

false values are posited for the answer, and the true value is found by a calculation involving the values and their respective errors. This method relies on proportion, so it is covered along with the rule of three in Chapter II.1.

Algebra is covered in Chapter II.2. The Arabic name for algebra is *al-jabr wa-l-muqābala*, meaning “restoration and confrontation”, or sometimes *al-jabr* for short. This is the origin of our word “algebra”. In an algebraic solution, an unknown is named in terms of designated names of the powers of the unknown, and an equation is set up in terms of these names and solved. In the other methods the calculations are worked out on known numbers, and without equations.

Ibn al-Bannāʾ and al-Hawārī explain the rules for each of the three methods, but they give sample problems only for double false position. We give examples translated from other books of the four proportional numbers, single false position, and algebra in Appendix B.

193.1 Chapter II.1. Working out [problems] with proportion.

The Arabic word for mathematical “ratio” is *nisba*. This word is also used in non-mathematical settings to mean any kind of relation between two objects, and even kinship between people. *Nisba* is also the Arabic word for “proportion”. The two meanings can be distinguished by the context.

195.2 Ibn al-Bannāʾ lists the following kinds of proportion in *Lifting the Veil*:[68]

1. Geometric. This is the usual proportion involving four numbers a, b, c, and d in which the ratio $a : b$ is equal to the ratio $c : d$, or, in notation, $a : b :: c : d$. Numerically, it is equivalent to $\frac{a}{b} = \frac{c}{d}$.
2. Arithmetic. “The difference between the first and the second is equal to the difference between the third and the fourth”.[69] The four numbers a, b, c, and d are in arithmetic proportion if $b - a = d - c$.
3. Harmonic. Ibn al-Bannāʾ writes “For three numbers, the ratio of the extremes is as the ratio of the two differences between the middle and each one of the extremes. This ratio is composed of the geometric and the arithmetic, since, on the one hand, it comes from the difference between the middle and each one of the extremes, resembling the arithmetic, and, on the other, from proportionality, resembling the geometric”.[70] Three (descending) numbers a, b, and c are in harmonic proportion if $a : c :: a - b : b - c$. Ibn al-Bannāʾ gives his source for the definition as Nicomachus’s *Arithmetical Introduction*.[71]
4. Ex-aequali. Given two sets of numbers $a_1, a_2, \ldots a_n$ and $b_1, b_2, \ldots b_n$, if $a_i : a_{i+1} :: b_i : b_{i+1}$ for each i, then $a_1 : a_n :: b_1 : b_n$. Although this reads like a proposition,

[68] (Ibn al-Bannāʾ 1994, 293.3-16).
[69] (Ibn al-Bannāʾ 1994, 293.5).
[70] (Ibn al-Bannāʾ 1994, 293.7).
[71] “And the third is harmonic proportion, which is described by the author of the *Arithmetic*…” (Ibn al-Bannāʾ 1984, 293.7). He then paraphrases (Nicomachus 1959, 102.1). Harmonic proportion is covered by Nicomachus in Book II Chapter 25, starting at (Nicomachus 1959, 101.20); (Nicomachus 1938, 274).

it is presented as Definition V.17 in Euclid's *Elements*. Nicomachus does not cover it.

5. Harmony. This one comes from Nicomachus, who describes it as "the most perfect proportion, that which is three-dimensional and embraces them all".[72] Both Nicomachus and Ibn al-Bannāʾ remark that this proportion is important in music and astronomy. Briefly, four numbers are "in harmony" if they are of the form a, $\frac{2ab}{a+b}$, $\frac{a+b}{2}$, and b. See Nicomachus's *Introduction*, Chapter II.29 for the different properties it satisfies.

As Nicomachus notes, geometric proportion "is the only one in the strict sense of the word to be called a proportion, because its terms are seen to be in the same ratio".[73]

In his list at 195.2 al-Hawārī omits geometric proportion, and he switches the order of the last two. Apparently he omitted geometric proportion by mistake. Not only does he dispense with all four proportions from his list in this passage, and then follow Ibn al-Bannāʾ by discussing geometric proportion, but the omission causes a misreading. Ibn al-Bannāʾ writes "[the] origin of these last three proportions is the first, which is in the four numbers mentioned in the book".[74] By leaving off geometric proportion al-Hawārī's paraphrase "all three derive from it and it does not derive from them" implies that the last three derive from arithmetic proportion instead of geometric proportion.

195.7 The four proportional numbers.

Ibn al-Bannāʾ focuses on geometric proportion for the rest of this chapter. Four numbers, a, b, c, and d, are proportional if $a : b :: c : d$, or, equivalently, if $ad = bc$.

195.9 Whenever three of the numbers are known, the fourth can be found. Continuing with modern notation, suppose we want to find x in the proportion $x : a :: b : c$ or $a : x :: c : b$.

195.14 In $x : a :: b : c$, the x and b are of the same kind because they have the same positions in their respective ratios. Likewise, a and c are of the same kind. The number b is isolated because it is not of the same kind as either of the other two known numbers. This is multiplied by a, the number in ratio with x. The product is then divided by the third number c to give the value of x.

195.16 Al-Hawārī gives examples of this rule for the proportion $3 : 6 :: 4 : 8$, treating each number in turn as the unknown and finding it from the other three by the rule.

196.16 A proportion can be converted to a number of other proportions. The four types mentioned by Ibn al-Bannāʾ are taken in order from Definitions V.12 through V.15 in Euclid's *Elements*. If $a : b :: c : d$, then the following proportions also hold:

[72] Translated in (Nicomachus 1938, 284). Both in Greek and in Arabic the words for "harmonic" and "harmony" have the same root.
[73] Translated in (Nicomachus 1938, 270).
[74] (Ibn al-Bannāʾ 1994, 293.17). One could argue that he meant "the first two", since he explicitly said

- Switching (*badala*): $a : c :: b : d$.
  In *Elements* Def. V.12 Heath translates this as “alternate ratio”.
- Reversing (*khalafa*): $b : a :: d : c$.
  In *Elements* Def. V.13 Heath translates this as “inverse ratio”.
- Combining (*rakiba*):
  $(a + b) : a :: (c + d) : c$ or $(a + b) : b :: (c + d) : d$.
  In *Elements* Def. V.14 Heath translates this as “composition of a ratio”.
- Separating (*faṣala*):
  $|a - b| : a :: |c - d| : c$ or $|a - b| : b :: |c - d| : d$.
  In *Elements* Def. V.15 Heath translates this as “separation of a ratio”.

One can also combine two or more of these operations. For example, combining after switching gives $(a + c) : a :: (b + d) : b$. Euclid gives one more in Definition V.16: the “conversion” of the proportion gives $a : (a - b) :: c : (c - d)$, presuming $a > b$ and $c > d$. But this can be obtained by reversing after separating, which may be why Ibn al-Bannāʾ did not include it.

Al-Hawārī may have chosen to copy this passage from *Lifting the Veil* because, as he mentions below at 198.4, one can derive the rule for double false position from a known proportion by switching, separating, and then switching again.

197.4 Al-Hawārī gives variations on Ibn al-Bannāʾ’s rule at 195.9 that switch the order of operations. The jurist mentioned at the end of this passage is otherwise unknown.

198.2 **The method of scales (double false position)**

The method of “scales”, or double false position, “comes from the art of geometry” because of its use in geometric proportion. The method is in fact purely numerical. In another part of *Lifting the Veil*, Ibn al-Bannāʾ explains the term “geometric proportion”: “It is said to be geometric proportion because it is more specific to the quantities that are considered in geometry”.[75]

Double false position works for problems in which the operations on the unknown described in the enunciation give a number proportional to the unknown. Sometimes a problem can be adjusted to satisfy this condition (our authors do not do this), and other times, as we shall see starting at 201.14, the method itself can be adjusted.

198.4 Al-Hawārī mentions that the proportion just stated by Ibn al-Bannāʾ is found by switching, separating, and then switching a known proportion. We explain this in terms of the specific numbers in the problem below at 199.1, so read that first and then come back here. The “first scale”, or the first posited value, is 15. The calculated “portion” is the result of performing the operations on the 15, and this is $6\frac{1}{4}$. The “assigned number” is the 10, and the value of the unknown is found at the end to be 24.

that the third, harmonic proportion, “is composed of the geometric and the arithmetic”. In any case, it is geometric proportion that is in “the four [proportional] numbers”.

[75] (Ibn al-Bannāʾ 1994, 214.12).

We begin with the ratio stated by Ibn al-Bannāʾ, "the ratio of the error of each scale to the difference between the scale and the unknown number is as the ratio of the assigned number to the unknown":

$$10 - 6\tfrac{1}{4} : 24 - 15 :: 10 : 24.$$

Switching gives

$$10 - 6\tfrac{1}{4} : 10 :: 24 - 15 : 24.$$

Separating gives

$$6\tfrac{1}{4} : 10 :: 15 : 24.$$

Switching again gives

$$6\tfrac{1}{4} : 15 :: 10 : 24.$$

This is now "the ratio of the portion to its scale is as the ratio of the assigned number to the unknown", which al-Hawārī took from *Lifting the Veil*.

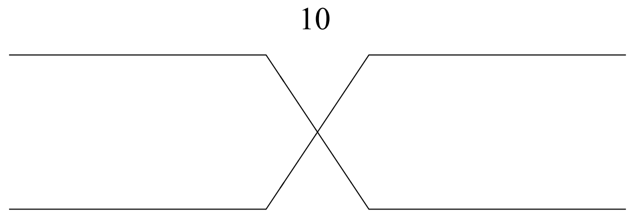
From here, Ibn al-Bannāʾ continues in *Lifting the Veil* to derive the rule for solving problems by double false position, though al-Hawārī does not copy it into his book. Like all his derivations/proofs, Ibn al-Bannāʾ expresses the steps in arithmetical terms, without the use of algebra. In other words, he does not *name* any of the quantities and then manipulate the subsequent algebraic expressions. For example, he says "so the two results from multiplying each error by the excess of the other scale are equal",[76] where by contrast we would name the unknown as, say, x, the two posited values or scales as x_1, x_2, and their respective errors as e_1, e_2, and then write $e_1 \cdot (x_2 - x) = e_2 \cdot (x_1 - x)$. The names of the powers of the unknown in premodern algebra were generally not used outside the context of algebraic problem-solving, and Ibn al-Bannāʾ's derivation is not an exception. We leave it to the reader to take our last equation and solve for x to find a modern rendering of the procedure.

198.7 Double false position is worked out with a diagram, shown below. It has the shape of a balance, or "scales".

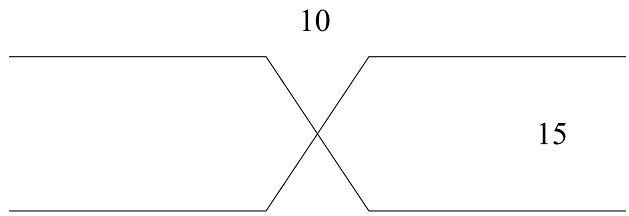
199.1 Al-Hawārī's first problem is: "A quantity: taking away its third and its fourth leaves ten. How much is the quantity?" See our commentary below at 211.8 for the different meanings of the word *māl*, translated here as "quantity", in Arabic arithmetic.

The first step is to draw the scales and put the assigned number above the dome:

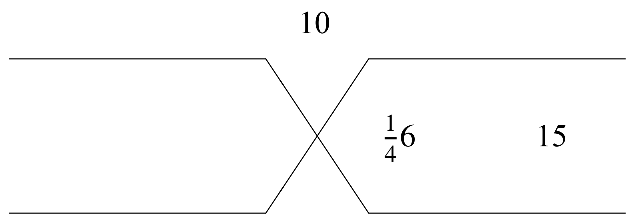
[76] (Ibn al-Bannāʾ 1994, 297.18).



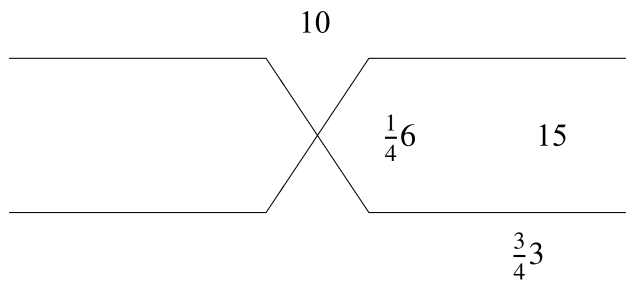
This method calls for two values to be posited. Al-Hawārī makes the first value 15. This is placed in the right scale like this:



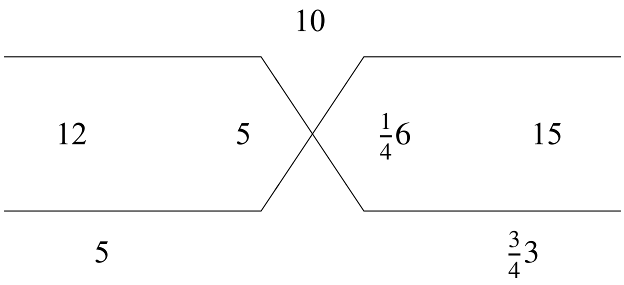
The operations are then performed on the posited value: subtracting 5 (a third of 15) and $3\frac{3}{4}$ (a fourth of 15) from the 15 leaves $6\frac{1}{4}$. This is placed next to the 15, inside the balance:



We confront this $6\frac{1}{4}$ with the desired 10, and we find that it falls short by an error of $3\frac{3}{4}$. Because it falls short, it is written below the scale:



Al-Hawārī then chooses 12 for the second value. He follows the same process, this time putting the numbers in the left scale. The 12 is written in, then the calculations are made: subtracting 4 (a third of 12) and 3 (a fourth of 12) from the 12 leaves 5. Confronting this 5 with the desired 10, we find that it falls short by 5. The second error, 5, is likewise written below the scale:

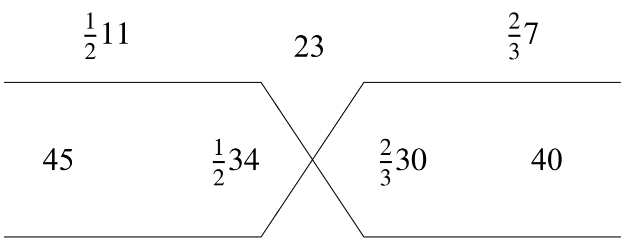


Now we are ready to calculate the value of the unknown. First, each posited value is multiplied by the other error, and their difference is taken. Multiplying the 15 by 5 gives 75, and $3\frac{3}{4}$ by 12 gives 45. Their difference is 30. This is divided by the difference between the two errors, which is $5 - 3\frac{3}{4} = 1\frac{1}{4}$. The result is 24, which is the answer.

200.1 The next example is: "A quantity: we take the sum of its third and its fifth, and we add to it half of the remainder, so it comes to twenty-three". To explain what is meant by "the remainder", we resort to modern algebraic notation. If we name the quantity x, then its third and its fifth are $\frac{1}{3}x + \frac{1}{5}x$. The remainder is what we get after subtracting this from the quantity, so let $r = x - (\frac{1}{3}x + \frac{1}{5}x)$. We could then express the enunciation as the equation $\frac{1}{3}x + \frac{1}{5}x + \frac{1}{2}r = 23$, but of course this is not how our author proceeds with his solution.

Al-Hawārī chooses 40 for the first value. Adding its third and its fifth gives $13\frac{1}{3}+8 = 21\frac{1}{3}$. The remainder from 40 is then $18\frac{2}{3}$. Adding to the $21\frac{1}{3}$ half the remainder, or $9\frac{1}{3}$, gives $30\frac{2}{3}$. This $30\frac{2}{3}$ is not the 23 we want. Since the calculated number this time exceeds the desired amount, we put the error $7\frac{2}{3}$ above the scale.

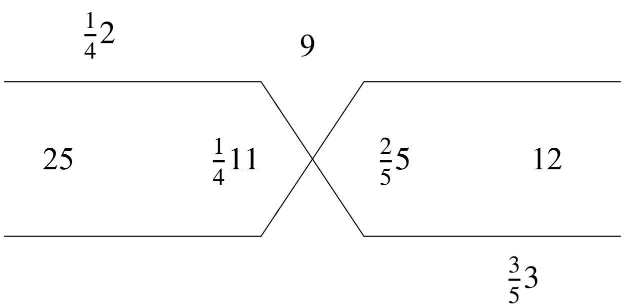
Next, al-Hawārī tries 45. Taking its third and its fifth and adding half the remainder gives $34\frac{1}{2}$, so the error is $11\frac{1}{2}$. Since the $34\frac{1}{2}$ is greater than the desired 23, the error is again put above the scale:



The same procedure is followed as in the last example. He calculates $7\frac{2}{3} \cdot 45 = 345$ and $40 \cdot 11\frac{1}{2} = 460$. Subtracting the smaller from the greater leaves 115. This is divided by the difference between the errors, which is $11\frac{1}{2} - 7\frac{2}{3} = 3\frac{5}{6}$, to get 30, which is the unknown quantity.

201.1 In the third example, al-Hawārī illustrates the case in which one value gives a result that falls short of the desired number and the other gives a result that exceeds it. The problem is to find a quantity such that if "we add its tenth to the difference between its fourth and three of its fifths…it comes to nine". We resort again to modern algebra to clarify what is being asked. If we call the quantity x and perform the operations, we get the equation $(\frac{3}{5}x - \frac{1}{4}x) + \frac{1}{10}x = 9$. Again, there is no algebra in the enunciation, and algebra plays no role in the solution.

The first posited value is 12, and the result of the operations is $5\frac{2}{5}$. This falls short of the desired 9 by $3\frac{3}{5}$. The second posited value is 25. The result of the operations is $11\frac{1}{4}$, which exceeds the desired 9 by $2\frac{1}{4}$. The two errors are placed below and above the scale like this:



Again we perform the two multiplications, but this time we add the results rather than subtract them, since one error is above the scale and the other is below. So $3\frac{3}{5} \cdot 25 = 90$ and $12 \cdot 2\frac{1}{4} = 27$. Their sum is 117, and this is divided by the sum of the errors, which is $2\frac{1}{4} + 3\frac{3}{5} = \frac{1}{4}\frac{4}{5}5$, which for us would be $5\frac{17}{20}$. The result of the division is 20, which is the unknown quantity.

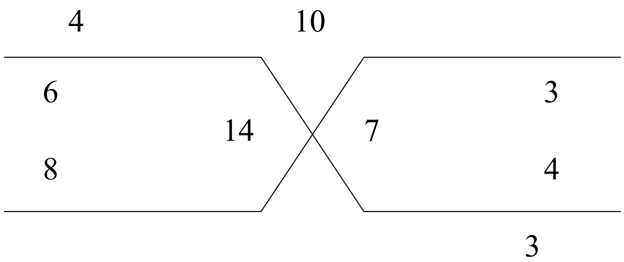
Al-Hawārī has now covered three cases: (a) both calculated portions fall short of the desired number, (b) both exceed it, and (c) one falls short and the other exceeds. We can, if we want, write a single formula to cover all three cases because our arithmetic allows for negative numbers.

201.14 Ibn al-Bannāʾ gives a second method of working with scales. The diagram and values are drawn as described above, but the calculations are now asymmetric with respect to the scales, so that it works even when the same value is posited for both scales (as in the example at 203.1). With only one error, it functions more like a complicated version of single false position. The asymmetry also opens it up to problems in which only one posited value and its error are given, but not the number above the dome! The example for this is given at 203.11.

"This second method is used only when there is a proportional relation", as al-Hawārī writes at 202.3. He means that in cases where two or more values are placed in each scale, they must be chosen proportionally according to the conditions of the problem. This method, in fact, will also work where there is only one value placed in each scale, like the three examples given above.

202.3 The first example is "ten: you divide it into two parts so that a third of one of them is a fourth of the other". The enunciation of this problem copies word-for-word the first problem in Ibn al-Bannāʾ's algebra book.[77] There, of course, it is solved by algebra, with the naming of one of the parts as "a thing", which plays the role of our x. It is easier for us to understand what is being asked in the enunciation by converting it to modern algebra: If we name the parts a and b, then $a+b=10$ and $\frac{1}{3}a=\frac{1}{4}b$. This second condition is equivalent to saying that the ratio of one part to the other is as the ratio of 3 to 4.

No naming occurs in double false position. The first step in al-Hawārī's solution is to put the 10 above the dome, and then to pick two numbers so that a third of one is a fourth of the other. He picks 3 and 4. They are placed in the right scale, with the 3 above the 4. Their sum is 7, and this is placed next to them. Confronting 7 with 10, we see that it falls short by 3, so a 3 is put below the scale. Next, 6 and 8 are posited for the second two numbers. Their sum is 14, which exceeds the 10 by 4. The completed diagram looks like this:

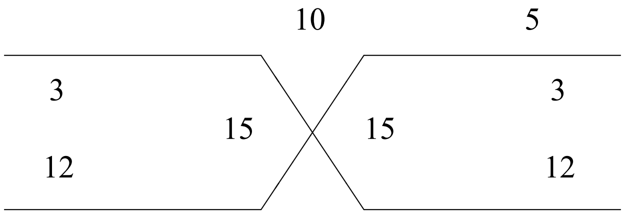


The rule for finding the answer will seem kind of bizarre. Al-Hawārī multiplies the 6 on the left by the error 3 on the right to get 18. Then he multiplies the 14 on the left by the 3 on the right, above the 4, to get 42. Because the first error is below the scale (whether the second error is above or below does not matter), he adds them to get 60. Dividing this by the 14 gives $4\frac{2}{7}$, which is the smaller number, and the greater is $10-4\frac{2}{7}=5\frac{5}{7}$. We leave it as an exercise for the reader to show with modern algebra, or by geometry if you want, that this method works.

203.1 Because of the asymmetry of the calculations, one can work with the same value twice rather than pick a different one. In this next example, al-Hawārī solves "ten: you divide it into two parts. You divide the greater by the smaller, so the result is four". This

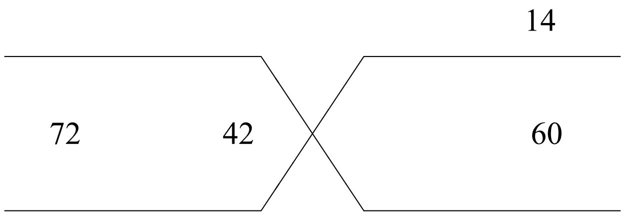
[77] (Saidan 1986, 557.7). Ibn al-Bannāʾ's first of two solutions by algebra is: "You make one of the parts a thing, so the other is ten less a thing. We take a third of a thing, giving a third of a thing, and you confront it with a fourth of ten less a thing, and that is two and a half less a fourth of a thing [this sets up the equation we write as $\frac{1}{3}x=2\frac{1}{2}-\frac{1}{4}x$]. You restore and confront [i.e., simplify the equation], resulting in the third type [of equation, which we would write as $\frac{7}{12}x=2\frac{1}{2}$], which gives the thing is four and two sevenths, which is the part whose third we took, and the other part is the remainder from the ten." In the second solution he switches the roles of "a thing" and "ten less a thing".

time the ratio of the parts is 4 instead of $\frac{3}{4}$. He picks as his first values the numbers 3 and 12. Their sum is 15, which exceeds the 10 by 5. So a 5 is put above the scale. With the same numbers on both sides, the diagram looks like this:



Now the 3 on the left is multiplied by the error 5 on the right to get 15, and the 15 on the left multiplied by the 3 on the right make 45. We take their difference because the error is above the scale. So $45 - 15 = 30$. We divide this by the 15 to get 2, which is the smaller part, and the greater part is 8.

203.11 Ibn al-Bannāʾ writes that this second method will also work when only one of the posited values and its error are given in the problem. He gives the example "a quantity: we subtract its third and its fourth from a third of sixty and its fourth, leaving fourteen". If we call the unknown x, this would yield the equation $(\frac{1}{3}60 + \frac{1}{4}60) - (\frac{1}{3}x + \frac{1}{4}x) = 14$. This derives from the problem "A quantity: you added its third and its fourth, so it gave <some number>". We can think of 14 as the error for the posited value 60. Pick another number for the second value, like 72, add its third and its fourth to get 42, and fill out the diagram:



Since the error 14 exceeds, we calculate $60 \cdot 42 - 72 \cdot 14 = 1512$, and we divide this by the 42 to get 36, which is the answer. The number above the dome would have been 21, but we were not asked to find that.

204.3 Ibn al-Bannāʾ dictated three alternatives to the basic rule of double false position illustrated in the examples above at 199.1, 200.1, and 201.1. These explanations would have been better placed just after those examples. Al-Hawārī does not give examples of these alternatives, so we give here the calculations from his previous examples. The diagrams are the same.

204.6 For the example at 199.1, the difference between (the numbers in) the scales is $15 - 12 = 3$. Multiplying this by one of the errors, 5, gives 15. Dividing this by the difference between the errors, or $1\frac{1}{4}$, gives 12. Adding to this the 12 in the scale associated with the error 5 gives 24, which is the answer.

For the example at 200.1, we calculate $45 - 40 = 5$, then $5 \cdot 11\frac{1}{2} = 57\frac{1}{2}$. This is divided by $11\frac{1}{2} - 7\frac{2}{3} = 3\frac{5}{6}$ to get 15. Subtracting this from the 45 gives 30, which is the answer.

For the example at 201.1, we calculate $25 - 12 = 13$, then $13 \cdot 2\frac{1}{4} = 29\frac{1}{4}$. Dividing this by $2\frac{1}{4} + 3\frac{3}{5} = 5\frac{17}{20}$ gives 5. Subtracting this from the 25 gives 20, which is the answer.

204.12 The calculations for these same three examples by the second alternative are as follows:

For 199.1: $15 - 12 = 3$; $5 + 3\frac{3}{4} = 8\frac{3}{4}$. Multiplying them, $3 \cdot 8\frac{3}{4} = 26\frac{1}{4}$; $26\frac{1}{4} \div (5 - 3\frac{3}{4}) = 26\frac{1}{4} \div 1\frac{1}{4} = 21$. Keep this in mind. One option is: $21 + 3 = 24$; its half is 12, and adding the 12 in the scale gives the answer 24. The other option is: $21 - 3 = 18$; its half is 9, and $9 + 15 = 24$.

For 200.1: $45 - 40 = 5$; $11\frac{1}{2} + 7\frac{2}{3} = 19\frac{1}{6}$. Multiplying them, $5 \cdot 19\frac{1}{6} = 95\frac{5}{6}$; $95\frac{5}{6} \div (11\frac{1}{2} - 7\frac{2}{3}) = 95\frac{5}{6} \div 3\frac{5}{6} = 25$. Keep this in mind. One option is: $25 + 5 = 30$, its half is 15, and $45 - 15 = 30$ is the answer. The other option is: $25 - 5 = 20$, its half is 10, and $40 - 10 = 30$.

For 201.1: $25 - 12 = 13$; $3\frac{3}{5} - 2\frac{1}{4} = 1\frac{7}{20}$. Multiplying them, $13 \cdot 1\frac{7}{20} = 17\frac{11}{20}$, $17\frac{11}{20} \div (3\frac{3}{5} + 2\frac{1}{4}) = 17\frac{11}{20} \div 5\frac{17}{20} = 3$. Keep this in mind. One option is: $3 + 13 = 16$, its half is 8, and $8 + 12 = 20$, which is the answer. The other option is: $13 - 3 = 10$, its half is 5, and $25 - 5 = 20$.

205.1 The calculations for these examples by the third alternative are as follows:

For 199.1: $15 - 12 = 3$; $3 \cdot 10 = 30$; $30 \div (5 - 3\frac{3}{4}) = 24$, which is the answer.

For 200.1: $45 - 40 = 5$; $5 \cdot 23 = 115$; $115 \div (11\frac{1}{2} - 7\frac{2}{3}) = 30$, which is the answer.

For 201.1: $25 - 12 = 13$; $13 \cdot 9 = 117$; $117 \div (3\frac{3}{5} + 2\frac{1}{4}) = 20$, which is the answer.

205.5 If an indeterminate problem does not restrict any number to a particular class such as squares or cubes, then to be non-trivial it must give conditions on at least three unknown numbers. In a solution by scales one of these numbers can reside above the dome, so there will be at least two values in each scale. Unlike problems with a proportional relation, two of these values can be chosen independent of each other. Thus the first method must be used.

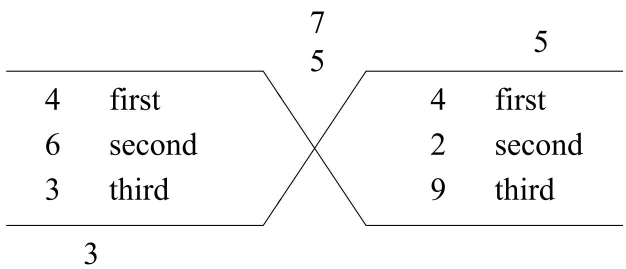
This first indeterminate problem is of a common type in Arabic and later Latin and Hebrew arithmetic, and it is found in abstract form in Diophantus.[78] In this type two or more men

78 See (Katz 2017).

want to buy a horse, with conditions on how much money each one has. Problems of this type are usually solved by algebra, like problem [5] in Appendix B.

Because the problem is indeterminate, any number can be chosen for the first man, i.e., the amount of money the first man has. Ibn al-Bannāʾ chooses 4 dirhams. He then first posits 2 dirhams for the second man. From these two numbers the price of the horse and the money of the third are found, 5 and 9 respectively. The 4, 2, and 9 are placed in the first scale, and the 5 is put above the dome. In the second scale the first man is given 4 again, and the value for the second man is posited as 6 dirhams. Then the price of the horse is calculated from the last condition as 7, so the third man must have 3 dirhams. The 4, 6, and 3 are placed in the second scale, and the 7 is placed above the dome.

The price of the horse is then calculated a second time from the last condition. For the right scale, the horse costs $9 + \frac{1}{4} \cdot 4 = 10$, which exceeds the calculated 5 by 5, so a 5 is put above the scale. In the left scale, the same condition gives the price of the horse as 4 dirhams, which is 3 less than the calculated 7, so the error of 3 is placed below the scale. Here is the diagram:



Now the money of each man can be calculated using the rule from the first method. The first man has 4 dirhams, as we decided in advance. The second man's money is found by the calculation $5 \cdot 6 + 3 \cdot 2 = 36$, divided by $5 + 3 = 8$, giving $4\frac{1}{2}$ dirhams. The third man's money is found the same way: $5 \cdot 3 + 3 \cdot 9 = 42$, and dividing by the same 8 gives $5\frac{1}{3}$ dirhams. By the first condition, the price of the horse is $4 + \frac{1}{2}(4\frac{1}{2}) = 6\frac{1}{4}$ dirhams.

Ibn al-Bannāʾ does not prove in *Lifting the Veil* that this variation with two numbers above the dome works. He could have derived it through a manipulation of proportions as he did for the standard version of the method, or he could have followed Qusṭā ibn Lūqā in proving it by geometry. We give an explanation in terms of modern elementary algebra. Regardless of how many men want to buy the horse, once the money of the first is chosen and the money of the second is posited, all other values are calculated. If we call the money of the second y, then the problem is equivalent to finding y where the price of the horse can be written two ways, each of them of the form $ay + b$. In our problem, these are $\frac{1}{2}y + 4$ and $\frac{3}{2}y - \frac{1}{2}$. When Ibn al-Bannāʾ calculates the values for the horse to put above the dome he gets the same values we get by plugging his numbers into $\frac{1}{2}y + 4$. When he recalculates the values, he gets the same results we get by plugging them into $\frac{3}{2}y - \frac{1}{2}$.

In the kinds of problems that are usually solved by double false position, the (posited) unknown is proportional to the desired value (the number placed above the dome), or, in modern algebra, $\alpha y = \beta$ for constants α and β. In the present problem, the condition is not stated as one of proportionality. It effectively equates the results of two calculations, which we can write as $ay+b = a'y+b'$. But this equation can be rearranged as $(a-a')y = b'-b$, and applying double false position to this proportional relation, with the single number $b'-b$ above the dome, gives the same errors for posited values of y, and thus the same answer, as Ibn al-Bannāʾ's way.

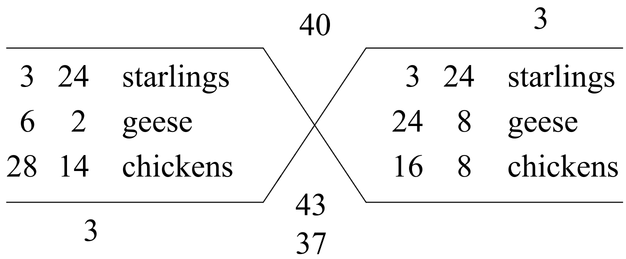
206.5 Since all solutions are proportional to this one, al-Hawārī notes that if we do not want fractions, we can multiply all the numbers by 4. This way, the first man has 16 dirhams, the second has 18 dirhams, the third has 21 dirhams, and the horse costs 25 dirhams.

207.1 We can also assign an arbitrary value for the price of the horse, and pick a value for what the first man has, and work through the calculations. Then we pick a different value for the horse, but the same value for the first man, and work through the calculations again. The same rule then gives what the second and third man have.

207.6 This next problem asks for the number of each kind of bird given the total cost and the price of each kind. Problems of this type were posed and solved in China and India before the rise of Islam, and they later appear in medieval Europe, including Fibonacci's *Liber Abaci*.[79] In our problem, 40 birds are purchased for 40 dirhams. There are three kinds of bird: starlings, which cost 1/8 of a dirham each; chickens, which cost 2 dirhams each; and geese, which cost 3 dirhams each. How many of each kind of bird was purchased?

207.9 Ibn al-Bannāʾ states two conditions for this problem. The first is that the number of each kind of bird must be a whole number. Then, since the total price is whole, and starlings are the only bird that cost a fraction of a dirham, their number must be a multiple of 8. The second condition is that the product of the price of the cheapest bird by the number of birds should be less than the total cost, and the product of the price of the most expensive bird by the number of birds should be greater than the total cost.

Working with these conditions, Ibn al-Bannāʾ finds that the number of starlings cannot be 8 or 16, but that 24 is a possibility. Then, two values are posited for the number of chickens, 8 and 14. The number of geese must be the difference from 40, or 8 and 2, respectively. The total cost is calculated, 37 and 43 dirhams, respectively, making the errors 3 and 3. The diagram is set up like this:



[79] (Caianiello 2018).

Next to the number of birds, al-Hawārī has put their cost. Only the Oxford manuscript shows the 43 and 37, and they were put below the scales because there was no room inside.

The number of geese is calculated as $\frac{2\cdot3+8\cdot3}{3+3} = 5$, and the number of chickens is $\frac{14\cdot3+8\cdot3}{3+3} = 11$. One can use modern algebra to show why this works, too. It is simpler than the case of problems with two numbers above the dome.

For an explanation as to why our authors write "the starlings", "the chickens", and "the geese" for what we would write as "the number of starlings", "the number of chickens", and "the number of geese", see the last three paragraphs of our commentary at 95.3 above.

208.12 Last, the possibility that the number of starlings is 32 is considered. This is shown to be impossible.

By "problems involving multiplication" Ibn al-Bannāʾ probably means problems in which unknowns are multiplied by unknowns. In such problems, the calculated portions will not have a proportional relation to the posited values, so the method of scales will not work. Problems [3] and [4] in Appendix B, solved by algebra, are of this type. Problems involving divisions of unknowns cannot be solved by scales, either. Problem [6] in Appendix B is of this type, and it too is solved by algebra.

209.1 **Chapter II.2. Algebra.**

Arabic algebra is the third and last method of numerical problem solving presented by Ibn al-Bannāʾ. As with single false position, al-Hawārī does not give any example of a problem worked out by the method. Instead, he only covers the kinds of calculations that crop up in the course of working through algebraic solutions. These include the solutions to the six simplified equations, operations on polynomials, and the simplification of equations. To make up for the lack of problems, we present some translations from other books in Appendix B.

211.1 **Section II.2.1. The meaning of algebra (*al-jabr wa-l-muqābala*).**

The name given to algebra in medieval Arabic is *al-jabr wa-l-muqābala*, literally "restoration and confrontation". Sometimes the name was shortened to just *al-jabr*, and via transliterations in medieval Latin and Italian it led to the English word "algebra". The words *al-jabr* and *al-muqābala* occur individually in the solutions to problems by algebra, specifically for particular steps in the simplification of equations, and it is these meanings that are briefly described in the next line, along with the term "equalization" (*muʿādala*). By "its types" Ibn al-Bannāʾ means the six types of simplified equation, described at 213.1.

211.2 Ibn al-Bannāʾ gives brief explanations of restoration (*al-jabr*), confrontation (*al-muqābala*), and equalization (*muʿādala*). Al-Hawārī will clarify these terms later, via examples in the section on addition and subtraction beginning at 219.1. It would not do much good for him to expound on them here anyway, before the basic elements of algebra are covered.

The rules for algebra, as explained by Ibn al-Bannāʾ and illustrated by al-Hawārī, are best understood in the context of the general structure of algebraic solutions. Three basic stages are followed in the solution of a problem by algebra in medieval Arabic:

Stage 1. An unknown number is named in terms of the names of the powers of the unknown, usually as a "thing" (like our "x"). Then the conditions of the problem are applied to the name to produce an equation;

Stage 2. The equation is simplified to one of six types (given below) via restoration and/or confrontation;

Stage 3. The simplified equation is solved using the specified rule for that type.

211.8 The name given to the first degree unknown is a "thing" (*shayʾ*), though it is sometimes called a "root" (*jidhr*, or, as Ibn al-Bannāʾ preferred, *jadhr* (see 163.11)). This is like our x, though whether rhetorically or in notation, it was interpreted differently from its modern counterpart. The second degree unknown, which corresponds to our x^2, is called a *māl*. This word ordinarily takes the meaning of "sum of money", "possession", or "wealth". Because there is no good English translation of the word, and because it was used in a technical sense in Arabic algebra unconnected with its everyday meanings, we leave it untranslated. And, because of the different plural forms for Arabic nouns, we write the plural of *māl* with the English suffix: *māl*s.

Units, things, and *māl*s were considered to be three different types or species of number, just like units, tens, hundreds, etc. are different species (see our description above at 68.18). This idea goes back at least to al-Khwārazmī in the early ninth century. He introduced the powers of the unknown by writing: "I found that the numbers which are needed in algebraic calculation are of three types, which are roots and *māl*s and simple number…"[80]

The word *māl* also commonly takes another meaning in arithmetic, where it can be an amount of money, or, in most cases, a generic "quantity". The word *māl* is used to mean a "quantity" in two problems translated in Appendix B: Ibn al-Bannāʾ solves problem [2] by single false position, and al-Ḥaṣṣār solves problem [7] by three different methods. The word also means "quantity" in four of the problems al-Hawārī solves by double false position, above at 199.1, 200.1, 201.1, and 203.11. For instance, leaving *māl* untranslated, the problem athyperlinkc199.1199.1 is "a *māl*: taking away its third and its fourth leaves ten. How much is the *māl*?" Such problems were also frequently solved by algebra. In the enunciation *māl* is a common noun meaning "quantity" or "amount", while in an algebraic solution it means instead the proper name given to the second power of the unknown "thing". Which meaning is intended is clear from the context.

The word "root" likewise takes different meanings in arithmetic and algebra. In arithmetic it means "square root", while in algebra it is also the name sometimes given to the first degree unknown. Again, the meaning is made clear by the context.[81]

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80 (al-Khwārizmī 2009, 97.9).
81 (Oaks and Alkhateeb 2005).

Units in Arabic arithmetic are often counted in dirhams, a denomination of silver coin. Sometimes the word refers to money, but often the problem is stated in more abstract terms, and it is clear, then, that “dirham” is a substitute for a generic, arithmetical “unit”. This is most evident in problems that are stated in terms that cannot possibly involve money, such as problems in al-Khwārazmī in which the dirham stands for one man, or in geometry problems in Abū Kāmil in which the dirham stands for a unit length.[82] In the passage at 158.1, “dirham” appears to refer to the arithmetical unit. In all other instances before the chapter on algebra it refers to the coin, and in the chapter on algebra it is the unit.

We shall encounter higher powers below, especially in Section 4 starting at 225.1. The cube of a thing is called a “cube” (*kaʿb*), the fourth power is a *māl māl*, the fifth power is usually a *māl* cube, and higher powers are some combination of “*māl*” and “cube”. These species can be written any way one likes, as long as the powers add up. For example, at 226.3 al-Hawārī writes that the eighth degree term can be called a *māl māl māl māl*, a cube *māl* cube, or a cube cube *māl*.

211.13 The most common ways of equating numbers and magnitudes in Arabic mathematics were with words deriving from *sawiya* (“to be equivalent, be equal”), *mithl* (“equivalent”), the prefix *ka-* (“as, like”), and the implied verb “to be”.[83] Algebraic equations, however, were stated in Arabic with the unusual verb *ʿadala*, meaning “equal” or “well-balanced”. Another feature that distinguishes equations from other kinds of equating is that the two sides of an equation are stated with the names of the algebraic powers. These two features are evident, for example, in the equation “a *māl* and five things equal (*taʿdil*, conjugated from *ʿadala*) ten dirhams and two *māl*s less a thing” at 224.1, which in modern notation would be $x^2 + 5x = 10 + 2x^2 - x$.

We explain below in our comments in the section on addition and subtraction, beginning at 219.1, that modern notation is not compatible with medieval algebraic expressions. For this reason, it is better to write the expressions and equations of Arabic algebra with a transliteration of the notation that was current in the western part of the Islamic world in the time of al-Hawārī, even if he does not show it in his book. This notation is an extension of Indian notation to cover algebraic expressions, operations, and equations.

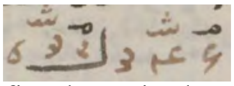
This last equation is a good example for introducing the Arabic notation. Instead of $x^2 + 5x = 10 + 2x^2 - x$, we write it as $\overset{m}{1}\,\overset{t}{5} = 10\,\overset{m}{2}\,\ell\,\overset{t}{1}$. In the first term, the $\overset{m}{1}$, the “m” stands for *māl*s and the 1 indicates that there is one of them. The “t” in $\overset{t}{5}$ stands for “things”, and the 5 below is how many there are. (Even when the first degree term is called “roots”, the letter in the notation is an abbreviation for “things”.) Simple numbers, here 10, do not need a designation, so there is no letter above them. There is no sign for addition on the left side or between the first two terms on the right because the terms are gathered together. This is the same kind of grouping we saw for distinct fractions at 136.8, where “five sixths and four fifths” is shown as $\frac{4}{5}\,\frac{5}{6}$, and for binomials in our commentary at 173.10, with examples from Ibn Qunfudh like “$\sqrt{60}\;8$”. Last, the ℓ is the same sign for “less” that we

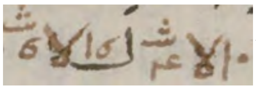
[82] Problems (24) and (28) in (al-Khwārazmī 2009, 185, 191); most of the 20 geometry problems in (Abū Kāmil 2012, 523-577).

[83] Definitions are taken from (Wehr 1994, 519, 943, 1047). See (Oaks 2010).

saw for excluded fractions and apotomes, and which we explain and show above at 86.1, 140.1, and 173.10. It indicates that the $\overset{t}{1}$ is lacking or missing from the $10\ \overset{m}{2}$.

The Arabic algebraic notation originated in the western part of the Islamic world (i.e., west of Egypt) no later than the end of the twelfth century.[84] Several textbooks with chapters on algebra show it, including those written by Ibn al-Yāsamīn (d. 1204), al-Mawāḥidī (14th c.), Ibn Qunfudh (1370), al-Qalaṣādī (d. 1486), and Ibn Ghāzī (1483), but it is not uncommon to find purely rhetorical presentations of algebra like al-Hawārī's, either. Like the Indian notation for numbers that al-Hawārī and others show, the algebraic notation only appears occasionally in books as figures to show what should be written on the dust-board. The notation was not part of the running text.

Here are a couple of sample equations from a manuscript of a book that shows the notation, Ibn Ghāzī's *Aim of the Students*. At one point he writes "five *māl*s and four things and three in number equal two *māl*s and three things and six in number, and its figure is  ".[85] Reading the notation right to left, the first term shows a *mīm*, the first letter in the word *māl*, above a 5, for the "five *māl*s". The next term shows a *shīn*, the first letter in *shay*ʾ ("thing") above a 4, and this is followed by a lone 3. The sign for "equals" is the black line shaped like a backwards "L". It is a *lām*, the last letter in *taʿdil* ("equals"). The left side of the equation is written similarly. Our transliteration, read left to right, is $\overset{m}{5}\ \overset{t}{4}\ 3 = \overset{m}{2}\ \overset{t}{3}\ 6$, which corresponds to the modern $5x^2 + 4x + 3 = 2x^2 + 3x + 6$.

An example from the same manuscript with "less" is: "ten less four things equals sixteen less six things, so the figure for this is  ".[86] We write it as $10\ \ell\ \overset{t}{4} = 16\ \ell\ \overset{t}{6}$, and in modern notation it is $10 - 4x = 16 - 6x$. Like we have seen for arithmetic, in the manuscripts either the whole word *illā* is written, or the initial *alif* is dropped so it looks like an upside-down ℓ. Note that here the Arabic "0" is written as a dot, so the "10" looks like "1·". Because it takes a little while to become accustomed to the Arabic notation, we will continue for a while to give the version in modern notation as well. See below at 229.8 for a brief description of how this notation was interpreted.

Today, we have just one form of simplified quadratic equation, $ax^2+bx+c = 0$, where our a, b, and c can be positive, negative, or zero. The solution to this equation, $x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$, is a formula expressed in terms of the coefficients. The corresponding solutions in medieval algebra also use the "coefficients", but because only positive numbers were acknowledged they had six types of simplified equation: three of them "simple" (*mufrad*), with a single term on each side, and the other three "composite" (*murakkaba*), where one side has two terms. These are listed below, with al-Hawārī's examples under each generic form:

Simple equations

Type 1: some *māl*s equal some roots

"three *māl*s equal seven things" ($\overset{m}{3} = \overset{t}{7}$; $3x^2 = 7x$).

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[84] See (Abdeljaouad 2002) and (Oaks 2012a) for a description of the notation.

[85] (Ibn Ghāzī manuscript, fol. 83a).

[86] (Ibn Ghāzī manuscript, fol. 84a).

Type 2: some *māl*s equal some numbers
    "five *māl*s equal twenty" ($\overset{m}{5} = 20$; $5x^2 = 20$).
Type 3: some roots equal some numbers
    "three roots equal twelve" ($\overset{t}{3} = 12$; $3x = 12$).

Composite equations
Type 4: some *māl*s and some roots equal some numbers
    "a *māl* and ten roots equal twenty-four" ($\overset{m}{1}\ \overset{t}{10} = 24$; $x^2 + 10x = 24$).
Type 5: some *māl*s and some numbers equal some roots
    "a *māl* and four equal five roots" ($\overset{m}{1}\ 4 = \overset{t}{5}$; $x^2 + 4 = 5x$).
Type 6: some *māl*s equal some roots and some numbers
    "a *māl* equals four roots and five" ($\overset{m}{1} = \overset{t}{4}\ 5$; $x^2 = 4x + 5$).

There is no notation in al-Hawārī's chapter on algebra, not even for numbers. Recall that in the first part of the book, on arithmetical calculation, he used the Indian figures only to illustrate work that was to be performed on the dust board or other surface. Like the previous chapter covering proportion and double false position, the calculations in the chapter on algebra are given entirely in words because their execution in notation has already been explained in the first part of the book.

213.1 **Section II.2.2. Working out the six types (of equation).**

213.2 The third stage in the solution to a problem by algebra is to solve the simplified equation. Each of the six types of equation is solved by its own rule. The solutions to the three simple types require a single division to get the *māl* or thing/root.

213.7 The example for type 1 is "three *māl*s equal fifteen things", or $\overset{m}{3} = \overset{t}{15}$. In modern notation, this is $3x^2 = 15x$.

Students who had not studied algebra before would not be familiar with the wording of algebraic equations. So like al-Khwārazmī and many others, al-Hawārī explains each of his equations by translating it into the enunciation of an arithmetic problem, taking advantage of the two meanings of the words "*māl*" and "root". In the equation "three *māl*s equal fifteen things", the word "*māl*" takes its algebraic meaning as the name of the second power of the unknown. Al-Hawārī reformulates it as an arithmetic problem: "the meaning of this problem is: what quantity (*māl*), if we take its (square) root fifteen times, gives a sum equal to three times the quantity?" Here the word *māl* takes the arithmetical meaning of "quantity" or "amount", and "root" means square root. Note how the equation and the enunciation are worded differently, too. The two sides of the equation are collections of the names of the powers that are declared equal to each other, while the enunciation expresses an operation ("we take its root…") with a specified outcome ("gives a sum…"). Also, the equation is stated with the algebraic "equal" (*taʿdil*, conjugated from *ʿadala*) while the arithmetical version uses an "equal" (*musāwiya*) common to arithmetic.[87]

Here al-Hawārī calls the equation a "problem" (*masʾala*), the same word used for other kinds of arithmetic problems or questions. This is also the word used for equations by

[87] (Oaks and Alkhateeb 2005, §4.5).

al-Khwārazmī, Abū Kāmil, and other early algebraists. The specific Arabic word for an algebraic "equation" (*muʿādala*) first appears in the latter ninth century in Qusṭā ibn Lūqā's Arabic translation of Diophantus's *Arithmetica*. It is common to find both "problem" and "equation" in later Arabic books and chapters on algebra, including Ibn al-Bannāʾ's and al-Hawārī's books. Other examples of "problem" are in the passages at 213.13, 214.1, 217.1, 217.10, etc., and "equation" is found in the passages at 217.1, 227.14, 227.17, and 228.1.[88]

213.13 Al-Hawārī's equation "two *māl*s equal eighteen", $\overset{m}{2} = 18$, corresponds to our $2x^2 = 18$. Al-Hawārī again translates the equation into an arithmetical enunciation: "What *māl*, if we add itself to it, becomes equal to eighteen?" Like in the first example, the equation "two *māl*s equal eighteen" does not invoke any operation, since it merely asserts the equality of a couple of "*māl*s" with 18. By contrast, the arithmetical version entails an operation: one *adds* a *māl* to itself, with an *outcome* equal to eighteen.

214.1 Al-Hawārī's equation "five things equal twenty", $\overset{t}{5} = 20$, corresponds to our $5x = 20$. In the reformulation, the "thing" becomes a root of the *māl*: "What *māl*, if we take five of its roots, gives a result equal to the twenty?" And again, the arithmetical version is framed as an operation and its outcome.

214.7 The rules for the composite equations are naturally more complicated, and one can easily show that they are equivalent to our quadratic formula. Many authors, like al-Khwārazmī, Abū Kāmil, and al-Khayyām, give proofs to these rules using geometric diagrams in their algebra books. Others justify the rules with purely arithmetical arguments, like Ibn al-Yāsamīn, Ibn al-Bannāʾ (in his *Algebra* and *Lifting the Veil*), and al-Fārisī.[89] But it is also common for an author not to bother with proofs at all, like ʿAlī al-Sulamī, Ibn al-Bannāʾ (in his *Condensed Book*), al-Hawārī, and Ibn Badr (ca. thirteenth century). In these books, only the arithmetical rules for the solutions are given.[90]

214.9 Al-Hawārī's type 4 example "a *māl* and two things equal fifteen", $\overset{m}{1}\ \overset{t}{2} = 15$, corresponds to our $x^2+2x = 15$. He again explains the equation in terms of an arithmetical enunciation framed with operations, in which "*māl*" and "root" revert to their meanings in arithmetic.

Some authors occasionally write out the whole phrase "half of the number of things", but it is usually written simply as "half of the things", like it is here. In the modern term $2x$, the 2 and the x are both numbers which are joined through scalar multiplication. But the medieval "two things" is simply a two of a particular type ("things"). In the present case "the things" is 2, and its half is 1. Then $1^2 = 1$, $1 + 15 = 16$, $\sqrt{16} = 4$, and $4 - 1 = 3$, which is the unknown thing/root. The unknown *māl* is then 9. This type of equation will have just one positive solution.

We can rewrite the solutions to each equation type as a modern formula. In this case, Ibn al-Bannāʾ's solution to the general equation $x^2 + bx = c$ can be transformed into

[88] For more on the word "equation" see (Oaks 2010, §5.2).
[89] For arithmetical proofs, see (Oaks 2018a).
[90] The books of these authors are named in Appendix C.

$x = \sqrt{(\frac{1}{2}b)^2 + c} - \frac{1}{2}b$. But keep in mind that like the rules for double false position, these rules are procedures consisting of a sequence of operations, and not static, written formulas. The rule is spoken, in time, with one step following another, while our formula is apprehended visually, and not necessarily left-to-right (or right-to-left) like rhetorical text (see our comments above at 198.4).

214.14 In most problems the desired unknown is the "thing", so the rule given above yields the answer to the problem. But, often one is searching instead for the *māl*. For instance, if we were asked to find a number given some condition involving its square root, we might name the number x^2 and create an equation with x^2 and its root x. This is why some algebraists give rules for finding the *māl* directly, even if in practice these rules were not used much. Rules for finding the *māl* date back to Abū Kāmil's late ninth century *Book of Algebra*, and Ibn al-Bannāʾ included them in his own *Book on the Fundamentals and Preliminaries in Algebra*. Al-Hawārī certainly learned these rules from Ibn al-Bannāʾ, and he explains the calculations well for this equation.

215.3 The fifth type of equation might have two, one, or no positive solutions. Al-Hawārī's example with two solutions is "a *māl* and eight equal six things" ($\overset{m}{1}\ 8 = \overset{t}{6}$; $x^2 + 8 = 6x$).

The rule to solve this equation is similar to the rule for the type 4 equation. Take half of the 6 to get 3, square it to get 9, subtract the 8 to get 1, and take its root to get 1. One of the solutions is obtained by adding the 3 to the 1 to get 4, and the other is obtained by subtracting the 1 from the 3 to get 2.

215.12 Ibn al-Bannāʾ gives the condition under which this equation will have exactly one solution: a square of half the number of roots/things equals the number. Al-Hawārī's example is "a *māl* and nine equal six things" ($\overset{m}{1}\ 9 = \overset{t}{6}$; $x^2 + 9 = 6x$). Half of the roots is 3, and its square 9 equals the 9 in the equation. The value of the thing is half the roots, or 3, and the *māl* is the number, 9.

215.14 Following the first rule at 215.3 for this kind of equation causes one to operate on "nothing" a number of times. This is reminiscent of the operations on zero we saw earlier in the book. It is not necessary to regard nothing or zero as being a number for these operations to be carried out. See our comments on "operating on zero" at 74.17 above.

216.1 Next, al-Hawārī gives the rule for finding the *māl* directly for equations with two solutions; in other words, when "the number is smaller than a square of half of the number of roots" (he had not given this condition earlier). He works this out for the first type 5 example, "a *māl* and eight equal six things". A square of the number of roots is 36, and its half is 18. Subtracting the 8 leaves 10. Keep this in mind. Then square the 8 to get 64, and subtract it from a square of the remembered 10, which is 100, leaving 36. Its square root is 6. One solution is that the *māl* is $10 + 6 = 16$, and the other is $10 - 6 = 4$.

216.11 Type 6 equations always have one positive solution. Al-Hawārī's example is "a *māl* equals two of its roots and three" ($\overset{m}{1} = \overset{t}{2}\ 3$; $x^2 = 2x + 3$). Al-Hawārī's explanation in terms of arithmetic is awkward, since it is the same as the equation, only with the arithmetical term *yusāwī* ("[is] equal") replacing the algebraic *yaʿdil* ("equals"): "what *māl* is equal to two of its roots and three?" This might be because the two terms in this equation are stated *after* the single term, and al-Hawārī did not see that he could simply switch them and write "what *māl*, if we add three to two if its roots, gives a result equal to the *māl*?" This confusion may also be why he wrote in the equation "two of its roots", which is the arithmetical formulation, instead of the usual "two roots".

In the solution, he takes half of the 2 and squares it to get 1. Adding this to the 3 gives 4. Its root is 2, and adding this to half of the 2 gives 3, "which is the unknown thing".

216.17 The rule for finding the *māl* directly is clear.

217.1 The rules that Ibn al-Bannāʾ gave for the three composite equations presume that the number of *māl*s is 1. Now he explains that if their number is greater or less than 1, we should set the number to 1 by restoring or reducing it, and we must also adjust the other terms of the equation in the same manner. Restoration and reduction have already been described outside the context of algebra for whole numbers in Section I.1.6 at 129.1, and for fractions in Section I.2.5 at 154.1. Once the number of *māl*s becomes 1, then the rules given above can be applied to find the solution(s).

The word we translate as "term" (*ism*) in this chapter is also the word for "name". The species "thing", "*māl*", and "cube" are often called "names" (first at 211.8, in the translation), and the same word is used to mean a term of an equation, like "three things", "a *māl*", and "half a cube". So "thing" as a name for the first degree unknown is an *ism*, just as "six things" in the equation below is an *ism*. In the present passage Ibn al-Bannāʾ uses another word, *alqāb*, also meaning "name", for a term in an equation. In common use *ism* is "given name", while *alqāb* is "family name".

217.10 Al-Hawārī's first example is "two *māl*s and six things equal thirty-six" ($\begin{smallmatrix} m & t \\ 2 & 6 \end{smallmatrix} = 36$; $2x^2 + 6x = 36$). Here the two *māl*s must be reduced to a single *māl*, so he multiplies it by a half. The other terms must be reduced by a half to compensate, so the equation becomes "a *māl* and three things equal eighteen" ($\begin{smallmatrix} m & t \\ 1 & 3 \end{smallmatrix} = 18$; $x^2 + 3x = 18$).

217.15 Another way to do this is to divide the two by two, and divide the other terms by two as well. Dividing by two and multiplying by a half may be mathematically equivalent, but they are conceptually different. One is division by a whole number, and the other is multiplication by a fraction.

218.1 The other example is "half a *māl* and two things equal six" ($\begin{smallmatrix} m & t \\ \frac{1}{2} & 2 \end{smallmatrix} = 6$; $\frac{1}{2}x^2 + 2x = 6$). Here the half of a *māl* must be restored to one *māl*, so he multiplies it and the other terms by two to get "a *māl* and four things equal twelve" ($\begin{smallmatrix} m & t \\ 1 & 4 \end{smallmatrix} = 12$; $x^2 + 4x = 12$). The other way is to divide each term by a half.

219.1 **Section II.2.3. Addition and subtraction.**

In this section, al-Hawārī explains how to add and subtract algebraic expressions. After giving some easy examples mainly taken from *Lifting the Veil* (219.5-220.4), he pays special attention to examples illustrating Ibn al-Bannāʾ's rule for simplifying a problem of subtraction in which one or both of the parts is a diminished algebraic amount. These amounts are of the form "A less B", and can be thought of as algebraic apotomes (220.5-222.4). Al-Hawārī follows this with a subsection on simplifying equations (223.1-224.8), since the techniques there are the same. In both subtractions and equations diminished amounts are "restored" (with *al-jabr*) and like terms in the two parts are settled by "confronting" them (with *al-muqābala*). We describe these terms along with "equalization" (*muʿādala*) below in our comments at 220.5.

And it is here, in the present section, that rewriting the Arabic calculations in modern notation breaks down. The cause of the problem goes back to the way arithmetical operations, and by extension algebraic operations as well, are worded in premodern mathematics. We have been able to overlook the matter so far because it really only becomes an issue for addition and subtraction of quantities that are already composed of more than one term, like the quantities "six eighths less a ninth", "two and a root of eight", or "ten less a thing". This wording is in turn linked with a difference in how multi-term quantities were conceived, and only when the medieval conceptions are addressed will the Arabic "restoration" of diminished amounts and other oddities of medieval practice make sense.

We begin our comments by examining the anatomy of Arabic arithmetical operations, keeping in mind that algebra is a part of arithmetic. In the process, we address how Arabic authors conceived of multi-term amounts, and we expose the inadequacy of modern notation. Later, in our remarks on al-Hawarī's examples, we continue to work with the transliteration of the Arabic algebraic notation, which is naturally well-suited to represent the calculations.

Al-Hawārī writes at 87.19 "We add the five to the three, giving eight". Today we write this as $3 + 5 = 8$, which is spoken as "three plus five equals eight". Mathematically, the medieval and modern operations are the same, but they are presented differently. The Arabic is phrased with a verb for the operation (in this example "add", conjugated from *jamaʿa*) and a word announcing the result (here "gives", from *bi-*). The operation is performed in time, with a specified outcome. By contrast, the verb in the modern version is "equals", and the word "plus" appears in its quasi-prepositional sense to indicate the operation. There was no word in Arabic that means "plus". The key difference between the Arabic and modern ways of expressing the operation is that the former is presented as a process that is worked through, while the latter is a static equivalence.

This difference becomes problematic for operations involving multi-term amounts, like al-Hawārī's "add a root of three to a root of fifteen… to get a root of three and a root of fifteen" (at 180.10). Translated into modern notation, the operation does not seem to accomplish anything: $\sqrt{3} + \sqrt{15} = \sqrt{3} + \sqrt{15}$. Like in the first example, the operation in the Arabic is expressed with the verb "add" (the first "+" in notation). Because the two numbers cannot be written as a single root, the result is a binomial expressed as "a root of three and a root of fifteen" using the common conjunction "and" (*wa*, the second "+"). This *wa* is not another word for the arithmetical operation of addition. The result "a root of three and a root of fifteen" is merely a collection of two objects, like saying "an apple and a banana". We see this kind of aggregation also in distinct fractions, explained above at 136.8 and 139.1. The fraction "five sixths and four-fifths" from 136.8, written $\frac{4}{5}\,\frac{5}{6}$, is a single number expressed as two numbers gathered together. This kind of joining of numbers into an aggregation is even present in the way numbers with different ranks are expressed, like 24, which is spoken as “four and twenty”. Where in Arabic a distinction is made between the operation of addition and the expression of the result with “and”, in our notation the plus sign “+” is used for both.

Another example, this one involving three multi-term amounts, is Ibn al-Hāʾim’s “add a root of eight and a root of twenty to a root of five and a root of two…so their sum is a root of eighteen and a root of forty-five”.[91] Translated into modern notation this would be

$$(\sqrt{5}+\sqrt{2})+(\sqrt{8}+\sqrt{20})=\sqrt{18}+\sqrt{45}.$$

The original Arabic calls for only one operation (the second “+”), but we see a total of *four* additions in notation. And where the right side should be the outcome of that operation, in our notation the two sides are symmetric, so it is equivalent to

$$\sqrt{18}+\sqrt{45}=(\sqrt{5}+\sqrt{2})+(\sqrt{8}+\sqrt{20}).$$

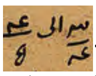
Ibn al-Hāʾim’s operation calls for the addition of two binomials that results in another binomial. We could adjust our notation to distinguish between aggregations and additions, and use an arrow rather than the “=” to show the direction of the operation, like this:

$$(\sqrt{5}\ \&\ \sqrt{2})+(\sqrt{8}\ \&\ \sqrt{20})\rightarrow\sqrt{18}\ \&\ \sqrt{45}.$$

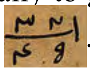
But this unhappy hybrid is no easier to read than our transliteration of the medieval notation. So we will use the latter, which in this case should look something like this:

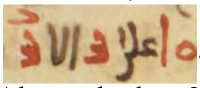
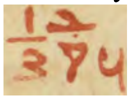
$$\frac{\sqrt{5}\ \sqrt{2}\ \text{to}\ \sqrt{8}\ \sqrt{20}}{\sqrt{18}\ \sqrt{45}\qquad\qquad}.$$

Here, the horizontal line serves to separate the steps in the working out of the problem. This notation for separating steps is rarely shown in manuscripts. Like we have seen for al-Hawārī’s calculations with Indian numerals, Arabic textbooks describe the operations rhetorically, and those that show the notation generally give only snippets to illustrate what should be written on the dust-board. Of all the medieval Arabic books that show the notation, we know of only one instance, from Ibn Ghāzī’s *Aim of the Students*, that shows operations and results without any intervening text. Ibn Ghāzī separates each of his steps with horizontal lines, whether there is an operation or not. So there may have been no special sign that was used to announce the result of an operation. Like our medieval sources, we only give snippets of the notation surrounded by text. And we will use the horizontal line only the one time, above, to indicate the outcome of an operation.

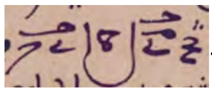
Operations themselves were designated in notation by prepositions. Al-Ḥaṣṣār’s problem “Add three fourths to four fifths” is shown in one late twelfth century manuscript as .[92] Between the fractions is the preposition *ilā*, “to”, the same word appearing in the rhetorical statement of the operation. The Arabic prepositions for addition, subtraction,

[91] (Ibn al-Hāʾim 2003, 199.22).
[92] (al-Ḥaṣṣār manuscript, fol. 68b).

multiplication, and division are all different, so they suffice to indicate which operation is intended. Al-Ḥaṣṣār performs this calculation rhetorically to get "one and two fifths and three fourths of a fifth", which is shown in notation as .

Another example is al-Mawāḥidī's problem "Divide ten by three less a root of three", shown in notation as .[93] Here the "by" (ʿ*alā*) and "less" are in black, and the numbers are in red. Above the last 3 is the dotless letter *jīm* (ـجـ) indicating a square root. We transcribe this as "10 by 3 ℓ $\sqrt{3}$". The result is given rhetorically as "five in number and a root of eight and a third", and is shown in notation as , or "5 $\sqrt{8\frac{1}{3}}$". Like with distinct fractions (at 136.8 and 139.1), the aggregation shows no sign for "and".[94]

In algebra, adding two terms of the same species is done by adding their "numbers", or what we call their coefficients. For example, adding three things to five things gives eight things (in modern notation, $5x + 3x = 8x$). But, if the species are different, one can only put the conjunction "and" (*wa*) between them, as we saw with binomials in the chapter on roots. In his abridgment of his own commentary on Ibn al-Yāsamīn's poem on algebra, Sibṭ al-Māridīnī (15th c.) gives the example "if you add two dirhams to three things, the answer is two dirhams and three things".[95] In notation, adding 2 to $\overset{t}{3}$ gives $2\ \overset{t}{3}$. The result of the addition is a collection of five items of two different species or types, and it is worded just like we would say "two apples and three bananas".

Because a term like "three things" was conceived as a collection of three objects, much like "three bananas", irrational coefficients were not allowed. One can have three bananas or two and a half bananas, but it makes no sense to speak of $\sqrt{12}$ bananas. Arabic algebraists avoided irrational "numbers" of terms by taking a root of the entire term. For example, in solving one problem Ibn al-Bannāʾ writes "you multiply a thing by a root of ten, to get a root of ten *māl*s". Expressing this in modern notation, he multiplies x by $\sqrt{10}$ to get $\sqrt{10x^2}$. This way the "number" (the coefficient 10) remains rational, while the root of the entire term is valid as an irrational number (the value of the "thing" in this problem is found to be $10\ \sqrt{2\frac{1}{2}}\ \ell\ \sqrt{2\frac{1}{2}\ \sqrt{1000}}$).[96] Here is an example of how such a root appears in notation in an Arabic equation, from Ibn Qunfudh's commentary: . In our transliterated version this is the equation $\overset{t}{3}\ \sqrt{\overset{m}{2}}\ =\ 18\ \sqrt{72}$, which corresponds to $3x + \sqrt{2x^2} = 18 + \sqrt{72}$ in modern notation.[97]

[93] (al-Mawāḥidī manuscript, fol. 77a).

[94] Occasionally the "and" will be written in the notation, usually to counterbalance a "less" in another part of the expression.

[95] (Sibṭ al-Māridīnī 1983, 59.8).

[96] Or, in modern notation, $10 + \sqrt{2\frac{1}{2}} - \sqrt{2\frac{1}{2} + \sqrt{1000}}$ (Saidan 1986, 581.6).

[97] (Ibn Qunfudh manuscript, 272.3). Here Ibn Qunfudh denotes "things" using only the three dots from the letter ش. The equation is stated verbally as "three things and a root of two *māl*s equal eighteen and a root of seventy-two in number". The example is given to explain that the term "a root of two *māl*s" is of the order of things, since it comes from the "multiplication of a root of two by a thing". For more on how medieval algebraists dealt with irrational numbers in algebra, see (Oaks 2017).

219.2 Al-Hawārī's example of adding appended expressions is to add together a *māl*, six things, and ten dirhams to get "a *māl* and six things and ten dirhams". This result is a collection or aggregation of seventeen items of three different species, and in notation we write it as $\overset{m}{1}\,\overset{t}{6}\,10$. Contrast al-Hawārī's polynomial with its modern equivalent, $x^2+6x+10$, where the arithmetical operation of addition appears twice along with scalar multiplication and exponentiation.

Al-Hawārī does not give an example where the addends themselves are already expressed with two or more appended names (i.e., where the terms are connected with "and"), so we give the explanation from al-Karajī's *al-Fakhrī*. He writes: "The way to do that is that you add each species to the same species. For example, add five things and four *māl*s to three things and three *māl*s. You add five things to three things to get eight things, and four *māl*s to three *māl*s to get seven *māl*s. The sum is then eight things and seven *māl*s."[98]

219.4 Where the word "and" (*wa*) is used to express the sum of two different species, the word "less" (*illā*) is used to express their difference. In Arabic grammar this word indicates that the noun that follows is absent from the preceding noun, like in the phrase "all the children except (*illā*) Amy". Here "all the children" is called *mustathnā minhu*, literally "excluded from it", or better, "diminished", and "Amy" is called *mustathnā*, "excluded" or "absent".

The same idea of something missing from something else applies also to mathematical expressions with *illā*. Al-Qalaṣādī gives this example in arithmetic: "subtract a root of five from a root of eight. You say the remainder is a root of eight less (*illā*) a root of five".[99] Again, in modern notation the operation seems to accomplish nothing: $\sqrt{8}-\sqrt{5} = \sqrt{8} - \sqrt{5}$. Instead, the result of the subtraction, "a root of eight less a root of five" ($\sqrt{8}\;\ell\;\sqrt{5}$), is an apotome, and apotomes by their nature contain no operation. This quantity is a diminished $\sqrt{8}$, from which $\sqrt{5}$ has been removed. In English, we have no word that functions quite like the Arabic *illā*. It is kind of like saying "an apple less a bite" or "three cents short of a dollar". We have already seen this use of "less" in the case of excluded fractions, like the example "six eighths less a ninth" at 140.8. The notational version of that example from one manuscript is shown above in our comments for 140.1. See also al-Mawāḥidī's notation for "three less a root of three" just above.

In mathematics, as in grammar, the two values A and B in expressions of the form "A less B" are called *mustathnā minhu* (diminished) and *mustathnā* (excluded), respectively. Al-Hawārī uses these terms in his explanation of excluded fractions beginning at 140.1, in the context of quadratic irrational apotomes at 183.15, and in algebra on pages 219-222 and 230. Sometimes the diminished term is called "appended" (*zāʾid*) and the excluded term "deleted" (*nāqiṣ*), as in the passages at 87.11, 183.15, 189.1, 211.2, etc. We translate these two words as "exceeds" and "falls short", respectively, in the solutions to problems by double false position (first at 198.8).

So where we simplify an equation like $x^2-3x = 2+x$ by adding the subtracted $3x$ to both sides, at 223.2 al-Hawārī restores (with *al-jabr*) the diminished *māl* in the left side of the equation $\overset{m}{1}\;\ell\;\overset{t}{3} = 2\;\overset{t}{1}$ to make it complete (restoration is explained more fully below at

98 (Saidan 1986, 118.5).
99 (al-Qalaṣādī 1999, 215.9, French 248.6).

220.5) and then adds the $\overset{t}{3}$ to the other side. Like with "plus", there is no medieval Arabic word with the meaning of the modern "minus". For this reason, neither word occurs in our translation.

219.5 On the previous line Ibn al-Bannāʾ had written "and the different excluded [species] cannot be subtracted". When an excluded species (a term following the word "less") has no counterpart from which it can be subtracted, that is, when it is "different" from the others, the exclusion must remain. Al-Hawārī's example is to "add a *māl* less a thing to ten dirhams", or, in notation, to add $\overset{m}{1}\ \ell\ \overset{t}{1}$ to 10. There are no other "things" in the problem from which we can remove the $\overset{t}{1}$, so the result of the addition is "a *māl* and ten dirhams less a thing", or $\overset{m}{1}\ 10\ \ell\ \overset{t}{1}$. The answer is eleven objects of two types (one *māl* and ten dirhams) that are diminished by a thing. Writing it entirely in modern notation makes the operation appear as if all we do is rearrange terms: $10 + (x^2 - x) = x^2 + 10 - x$.

219.7 When adding expressions with deleted terms, subtract when possible the deleted term from an appended term of the same power. The first example is to add $\overset{m}{2}\ \ell\ \overset{m}{1}$ to 10. This is not a well-formed problem, since the first term, the "two *māl*s less a *māl*", is itself not fully resolved. It is of course the same as "a *māl*", and the answer to the addition is $\overset{m}{1}\ 10$.

In the second example $\overset{m}{1}\ \ell\ \overset{t}{2}$ is added to $\overset{t}{10}$, resulting in $\overset{m}{1}\ \overset{t}{8}$. The third example is similar: add $\overset{m}{1}\ \ell\ \overset{t}{5}$ to $\overset{t}{10}$ to get $\overset{m}{1}\ \overset{t}{5}$. In modern notation this corresponds to adding $x^2 - 5x$ to $10x$ to get $x^2 + 5x$.

220.1 Here al-Hawārī explains the particle "less" to designate the result of a subtraction of different species. The result of subtracting a thing from a *māl* is "a *māl* less a thing", or $\overset{m}{1}\ \ell\ \overset{t}{1}$. The result in the second example is $\overset{m}{2}\ \overset{t}{3}\ \ell\ 10$.

220.5 Al-Hawārī has postponed until this section his explanations of restoration (*al-jabr*), confrontation (*al-muqābala*), and equalization (*muʿādala*) that were briefly described by Ibn al-Bannāʾ above at 211.2. It is here, after all, that he applies them in problems of subtraction, and in the following subsection on simplifying equations.

The subtraction of appended terms in algebraic expressions (those connected with the word "and") is easy: one simply subtracts like terms. Subtraction is less straightforward when it involves an expression of the form "A less B", especially when this occurs in the subtrahend. Ibn al-Bannāʾ and al-Hawārī cover these here. Like other Arabic algebraists, al-Hawārī transforms the problem of subtraction into a simpler problem of subtraction by *restoring* diminished amounts (of the form "A less B"), and *confronting* like terms in the two sides. These are also the same steps one takes in simplifying equations, which is covered in the next subsection.

As Ibn al-Bannāʾ remarked above at 211.2, the restoration of a diminished term (the "A" in "A less B") is the same restoration he explained of smaller numbers to greater numbers described at 129.1 and 154.1, and it is also the same restoration we saw of fractions of a

[100] (Oaks and Alkhateeb 2007, §3.3).

māl at 217.1. To pick two examples, both "half a *māl*" ($\overset{m}{\frac{1}{2}}$) and "a *māl* less three things" ($\overset{m}{1}\ \ell\ \overset{t}{3}$) were considered to be incomplete *māl*s. To restore the first of these one multiplies by two, and to restore the second one adds back the missing "three things".[100]

In his problem (T1), Abū Kāmil explains restoration in the context of simplifying the equation "fifteen things less a *māl* and a half equal a *māl*" ($\overset{t}{15}\ \ell\ \overset{m}{\frac{1}{2}1} = \overset{m}{1}$): "And restore the fifteen [things] by the *māl* and a half so that it is equivalent to fifteen things. Then add the *māl* and a half to the *māl* to get: two *māl*s and half a *māl* equal fifteen things"[101] ($\overset{m}{\frac{1}{2}2} = \overset{t}{15}$). Two distinct steps are performed. First, the "fifteen things less a *māl* and a half", being a deficient "fifteen things", is restored to a full "fifteen things". Then, to balance the equation, "a *māl* and a half" is added to the other side. The restoration is the first of these two steps.

In the early eleventh century al-Karajī applied the same wording in his explanation of how to simplify a subtraction problem with a diminished subtrahend by restoration in his *al-Fakhrī*, just before working it out for a few examples: "The way [to do it] is you restore the subtrahend by what is excluded from it, and you add its same to the minuend".[102]

By the twelfth century many authors had begun to also call the second step, the addition of the same amount to the other side, a "restoration". This new way of wording it as two restorations was applied to the simplification of both subtractions and equations. Ibn al-Yāsamīn simplifies the problem of subtracting ten less a thing from a thing ($10\ \ell\ \overset{t}{1}$ from $\overset{t}{1}$): "you restore the ten by the deleted thing to get ten. Then restore the thing, which is the greater part, by the same [amount] you restored the ten, to get two things. So it is as if someone had said, subtract ten from two things".[103] Note that the restorations do not give the answer to the subtraction, but result in an equivalent subtraction problem from which the remainder will be found.

The modern version of Ibn al-Yāsamīn's problem is $x - (10 - x)$, which we work out by distributing the minus sign to get $x - 10 + x$, and which reduces to $2x - 10$. Although medieval algebraists knew that "when subtracting the deleted from another deleted, it is appended" (from above at 87.11), they usually handled the problem of subtracting an apotome through a restoration.

Like Ibn al-Yāsamīn, Ibn al-Bannāʾ also calls both steps restorations. His explanation in *Lifting the Veil* of his remark from 211.2 applies to equations:

> Restoring the deleted to the appended always entails the two sides. This is because whenever you restore the appended by what is excluded from it on one side, once your restoration is completed, that side has become greater than the other side. Thus it is different from what is equated to it, so you restore it by the excess.[104]

[101] (Abū Kāmil 2012, 321.11).
[102] (Saidan 1986, 120.14).
[103] (Zemouli 1993, 229.1).
[104] (Ibn al-Bannāʾ 1994, 309.3).

Confrontation is performed whenever there are like terms in the two sides. The verb *qabila* ("to confront") is rarely used in the course of simplifying an equation or subtraction, however. The tenth-century CE lexicographer Muḥammad ibn Aḥmad al-Khwārazmī explains the step:

> And for an example of confrontation, suppose one encounters in a problem: a *māl* and two roots equal five roots [$\overset{m}{1}\ \overset{t}{2} = \overset{t}{5}$]. You cast away the two roots which are with the *māl*, and you cast away the same from what it is equal to, to get: a *māl* equals three roots, and that is nine.[105]

The idea is that the like terms are confronted, or put face-to-face, and their difference is placed on the side of the greater. Like restoration, confrontation is performed in two steps. The smaller amount is deleted from one side, and the smaller is subtracted from the larger on the other side.

Ibn al-Bannāʾ explains confrontation similarly, though more tersely, in *Lifting the Veil*.[106] What is new in Ibn al-Bannāʾ is the word meaning "equalization" (*muʿādala*). He writes:

> Restoring the deleted to the appended is equalization with restoration, and subtracting the appended from the appended or the deleted from the deleted of things of the same species is equalization with confrontation.[107]

In both cases the "equalization" is the necessary compensation in the other part so that the new subtraction or equation yields the same answer as the original. "Restoring the deleted to the appended" is the adding of the deleted amount to the other side, and "subtracting the appended from the appended or the deleted from the deleted" is apparently what one does to compensate for deleting the smaller of the two like terms (the meaning is not presented clearly for confrontation). In confrontation, one typically subtracts "the appended from the appended". Al-Hawārī gives an example of subtracting "the deleted from the deleted" in the second solution to the problem at 223.7.

Restoration was also applied to the subtraction of apotomes. Al-Baghdādī explains that one must "restore the exception by what it is deleted from on that side, and add the same to the other side. Then subtract the subtrahend from the minuend", just before giving the example of subtracting "ten less a root of eighty from a root of eighty less five".[108]

220.9 In the first of four examples of subtraction, the minuend is diminished. Here al-Hawārī subtracts "two and a thing" ($2\ \overset{t}{1}$) from "a *māl* less three things" ($\overset{m}{1}\ \ell\ \overset{t}{3}$). Adding the $\overset{t}{3}$ to the minuend *restores* the $\overset{m}{1}$, and adding the same to the subtrahend *equalizes* the operation. Thus it is called "equalization with restoration". The problem then becomes to "subtract two and four things from a *māl*", or $2\ \overset{t}{4}$ from $\overset{m}{1}$. The answer is $\overset{m}{1}\ \ell\ \overset{t}{4}\ \ell\ 2$.

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105 (M. i. A. al-Khwārizmī 1895, 201.3). The "what it [i.e., the left side] is equal to" is the "five roots" on the other side of the equation.
106 (Ibn al-Bannāʾ 1994, 309.6).
107 (Ibn al-Bannāʾ 1994, 309.7).
108 (al-Baghdādī 1985, 221.1).

221.1 In the second example, the diminished term is in the subtrahend: subtract $52\;\ell\;{}^{t}_{5}$ from ${}^{c}_{2}\;30$ (we write “c” for “cube”). Restoring the 52 and adding ${}^{t}_{5}$ to the other part changes the question into one of subtracting 52 from ${}^{c}_{2}\,{}^{t}_{5}\;30$. So far, this is equalization with restoration. Now both parts have simple numbers, 30 and 52, and these should be confronted. Al-Hawārī subtracts the smaller of the two from both sides, making the problem one of subtracting 22 from ${}^{c}_{2}\,{}^{t}_{5}$.

221.8 The step just described is called “confrontation with equalization” (he reverses Ibn al-Bannāʾ’s words) since the 30 is dropped from one part, and is subtracted from 52 in the other. The answer is then ${}^{c}_{2}\,{}^{t}_{5}\;\ell\;22$. In the modern version of this problem we would not need to transform the subtraction of 52 from $2x^3 + 5x + 30$ into the problem of the subtraction of 22 from $2x^3 + 5x$, since for us the latter “problem” $2x^3 + 5x - 22$ is also its solution.

221.13 In this next example, both terms are diminished. To subtract $12\;\ell\;{}^{t}_{4}$ from ${}^{m}_{3}\;\ell\;{}^{t}_{2}$ al-Hawārī adds ${}^{t}_{2}$ and ${}^{t}_{4}$, totaling ${}^{t}_{6}$, to both sides to turn the problem into one of subtracting $12\;{}^{t}_{2}$ from ${}^{m}_{3}\,{}^{t}_{4}$. We can see that he could have just added ${}^{t}_{4}$ to both sides, but our author is following the rule to the letter: add to both sides all amounts that are lacking from diminished terms. From here, confronting the ${}^{t}_{4}$ with the ${}^{t}_{2}$ yields the problem of subtracting 12 from ${}^{m}_{3}\,{}^{t}_{2}$, giving ${}^{m}_{3}\,{}^{t}_{2}\;\ell\;12$.

222.1 The last example is worked out like the previous one. The problem is to subtract ${}^{c}_{1}\;\ell\;{}^{m}_{2}$ from $30\;\ell\;{}^{t}_{4}$. Restoring with equalizing means adding ${}^{t}_{4}\,{}^{m}_{2}$ to both sides to get the equivalent problem of subtracting ${}^{c}_{1}\,{}^{t}_{4}$ from ${}^{m}_{2}\;30$. There is no confrontation to be done here, since all terms are of different species. The answer is then $30\;{}^{m}_{2}\;\ell\;{}^{c}_{1}\;\ell\;{}^{t}_{4}$.

223.1 Subsection concerning examples with the two sides of an equation.

Now al-Hawārī shows how to apply restoration (*al-jabr*) and confrontation (*al-muqābala*) to simplify equations in stage 2 to one of the six types. The phrase *al-jabr wa-l-muqābala*, taken from these steps in stage 2, was the name given to the art of algebra. As mentioned above, some algebraists would shorten the name to just *al-jabr*, and via transliterations in medieval Latin and Italian it became the origin of our word “algebra”.

223.2 The first example is ${}^{m}_{1}\;\ell\;{}^{t}_{3} = 2\;{}^{t}_{1}$. Restoring the ${}^{m}_{1}$ amounts to turning the ${}^{m}_{1}\;\ell\;{}^{t}_{3}$ into ${}^{m}_{1}$. To equalize, ${}^{t}_{3}$ must be added to the $2\;{}^{t}_{1}$, so the equation becomes ${}^{m}_{1} = 2\;{}^{t}_{4}$. This is a type 6 equation, so one should next follow the rule at 216.11 to solve it.

223.7 To simplify ${}^{m}_{1}\;\ell\;{}^{t}_{3} = 24\;\ell\;{}^{t}_{5}$ one restores first. Adding both ${}^{t}_{3}$ and ${}^{t}_{5}$ to both sides gives ${}^{m}_{1}\,{}^{t}_{5} = 24\;{}^{t}_{3}$. Now the ${}^{t}_{5}$ is confronted with the ${}^{t}_{3}$ to get ${}^{m}_{1}\,{}^{t}_{2} = 24$. This is a type 4 equation, and the rule for its solution is given above at 214.7.

Another way to do it is to confront the deleted ${}^{t}_{3}$ with the deleted ${}^{t}_{5}$ on the other side to get a deleted ${}^{t}_{2}$: ${}^{m}_{1} = 24\;\ell\;{}^{t}_{2}$. Now the 24 is restored and ${}^{t}_{2}$ is added to the other side to get ${}^{m}_{1}\,{}^{t}_{2} = 24$.

223.14 The next example is to simplify $\overset{m}{1}\ \ell\ 10 = \overset{m}{1}\ \ell\ \overset{t}{2\frac{1}{2}}$. Adding $10\ \overset{t}{2\frac{1}{2}}$ to both sides gives $\overset{m}{1}\ \overset{t}{2\frac{1}{2}} = \overset{m}{1}\ 10$; confronting the two $\overset{m}{1}$s results in the equation $10 = \overset{t}{2\frac{1}{2}}$. Al-Hawārī cannot begin this one by confronting the $\overset{m}{1}$s, since it would leave terms deleted from nothing on both sides, which would be impossible.

223.17 In the equation $\overset{m}{1}\ 10 = 51\ \ell\ \overset{t}{4}$, adding the $\overset{t}{4}$ to both sides gives $\overset{m}{1}\ \overset{t}{4}\ 10 = 51$, and confronting the 10 with the 51 results in the simplified equation $\overset{m}{1}\ \overset{t}{4} = 41$. Al-Hawārī does not mention the confrontation. The other equations he gives are simplified similarly.

225.1 **Section II.2.4. Multiplication and knowing the power and the term.**

The power of a term in modern algebra is easy to determine, since it is the exponent. The power of x^3 is 3, the power of x^{12} is 12, etc. Knowing the power is necessary for multiplying or dividing terms, such as figuring that $x^2 \cdot x^6 = x^8$ or $6x^3 \div 2x^2 = 3x$. In Arabic algebra the terms are expressed with the names "thing", "*māl*", and "cube", so the exponent must be calculated from these names. Ibn al-Bannāʾ tells us that the power of the things is 1, of *māl*s is 2, and of cubes is 3, and that for higher powers one collects these numbers together. For example, the power of a *māl* cube *māl māl* is $2+3+2+2 = 9$. From 225.2 to the end of the chapter the text has "[a] power (*uss*)" with the implied indefinite article. We translate it as "the power" to make the reading easier, and because the power of a term is unique.

226.1 To find a name of the term from the exponent, one need only come up with some combination of *māl* and cube so the powers add up right. There was no standard way of writing the powers greater than the fourth in Arabic, as we already knew from variations offered by such authors as Abū Kāmil and al-Karajī. Al-Hawārī makes this terminological flexibility explicit.

226.9 To multiply two monomials, multiply their numbers (i.e., coefficients) and add the powers. If one of the terms is a simple number, then the power of the product is the power of the other term.

226.12 Al-Hawārī explains his examples well. In notation they are $\overset{t}{5}$ by $\overset{t}{7}$ gives $\overset{m}{35}$, $\overset{t}{10}$ by $\overset{m}{6}$ gives $\overset{c}{60}$, $\overset{t}{1}$ by $\overset{c}{1}$ gives $\overset{mm}{1}$, 6 by $\overset{m}{4}$ gives $\overset{m}{24}$, and 7 by $\overset{mc}{3}$ gives $\overset{mc}{21}$.

227.10 In Arabic algebra there was no standard way of making the plurals for species expressed with more than one term (*māl*, cube). Al-Hawārī makes the first term plural when there is more than one, but because the plural and singular forms of Arabic nouns are the same for numbers greater than ten, we often cannot tell if the second term is intended to be plural. What appears to be "three *māl*s cube" could be "three *māl*s cubes" if we take the "three *māl*s" to be the number of cubes, and this number is greater than ten. See our remarks above at 70.8 for a similar situation with "thousands". In our translation we leave these potentially plural forms singular. Perhaps Ibn al-Bannāʾ can make all the terms plural at 227.14 because he has not specified the number of the term.

227.14 Here Ibn al-Bannāʾ explains how to reduce the degree of an equation. This passage seems to be misplaced. It fits better in the section below on division.

If every term in an equation has degree of at least one, or, as al-Hawārī says, "you do not have a number", then drop the power of each term by the lowest power in the equation. For a modern example, if we had $4x^3 - 2x^2 = x^5$, the smallest power is 2, so we drop all powers by two to get $4x - 2 = x^3$. Zero cannot be a solution in medieval algebra, so for al-Hawārī this gives an equivalent equation. Again, al-Hawārī explains his three examples well. In notation they are: $\overset{mm}{3} = \overset{c}{4}\,\overset{m}{10}$ reduces to $\overset{m}{3} = \overset{t}{4}\,10$, $\overset{c}{3} = \overset{m}{10}\,\overset{t}{20}$ reduces to $\overset{m}{3} = \overset{t}{10}\,20$, and $\overset{c}{1}\,\overset{m}{10} = \overset{t}{39}$ reduces to $\overset{m}{1}\,\overset{t}{10} = 39$.

228.8 Before Italian algebraists in the sixteenth century found solutions to irreducible cubic and quartic equations, the only equation types that could be solved generally were those that simplify, or could in some way be reduced, to one of the six types of degree 1 or 2.

228.9 When multiplying polynomials, some terms are appended and some are deleted. For example, in "a hundred and a *māl* less twenty things" ($100\,\overset{m}{1}\,\ell\,\overset{t}{20}$) the "hundred" and the "*māl*" are appended, while the "twenty things" is deleted. Be careful not to interpret "appended" and "deleted" as "positive" and "negative". See the discussions at 219.1 and 219.4 above.

Al-Hawārī does not explain why a deleted term multiplied by a deleted term yields an appended term, but the rule was well known. It would have been easy to derive it by geometry, among other possible ways. Abū Kāmil, in fact, gives a geometric proof in the context of the multiplication of "ten dirhams less a thing by ten dirhams less a thing" ($10\,\ell\,\overset{t}{1}$ by $10\,\ell\,\overset{t}{1}$).[109]

228.15 The example in notation is to multiply $8\,\ell\,\overset{t}{2}$ by $7\,\ell\,\overset{m}{4}$. First al-Hawārī multiplies the terms that will give appended amounts: 8 by 7 gives 56, and $\overset{t}{2}$ by $\overset{m}{4}$ gives $\overset{c}{8}$. The other two products yield deleted terms: 8 by $\overset{m}{4}$ gives $\overset{m}{32}$ and $\overset{t}{2}$ by 7 gives $\overset{t}{14}$. The product is then $\overset{c}{8}\,56\,\ell\,\overset{t}{14}\,\ell\,\overset{m}{32}$.

229.1 **Section II.2.5. Division.**

229.2 Ibn al-Bannāʾ first explains how to divide a monomial by a monomial of lower degree. Al-Hawārī's first example is to divide $\overset{m}{10}$ by $\overset{t}{2}$. The 10 is divided by 2 to get 5. The power of the *māl*s is 2, and the power of the things is 1, so the power of the quotient is their difference, 1. The answer is then "five things". His other example is to divide $\overset{c}{15}$ by $\overset{t}{3}$, from which he gets $\overset{m}{5}$.

229.8 Note the wording: "The resulting five is then five *māl*s". In this term the "*māl*s" is the kind, and the "five" tells how many there are. The two words are the two aspects of a single number or value, like saying "five dollars". When there is only one of them, the

[109] (Abū Kāmil 2012, 293.7).

notation must still specify the number. If, instead of “$\overset{m}{1}\ \overset{t}{5}$” we write “m $\overset{t}{5}$”, it would be like saying “dollars and five pesos”. This leaves unanswered how many dollars/*māl*s there are. For this reason, throughout premodern algebra a “1” was always written in notation when there is one of a term. By contrast, the 5 and the x^2 in our $5x^2$ are both numbers, one known and the other not, that are multiplied together. Our x^2 itself, without any coefficient, already designates a value, and so does not need a 1 in front.[110]

229.12 In the case that the powers of the two terms are the same, the result is a number. Al-Hawārī’s example is to divide $\overset{m}{12}$ by $\overset{m}{3}$ to get 4.

229.17 Dividing by a number gives the same power as the dividend. The example is to divide $\overset{t}{12}$ by 4 to get $\overset{t}{3}$. This and the previous rule are given separately because the power of a number was not regarded as being zero. Zero was a place holder in Indian numerical notation, and not a number.

230.1 If the dividend is diminished, divide both terms by the divisor. By this rule, dividing $\overset{c}{12}\ \ell\ \overset{m}{3}$ by $\overset{t}{2}$ gives $\overset{m}{6}\ \ell\ \overset{t}{\frac{1}{2}1}$. A second example is dividing $\overset{m}{10}\ \ell\ \overset{t}{3}$ by 2 to get $\overset{m}{5}\ \ell\ \overset{t}{\frac{1}{2}1}$. Our authors do not mention that the same rule works if “less” is replaced by “and”, but this may be because the procedure is easily understood in the case of terms gathered together.

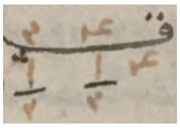
230.11 Ibn al-Bannāʾ writes that one cannot divide a lower power by a higher power. But one can at least eliminate common divisors. The example is to divide $\overset{m}{6}$ by $\overset{c}{3}$ to get “two divided by a thing”. The division is still unresolved, but at least the common divisors have been dealt with. Ibn al-Bannāʾ explains the same rule in his book on algebra. In one example he writes “divide eight cubes by three *māl*s *māl*”. The answer is given as “two and two thirds divided by a thing”.[111]

When working with polynomials, the three operations of addition, subtraction, and multiplication can all be performed to yield another polynomial. But the division of a polynomial by a polynomial often cannot be performed, which is why an algebraist would sometimes admit an unresolved division in an equation. But writing the result of a division in the form “*A* divided by *B*” was generally avoided in algebraic problem-solving. Ibn al-Bannāʾ’s problem (I.10) from his book on algebra is a good example. He solves this particular problem five different ways, each time coming up with a clever way to avoid unresolved divisions. (The enunciation and first solution are translated as problem [6] in Appendix B.) Al-Karajī solves this same problem, but with the number $2\frac{1}{6}$ in place of $4\frac{1}{4}$. Only in his fourth and last solution does he set up the equation with divisions: “ten less a thing divided by a thing and a thing divided by ten less a thing, and that equals two dirhams and a sixth”.[112]

In modern notation, al-Karajī’s equation is $\frac{10-x}{x} + \frac{x}{10-x} = 2\frac{1}{6}$. We know only one book that shows notation for divisions of algebraic expressions. In place of the division bar, Ibn Ghāzī writes them with an elongated version of the Arabic letter *qāf* (ق), the first letter

110 See (Oaks 2009) and (Oaks 2017) for more on this difference.
111 (Saidan 1986, 541.2).
112 (Saidan 1986, 213.2).

in the word *qasama* ("divide"). Here is the notational version of his "twenty-four divided by half a thing and four and a half in number", followed by our transcription (with a "d" for "divided by"), and then the modern version:



$$d\frac{24}{4\frac{1}{2}\quad \overset{t}{\tfrac{1}{2}}} \qquad \frac{24}{4\frac{1}{2}+\frac{1}{2}x}$$

The manuscript indicates "things" with only the three dots from the letter ش. The similarity between the simple bar for fractions and the elongated *qāf* for algebraic expressions is deliberate, but because divisions are not fractions they are differentiated.

230.16 One cannot perform a division if the divisor is diminished. To "divide ten *māl*s by three less a thing" can only yield "ten *māl*s divided by three less a thing". Of course, one cannot perform the division if the divisor consists of two or more appended terms, either. Dividing "ten *māl*s" by "three and a thing" can only give "ten *māl*s divided by three and a thing". Neither Ibn al-Bannāʾ nor al-Hawārī make note of this.

231.1 **Section [on secret numbers].**

It was not uncommon for authors of arithmetic books to include problems on finding secret numbers. Other medieval authors who did so include Ibn al-Yāsamīn (d. 1204), Ibn al-Hāʾim (1389), and Ibn Haydūr (ca. 1400). We have not found the particular problems dictated to al-Hawārī in other books. It is probably by intent that there is one secret number in the first problem, two secret numbers in the second problem, and three secret numbers in the third problem. We describe the conditions using modern algebraic notation, in which the secret numbers are x, y, and z, and a and b are calculated values revealed to the guesser. Keep in mind that there is no algebra (*al-jabr wa-l-muqābala*) in the questions or solutions.

231.4 First problem: the person is asked to secretly think of a number, x, and then to calculate x^2 and $(10-x)^2$. If $(10-x)^2 < x^2$, the remainder $a = x^2 - (10-x)^2$ is revealed. Then $x = \frac{1}{2}(\frac{a}{10} + 10)$. If $x^2 < (10-x)^2$ then $a = (10-x)^2 - x^2$ is revealed, and $x = \frac{1}{2}(10 - \frac{a}{10})$.

231.13 Second problem: the person is asked to secretly think of two numbers, x and y, that add to 10. This time the revealed number is $a = x^2/xy$, which of course is the same as x/y. Given the sum and ratio of two numbers, the numbers are also given. Al-Hawārī does not give the rule past this point. We can solve our equations to get $x = \frac{10a}{1+a}$, but if the person chose whole numbers they are easy to guess without any particular rule. If the revealed number is 9 or 1/9, the numbers are 1 and 9. If they are 4 or 1/4, the numbers are 2 and 8, etc.

232.1 Third problem: A secret number is divided into two other secret numbers. In notation, suppose $z = x + y$ with $x > y$. Two numbers are revealed: $a = xy - y^2$ and $b = x^2 - xy$. Then $x - y = \sqrt{b-a}$ and $z = x + y = \frac{a+b}{x-y}$, so $2x = (x+y) + (x-y)$ and $2y = (x+y) - (x-y)$.

233.2 The date 18 Dhū al-Qaʿda 704 AH corresponds to the Julian date June 12, 1305.

Appendices

A. Conspectus of problems

For the following list of calculations and problems in al-Hawārī's book, we have adopted our transcription of the Arabic notation. This is explained in our commentary, especially at 219.1. We had introduced the algebraic notation in our commentary at 211.2, and notation for fractions is explained in our commentary in the chapter on fractions, beginning at 135.1. Here is a quick guide to the notation:

- The notation is a transcription of how it appears, or in some cases would have appeared, in the manuscripts. With the exception of the figures for double false position, we put the notation in red, as it is in many Arabic manuscripts. It should be read right to left.

- The reversed letter "ﻻ" stands for "less", and indicates that the number to its left is removed from the number to its right. So, an apotome that we would write as $\sqrt{5} - \sqrt{3}$ is shown as $\sqrt{3}$ ﻻ $\sqrt{5}$. See our commentary at 86.1. Binomials have no sign for "+", so the modern $5 + \sqrt{21}$ is shown as $\sqrt{21}\ 5$.

- Instead of the letter *jīm* above a number to indicate square root, we use the modern "$\sqrt{\ }$" which functions similarly.

- We write "=" for the "equals" in algebraic equations, and for no other purpose. This sign functions like the elongated *lām*, its counterpart in Arabic manuscripts.

- Because the transcribed notation for algebra is entirely different from our notation, we include the versions of the calculations in modern notation as well, even if it sometimes makes little sense (as in the calculations from 219.2 through 222.1).

- An "L" is placed after the reference for examples taken from Ibn al-Bannāʾ's *Lifting the Veil*.

Part 1. On known numbers

Chapter 1. On whole numbers

Passage	Example
65.2	Examples of whole numbers: 15, 18, etc.
65.2	Examples of fractions: $\frac{1}{2}$, $\frac{1\ 3}{2\ 8}$, $\frac{1\ 1}{4\ 9}$, $\frac{7 \bullet 6}{8 \bullet 7}$, $\frac{1}{9}$ ﻻ $\frac{5}{6}$

Passage	Example
65.6,10,17, 66.1	Even: 10, 50; evenly-even: 32; evenly-odd: 14; evenly-evenly-odd: 28
66.7	Odd prime: 11, 29; Oddly-odd: 15, composed of 3 by 5
66.17	Even square: 36 is 6 by 6
67.3	Even, composed of two unequal numbers (a surface): 18 is 3 by 6, and also 2 by 9
67.3	Even, composed of three unequal numbers (a surface): 24 is 3 by 4 by 2
67.12	Even cube: 64 is 4 by 4 by 4
67.19	Odd square: 25 is 5 by 5
68.1	Odd, composed of two unequal numbers (a surface): 35 is 5 by 7; 105 is 3 by 5 by 7
68.8	Odd cube: 27 is composed of 3 by 3 by 3
70.8, 70.23	Sample figures for numbers: 9367184225 and 84725
71.6	143 has three places
71.16	The rank of 10000000 is 8
72.4	The name of 1000000000 is thousands thousands of thousands
74.17	Adding from the units place: add 4043 to 2685 to get 6728
75.9	Adding from the highest place: add 978 to 456 to get 1434
75.20	The most one can get by adding is one extra place: add 9 to 9 to get 18
76.15	Chessboard: add the first sixteen squares to get 65535
77.11	Chessboard: add the first eight squares with 4 as the first square to get 1020
78.4	Adding five numbers with a ratio of $\frac{2}{3}$: add 16, 24, 36, 54, 81 to get 211
79.4	Adding six numbers starting at 10, with a difference of three: add 10, 13, 16, 19, 22, 25 to get 105
79.13	Add 1, 2, 3, …, 10 to get 55

Passage	Example
79.18	Add the squares of 1, 2, 3, …, 10 to get 385
80.1	Add the cubes of 1, 2, 3, …, 10 to get 3025
80.5	Add 1, 3, 5, 7, 9 to get 25
80.10	Add the squares of 1, 3, 5, 7, 9 to get 165
80.15	Add the cubes of 1, 3, 5, 7, 9 to get 1225
80.20	Add 2, 4, 6, 8, 10 to get 30
81.4	Add the squares of 2, 4, 6, 8, 10 to get 220
81.9	Add the squares of 2, 4, 6, 8, 10, 12 to get 364
81.15	Add the cubes of 2, 4, 6, 8, 10 to get 1800
83.16	Subtracting from the highest place: subtract 4968 from 5035 to get 67
84.13	Subtracting from the units place: subtract 3469 from 6543 to get 3074
85.8	The most one can get by subtracting is one fewer place: subtract 1 from 10 to get 9
86.9 L	2 ⸝ 5 ⸝ 7 ⸝ 8 ⸝ 10 is 6
87.19	Casting out nines from 6435 gives nothing
88.5	Casting out eights from 5393 gives 1
89.4	Casting out sevens from 23786435 gives 1
90.7	Casting out sevens from 58064 gives 6
91.4	Add 43 to 64 to get 107 Cast out sevens to check: add 1 to 1 to get 2
91.16	Subtract 74 from 96 to get 22 Cast out sevens to check: subtract 4 from 5 to get 1
92.10	Multiply 12 by 16 to get 192 Cast out sevens to check: multiply 5 by 2 to get 3
92.17 L	Multiply $\frac{1}{3}$ by $\frac{1}{4}14$ to get $\frac{3}{4}4$ Cast out sevens to check: multiply $\frac{1}{3}$ by $\frac{1}{4}$ to get $\frac{1\ 0}{3\ 4}$

Passage	Example
93.10	Divide 1488 by 12 to get 124 Cast out sevens to check: multiply 5 by 5 to get 4
93.15 L	Divide $\frac{3}{4}\,\frac{5}{6}$ by $\frac{1}{2}$ to get $\frac{1}{6}3$. Cast out sevens to check: multiply $\frac{5}{6}$ by $\frac{1}{2}$, then convert to 4ths of 6ths to get $\frac{3\ 0}{4\ 6}$
94.1	Denominate 11 with 15 to get $\frac{2\ 3}{3\ 5}$. Cast out sevens to check: multiply 4 by 1 to get 4
94.5 L	Denominate $\frac{2\ 2}{3\ 6}$ with $\frac{1\ 5}{3\ 8}$ to get $\frac{2}{3}$. Cast out sevens to check: multiply 2 by 2, adjust for the denominators to get 3
95.10 L	Meanings of multiplication: 3 men, each has 5 dirhams; 5 dirhams, how many thirds?
96.3	Multiplication by shifting: multiply 43 by 54 to get 2322
97.4	Vertical multiplication: multiply 42 by 37 to get 1554
98.15	Multiplication by half-shifting: multiply 463 by itself to get 214369
100.5	Lattice multiplication: multiply 435 by 287 to get 124845
102.1	Vertical multiplication (no shifting): multiply 183 by 347 to get 63501
104.10	Sleeper multiplication (no shifting): multiply 253 by 987 to get 249711
107.6	Multiply 444 by 333 to get 147852
108.13	Multiplication by excess: multiply 12 by 15 to get 180
109.1	Multiply 13 by 17 to get 221
109.10	Multiplication by denomination: multiply 6 by 12 to get 72
110.6	Another method of multiplication by denomination: multiply 24 by 8 to get 192
110.12	Multiply 12 by 15 to get 180
111.1	Multiply 3 by 15 to get 45
111.15	Multiplication by nines: multiply 444 by 999 to get 443556

Passage	Example
112.10	Another method of multiplication by nines: multiply 999 by 9354 to get 9344646
113.1	Multiplication by squaring: multiply 17 by 19 to get 323
113.9	Another squaring method: multiply 25 by 15 to get 375
113.19	Another squaring method: multiply 36 by 14 to get 504
114.8	Multiplication with zeros: multiply 30 by 140 to get 4200
117.16 L, 118.1 L	Meanings of division: Divide 15 dirhams among 3 men; Divide a piece of wood of 15 spans by a piece of wood of 3 spans
119.1	Divide 245 by 12 to get $\frac{1\ \ 2}{2\ \ 6}\,20$
120.5	Divide 44 among 11 men to get 4
120.10	Divide 96 by 12 to get 8
120.16	Divide 35 by 15 to get $\frac{1}{3}2$
121.4	Apportionment. Wealth of donors: 4, 5, 6 dinars, amount of loan: 10 dinars
122.5	Apportionment. Wealth of donors: $\frac{1}{3}4$, $\frac{1}{4}5$, $\frac{1}{6}6$ dinars, amount of loan: 12 dinars
123.22	Common denomination: denominate 11 with 15 to get $\frac{2\ \ 3}{3\ \ 5}$
124.12,14,17	Other denominations: denominate 4 with 12 to get $\frac{1}{3}$; denominate 9 with 15 to get $\frac{3}{5}$; denominate 10 with 16 to get $\frac{5}{8}$
124.20ff	Finding divisors: 50, 36, 66, 42, 64, 68, 14, 26, 81, 39, 123, 77, 221
129.6	Restore 8 to 19; reduce 50 to 6
129.8	Restore 3 to 6: divide 6 by 3 to get 2
129.12	Reduce 8 to 3. Denominate 3 with 8 to get $\frac{3}{8}$

Part 1, Chapter 2. On fractions

Passage	Example
134.2 L	Language of parts: $\frac{1}{11}$, $\frac{1}{17}$
134.8	Numerator < denominator: $\frac{1}{4}$, $\frac{2}{4}$, $\frac{3}{4}$; and $\frac{1}{7}$ through $\frac{6}{7}$
135.1	Fractions with two or more names: $\frac{1\ 2}{7\ 8}$; $\frac{1\ 2\ 4\ 8}{3\ 6\ 7\ 9}$
135.10 L	A related fraction: $\frac{2\ 4\ 5}{3\ 5\ 6}$
136.8 L	A distinct fraction: $\frac{4}{5}\ \frac{5}{6}$
137.1 L	A portioned fraction: $\frac{5\ \bullet\ 3}{6\ \bullet\ 4}$
137.13	The numerator of $\frac{1}{7}$ is 1
138.5	The numerator of $\frac{2\ 3\ 4\ 5}{3\ 5\ 7\ 8}$ is 596
139.2	The numerator of $\frac{4}{6}\ \frac{1\ 5}{2\ 7}$ is 122
139.11	The numerator of $\frac{3\ \bullet\ 5\ \bullet\ 7}{10\ \bullet\ 6\ \bullet\ 9}$ is 105
140.8	The numerator of $\frac{1}{9}$ ⟍ $\frac{6}{8}$ is 46
141.1	The numerator of $\frac{1}{3}$ ⟍ $\frac{1\ 6}{2\ 7}$ is 26
141.11	The numerator of the connected fraction $\frac{1}{5}$ ⟍ $\frac{1}{7}$ ⟍ $\frac{1}{4}$ ⟍ $\frac{1}{3}$ 5, also written $\frac{1}{5}\ \frac{1}{7}\ \frac{1}{4}$ ⟍ $\frac{1}{3}$ 5, is 912
142.6	The numerator of the distinct fractions $\frac{1}{5}$ ⟍ $\frac{1}{7}$ ⟍ $\frac{1}{4}$ ⟍ $\frac{1}{3}$ 5 is 1991
143.5	The numerator of $\frac{3\ 5}{4\ 6}$ 5 is 143
143.13	The numerator of 10 $\frac{6}{8}\ \frac{4}{7}$ is 740
144.10	The numerator of $\frac{3}{6}$ 5 $\frac{4}{9}$ is 147
145.4	The numerator of $\frac{4}{7}$ 7 $\frac{2}{3}$ is 106
147.4	Add $\frac{6}{8}\ \frac{4}{5}$ 3 to $\frac{1\ 3\ \ 4}{2\ 8\ 10}$ to get $\frac{1\ 7\ \ 9}{2\ 8\ 10}$4
147.15	Subtract $\frac{1}{3}$ ⟍ 2 $\frac{7}{10}$ from $\frac{5\bullet 3}{6\bullet 4}$ 4 to get $\frac{1\ 3\ \ 5}{2\ 6\ 10}$3
149.8	Multiply $\frac{1\ 3}{3\ 4}$ by $\frac{1\ 4\ 3}{5\ 6\ 9}$ to get $\frac{1\ 0\ 0\ 4}{4\ 5\ 6\ 9}$
150.2	Multiply $\frac{1}{8}$ 4 $\frac{1}{3}$ by 10 $\frac{2\bullet 1}{3\bullet 5}$ to get $\frac{5}{6}$1
151.4	Divide $\frac{1}{3}$ 6 by 3 $\frac{7\bullet 4}{8\bullet 5}$ to get $\frac{1\ 0}{7\ 9}$ 3

Passage	Example
151.14	Denominate $\frac{2}{9}$ ﻻ $\frac{1}{4}3$ with $\frac{3}{5}\,\frac{2}{8}\,6$ to get $\frac{5\ \ 50}{9\ 137}$
152.11	Divide $\frac{2\ \ 9}{3\ 10}8$ by $\frac{1\ \ 5}{3\ 10}$ to get $\frac{1\ 6}{2\ 8}16$
153.3	Denominate $\frac{1}{3}2$ with $\frac{2}{3}6$ to get $\frac{1\ \ 3}{2\ 10}$
153.12 L	Divide 5 by $\frac{5}{6}$ to get 6; denominate $\frac{5}{6}$ with 5 to get $\frac{1}{6}$
154.6ff	Restore $\frac{1}{2}$ to $\frac{9}{10}$; $\frac{1\ 2}{2\ 7}$ to $\frac{1}{2}5$; $\frac{5\bullet 2}{7\bullet 3}$ to 10; 5 to $\frac{4}{6}10$; $\frac{1\ \ 3}{2\ 10}4$ to 8; $\frac{1}{3}$ ﻻ $\frac{3}{5}3$ to $\frac{3}{5}12$
155.11ff	Reduce $\frac{7}{10}$ to $\frac{1}{3}$; 8 to $\frac{1}{2}2$; 10 to $\frac{3}{4}$; $\frac{1}{4}7$ to $\frac{4}{6}3$; $\frac{4\ \ 9}{7\ 10}11$ to 5; $\frac{1}{3}2$ to $\frac{1}{9}$
157.2 L	Convert $\frac{3}{4}\,\frac{5}{6}$ to tenths. Answer: $\frac{5\ \ 5}{6\ 10}1$
157.12 L	How many tenths are in $\frac{3}{4}\,\frac{5}{6}$? Again, it is $\frac{5\ \ 5}{6\ 10}1$
158.1 L	How many tenths are in 5? Answer: 50
158.11	How many ninths are in $\frac{4}{10}\,\frac{6}{8}$? Answer: $\frac{4\ \ 3\ \ 1}{8\ 10\ 9}1$

Part 1, Chapter 3. On roots

Passage	Example
163.4 L	Examples: $\sqrt{10}$, $\sqrt{\frac{1}{2}}$, $\sqrt{\frac{1}{2}10}$, $\sqrt{\sqrt{10}}$
166.13	$\sqrt{625}$ is 25
167.8	$\sqrt{20}$ is approximately $\frac{1}{2}4$
167.14	$\sqrt{54}$ is approximately $\frac{1\ 2}{2\ 7}7$
168.1	$\sqrt{92}$ is approximately $\frac{3}{5}9$
168.16	$\sqrt{92}$ is approximately $\frac{1\ 5\ \ 5}{2\ 6\ 10}9$
169.4	$\sqrt{12}$ is approximately $\frac{1\ 3}{4\ 7}3$
169.10,17 L	$\sqrt{625}$ is 25; $\sqrt{729}$ is 27
170.6	$\sqrt{100}$ is 10
170.16	$\sqrt{\frac{1\ 4}{6\ 6}}$ is $\frac{5}{6}$

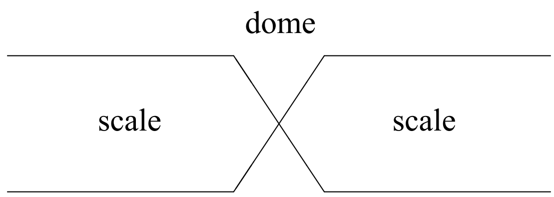
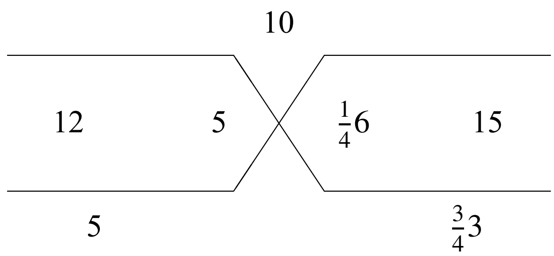
Passage	Example
171.1	$\sqrt{\tfrac{1}{4}12}$ is $\tfrac{1}{2}3$
171.6	$\sqrt{\frac{3\;4}{6\;9}}$ is $\frac{1\;5\;3\;13}{2\;6\;9\;19}$
171.15	$\sqrt{\frac{1\;7}{2\;8}10}$ is approximately $\frac{1\;2\;16}{4\;8\;53}3$
172.6	$\sqrt{\frac{1\;7}{2\;8}10}$ is approximately $\frac{4}{13}3$
172.12	$\sqrt{\frac{1\;4}{2\;7}}$ is approximately $\frac{3\;5\;8}{4\;7\;11}$
173.13 L	Binomials: $\sqrt{3}\;5$; $\sqrt{3}\;\sqrt{5}$
173.16 L	Apotomes: $\sqrt{3} \smallsetminus 5$; $\sqrt{3} \smallsetminus \sqrt{5}$
174.5	1st & 4th binomials: $\sqrt{21}\;5$, $\sqrt{2}\;2$
174.8	2nd & 5th binomials: $\sqrt{45}\;5$, $\sqrt{72}\;5$
174.11	3rd & 6th binomials: $\sqrt{18}\;\sqrt{10}$, $\sqrt{8}\;\sqrt{7}$
175.11	$\sqrt{\sqrt{60}\;8}$ is $\sqrt{3}\;\sqrt{5}$
175.19	$\sqrt{\sqrt{60} \smallsetminus 8}$ is $\sqrt{3} \smallsetminus \sqrt{5}$
176.6	$\sqrt{\sqrt{55}\;8}$ is $\sqrt{\tfrac{1}{2}2}\;\sqrt{\tfrac{1}{2}5}$; $\sqrt{\sqrt{55} \smallsetminus 8}$ is $\sqrt{\tfrac{1}{2}2} \smallsetminus \sqrt{\tfrac{1}{2}5}$
176.10	$\sqrt{\sqrt{112}\;7}$ is $\sqrt{\sqrt{\tfrac{3}{4}1}}\;\sqrt{\sqrt{\tfrac{3}{4}85}}$; $\sqrt{\sqrt{112} \smallsetminus 7}$ is $\sqrt{\sqrt{\tfrac{3}{4}1}} \smallsetminus \sqrt{\sqrt{\tfrac{3}{4}85}}$
176.15	$\sqrt{\sqrt{14}\;\sqrt{32}}$ is $\sqrt{\sqrt{\tfrac{1}{2}}}\;\sqrt{\sqrt{\tfrac{1}{2}24}}$; $\sqrt{\sqrt{14} \smallsetminus \sqrt{32}}$ is $\sqrt{\sqrt{\tfrac{1}{2}}} \smallsetminus \sqrt{\sqrt{\tfrac{1}{2}24}}$
176.20	$\sqrt{\sqrt{30}\;7}$ is $\sqrt{\sqrt{\tfrac{3}{4}4} \smallsetminus \tfrac{1}{2}3}\;\sqrt{\sqrt{\tfrac{3}{4}4}\;\tfrac{1}{2}3}$; $\sqrt{\sqrt{30} \smallsetminus 7}$ is $\sqrt{\sqrt{\tfrac{3}{4}4} \smallsetminus \tfrac{1}{2}3} \smallsetminus \sqrt{\sqrt{\tfrac{3}{4}4}\;\tfrac{1}{2}3}$
177.5	$\sqrt{\sqrt{20}\;3}$ is $\sqrt{\sqrt{\tfrac{1}{4}\;\tfrac{1}{2}2} \smallsetminus \sqrt{5}}\;\sqrt{\sqrt{\tfrac{1}{4}\;\tfrac{1}{2}2}\;\sqrt{5}}$; $\sqrt{3 \smallsetminus \sqrt{20}}$ is $\sqrt{\sqrt{\tfrac{1}{4}\;\tfrac{1}{2}2} \smallsetminus \sqrt{5}} \smallsetminus \sqrt{\sqrt{\tfrac{1}{4}\;\tfrac{1}{2}2}\;\sqrt{5}}$
177.11	$\sqrt{\sqrt{11}\;\sqrt{10}}$ is $\sqrt{\tfrac{1}{2} \smallsetminus \sqrt{\tfrac{3}{4}2}}\;\sqrt{\sqrt{\tfrac{3}{4}2}\;\tfrac{1}{2}}$; $\sqrt{\sqrt{10} \smallsetminus \sqrt{11}}$ is $\sqrt{\tfrac{1}{2} \smallsetminus \sqrt{\tfrac{3}{4}2}} \smallsetminus \sqrt{\sqrt{\tfrac{3}{4}2}\;\tfrac{1}{2}}$

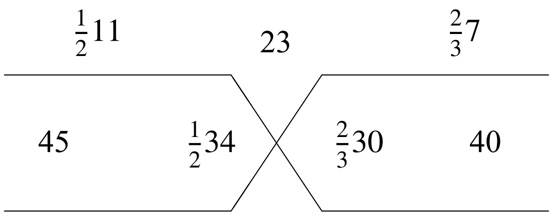
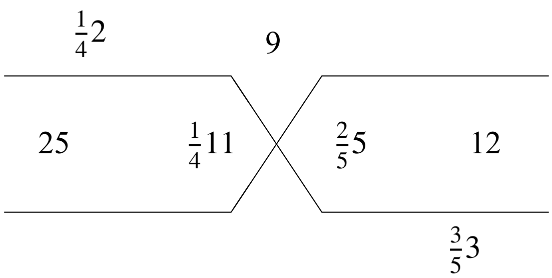
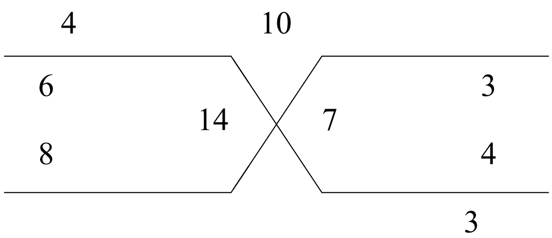
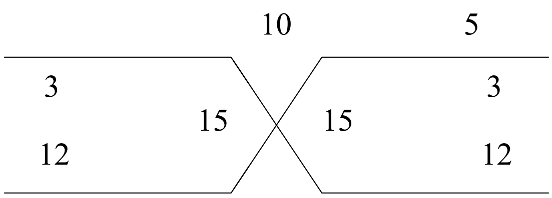
Passage	Example
179.7,11	Add $\sqrt{3}$to $\sqrt{27}$ to get $\sqrt{48}$
179.16	Add $\sqrt{2}$ to $\sqrt{8}$ to get $\sqrt{18}$
179.20	Add half of $\sqrt{20}$ to two $\sqrt{5}$s to get $\sqrt{45}$
180.10	Add $\sqrt{3}$ to $\sqrt{15}$ to get $\sqrt{15}\ \sqrt{3}$
180.15	Add half of $\sqrt{\sqrt{80}}$ to $\frac{1 \bullet 1}{4 \bullet 3}$ of $\sqrt{684}$ to get $\sqrt{\sqrt{\frac{1\ 4}{2\ 8}22}}\ \sqrt{\sqrt{5}}$
181.6	Subtract $\sqrt{8}$ from $\sqrt{32}$ to get $\sqrt{8}$
181.12,16	Subtract $\sqrt{12}$ from $\sqrt{27}$ to get $\sqrt{3}$
182.4	Subtract $\sqrt{8}$ from $\sqrt{10}$ to get $\sqrt{8} \wr \sqrt{10}$
183.4	Multiply $\sqrt{8}$ by $\sqrt{9}$ to get $\sqrt{72}$
183.7	Multiply $\sqrt{\sqrt{5}}$ by $\sqrt{\sqrt{7}}$ to get $\sqrt{\sqrt{35}}$
183.11	Multiply $\sqrt{\sqrt{\sqrt{3}}}$ by $\sqrt{\sqrt{\sqrt{8}}}$ to get $\sqrt{\sqrt{\sqrt{24}}}$
183.15	Multiply 3 by $2 \wr \sqrt{7}$ to get $6 \wr \sqrt{63}$
184.1	Multiply 3 by $\sqrt{7}$ to get $\sqrt{63}$
184.4	Multiply 2 by $\sqrt{\sqrt{3}}$ to get $\sqrt{\sqrt{48}}$
184.11	Multiply 2 by two $\sqrt{7}$s to get $\sqrt{112}$
184.18	Multiply 5 by three $\sqrt{\sqrt{2}}$s to get $\sqrt{\sqrt{101250}}$
185.6	Multiply $\frac{2}{3}$ by half of $\sqrt{20}$ to get $\sqrt{\frac{2}{9}2}$
185.12	Multiply $\sqrt{5}$ by half of $\sqrt{\sqrt{40}}$ to get $\sqrt{\frac{1}{2}62}$
186.1	Duplicate $\sqrt{3}$ twice to get $\sqrt{12}$
186.4	Duplicate $\sqrt{7}$ five times to get $\sqrt{175}$
186.8	Half of $\sqrt{10}$ is $\sqrt{\frac{1}{2}2}$
186.11	$\frac{4 \bullet 1}{8 \bullet 3}$ of $\sqrt{\sqrt{60}}$ is $\sqrt{\sqrt{\frac{1\ 2\ 0}{2\ 6\ 9}}}$
187.4	Divide $\sqrt{20}$ by $\sqrt{3}$ to get $\sqrt{\frac{2}{3}6}$
187.7	Denominate $\sqrt{3}$ with $\sqrt{8}$ to get $\sqrt{\frac{3}{8}}$

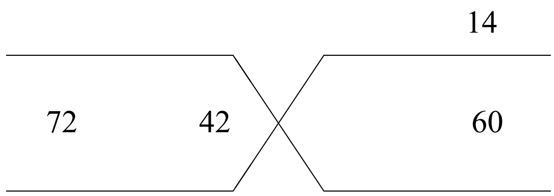
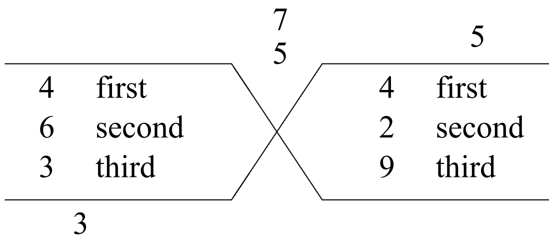
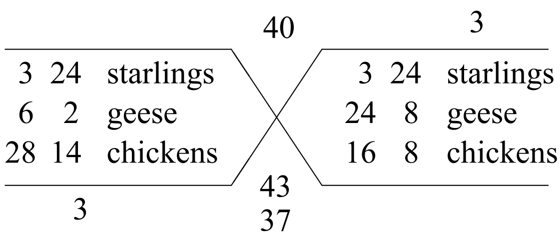
Passage	Example
187.10	Divide $\sqrt{\sqrt{6}}$ by $\sqrt{\sqrt{2}}$ to get $\sqrt{\sqrt{3}}$
187.14	Denominate $\sqrt{\sqrt{18}}$ with $\sqrt{\sqrt{32}}$ to get $\sqrt{\sqrt{\frac{1\ 4}{2\ 8}}}$
188.6	Divide $\sqrt{\sqrt{14}}$ by $\sqrt{2}$ to get $\sqrt{\sqrt{\frac{1}{2}3}}$
188.11	Divide two $\sqrt{15}$s by 2 to get $\sqrt{15}$
188.15	Divide half of $\sqrt{24}$ by $\sqrt{2}$ to get $\sqrt{3}$
189.1	Divide 12 by $\sqrt{3}$ 5 to get $\sqrt{\frac{9\ 9}{11\ 11}}$ − $\frac{8}{11}2$
189.11	Divide 10 by $\sqrt{7}$ − 3 to get $\sqrt{175}$ 15

Part 2. Finding unknown numbers

Chapter 1. Solving problems by proportion

Passage	Example
195.16	Example of four proportional numbers: 3 : 6 :: 4 : 8
198.7 L	
199.1	

Passage	**Example**
200.1	
201.1	
202.3	
203.1	

Passage	Example
203.11 L	
205.5 L	
207.6 L	

Part 2, Chapter 2. Solving problems by algebra

Passage	Example
211.15	Simple equations: $\frac{t}{7} = \frac{m}{3}$; $20 = \frac{m}{5}$; $12 = \frac{t}{3}$ ($3x^2 = 7x$; $5x^2 = 20$; $3x = 12$)
212.6	Composite equations: $24 = \frac{t}{10}\ \frac{m}{1}$; $\frac{t}{5} = 4\ \frac{m}{1}$; $5\ \frac{t}{4} = \frac{m}{1}$ ($x^2 + 10x = 24$; $x^2 + 4 = 5x$; $x^2 = 4x + 5$)
213.7	Solve $\frac{t}{15} = \frac{m}{3}$ to get $\frac{t}{1}$ is 5 and $\frac{m}{1}$ is 25 ($3x^2 = 15x \Rightarrow x = 5$, $x^2 = 25$)

Passage	Example
213.13	Solve 18 = $^{m}_{2}$ to get $^{m}_{1}$ is 9 and $^{t}_{1}$ is 3 $(2x^2 = 18 \Rightarrow x^2 = 9,\ x = 3)$
214.1	Solve 20 = $^{t}_{5}$ to get $^{t}_{1}$ is 4 and $^{m}_{1}$ is 16 $(5x = 20 \Rightarrow x = 4,\ x^2 = 16)$
214.9	Solve 15 = $^{t}_{2}$ $^{m}_{1}$ to get $^{t}_{1}$ is 3 and $^{m}_{1}$ is 9 $(x^2 + 2x = 15 \Rightarrow x = 3,\ x^2 = 9)$
215.6	Solve $^{t}_{6}$ = 8 $^{m}_{1}$ to get $^{t}_{1}$ is 4 and $^{m}_{1}$ is 16, or $^{t}_{1}$ is 2 and $^{m}_{1}$ is 4 $(x^2 + 8 = 6x \Rightarrow x = 4,\ x^2 = 16$ or $x = 2,\ x^2 = 4)$
215.14	Solve $^{t}_{6}$ = 9 $^{m}_{1}$ to get $^{m}_{1}$ is 9 and $^{t}_{1}$ is 3 $(x^2 + 9 = 6x \Rightarrow x^2 = 9,\ x = 3)$
216.13	Solve 3 $^{t}_{2}$ = $^{m}_{1}$ to get $^{t}_{1}$ is 3 and $^{m}_{1}$ is 9 $(x^2 = 2x + 3 \Rightarrow x = 3,\ x^2 = 9)$
217.10	36 = $^{t}_{6}$ $^{m}_{2}$ simplifies to 18 = $^{t}_{3}$ $^{m}_{1}$ $(2x^2 + 6x = 36 \Rightarrow x^2 + 3x = 18)$
218.1	6 = $^{t}_{2}$ $^{m}_{\frac{1}{2}}$ simplifies to 12 = $^{t}_{4}$ $^{m}_{1}$ $(\frac{1}{2}x^2 + 2x = 6 \Rightarrow x^2 + 4x = 12)$
219.2	Add $^{m}_{1}$, $^{t}_{6}$, and 10 to get 10 $^{t}_{6}$ $^{m}_{1}$ $(6x + x^2 + 10 \Rightarrow x^2 + 6x + 10)$
219.5 L	Add $^{t}_{1}$ ⟍ $^{m}_{1}$ to 10 to get $^{t}_{1}$ ⟍ 10 $^{m}_{1}$ $((x^2 - x) + 10 \Rightarrow x^2 + 10 - x)$
219.7 L	Add $^{m}_{1}$ ⟍ $^{m}_{2}$ to 10 to get 10 $^{m}_{1}$ $((2x^2 - x^2) + 10 \Rightarrow x^2 + 10)$
219.10 L	Add $^{t}_{2}$ ⟍ $^{m}_{1}$ to $^{t}_{10}$ to get $^{t}_{8}$ $^{m}_{1}$ $((x^2 - 2x) + 10x \Rightarrow x^2 + 8x)$
219.12 L	Add $^{t}_{5}$ ⟍ $^{m}_{1}$ to $^{t}_{10}$ to get $^{t}_{5}$ $^{m}_{1}$ $((x^2 - 5x) + 10x \Rightarrow x^2 + 5x)$
220.1	Subtract $^{t}_{1}$ from $^{m}_{1}$ to get $^{t}_{1}$ ⟍ $^{m}_{1}$ $(x^2 - x \Rightarrow x^2 - x)$
220.3	Subtract 10 from $^{t}_{3}$ $^{m}_{2}$ to get 10 ⟍ $^{t}_{3}$ $^{m}_{2}$ $((2x^2 + 3x) - 10 \Rightarrow 2x^2 + 3x - 10)$
220.9	Subtract $^{t}_{1}$ 2 from $^{t}_{3}$ ⟍ $^{m}_{1}$ to get 2 ⟍ $^{t}_{4}$ ⟍ $^{m}_{1}$ $((x^2 - 3x) - (2 + x) \Rightarrow x^2 - 4x - 2)$
221.1	Subtract $^{t}_{5}$ ⟍ 52 from 30 $^{c}_{2}$ to get $^{t}_{22}$ ⟍ $^{t}_{5}$ $^{c}_{2}$ $((2x^3 + 30) - (52 - 5x) \Rightarrow 2x^3 + 5x - 22)$

Passage	Example
221.13	Subtract $\overset{t}{4}$ ل 12 from $\overset{t}{2}$ ل $\overset{m}{3}$ to get 12 ل $\overset{t}{2}\ \overset{m}{3}$ $((3x^2 - 2x) - (12 - 4x) \Rightarrow 3x^2 + 2x - 12)$
222.1	Subtract $\overset{m}{2}$ ل $\overset{c}{1}$ from $\overset{t}{4}$ ل 30 to get $\overset{t}{4}$ ل $\overset{c}{1}$ ل $\overset{m}{2}\ 30$ $((30 - 4x) - (x^3 - 2x^2) \Rightarrow 30 + 2x^2 - x^3 - 4x)$
223.2	$\overset{t}{1}\ 2 = \overset{t}{3}$ ل $\overset{m}{1}$ simplifies to $\overset{t}{4}\ 2 = \overset{m}{1}$ $(x^2 - 3x = 2 + x \Rightarrow x^2 = 2 + 4x)$
223.7	$\overset{t}{5}$ ل $24 = \overset{t}{3}$ ل $\overset{m}{1}$ simplifies to $24 = \overset{t}{2}\ \overset{m}{1}$ $(x^2 - 3x = 24 - 5x \Rightarrow x^2 + 2x = 24)$
223.14	$\overset{t}{\tfrac{1}{2}2}$ ل $\overset{m}{1} = 10$ ل $\overset{m}{1}$ simplifies to $\overset{t}{\tfrac{1}{2}2} = 10$ $(x^2 - 10 = x^2 - 2\tfrac{1}{2}x \Rightarrow 10 = 2\tfrac{1}{2}x)$
223.17	$\overset{t}{4}$ ل $51 = 10\ \overset{m}{1}$ simplifies to $41 = \overset{t}{4}\ \overset{m}{1}$ $(x^2 + 10 = 51 - 4x \Rightarrow x^2 + 4x = 41)$
224.1	$\overset{t}{1}$ ل $\overset{m}{2}\ 10 = \overset{t}{5}\ \overset{m}{1}$ simplifies to $10\ \overset{m}{1} = \overset{t}{6}$ $(x^2 + 5x = 10 + 2x^2 - x \Rightarrow 6x = x^2 + 10)$
224.4	$\overset{t}{7}$ ل $7\ \overset{m}{3} = \overset{t}{2}$ ل $10\ \overset{m}{2}$ simplifies to $\overset{t}{5}\ 3 = \overset{m}{1}$ $(2x^2 + 10 - 2x = 3x^2 + 7 - 7x \Rightarrow x^2 = 3 + 5x)$
225.8	The power of the *māl māl* is 4; of the *māl* cube is 5; of the *māl māl māl* is 6; of the *māl* cube *māl* cube is 10
225.13	The power of the *māl* cube *māl māl* is 9; of the cube *māl* cube cube *māl māl* is 15
226.1	A term for 4 is a *māl māl*; for 7 is a cube *māl māl*; for 6 is a *māl māl māl* or a cube cube
226.3	A term for 8 is a *māl māl māl māl* or a cube *māl* cube, or a cube cube *māl*, or a *māl* cube cube, etc.
226.7	A term for 9 is a cube cube cube or a cube *māl māl māl*, etc.
226.12	Multiply $\overset{t}{5}$ by $\overset{t}{7}$ to get $\overset{m}{35}$ $(5x \times 7x \Rightarrow 35x)$
226.17	Multiply $\overset{t}{10}$ by $\overset{m}{6}$ to get $\overset{c}{60}$ $(10x \times 6x^2 \Rightarrow 60x^3)$
227.4	Multiply $\overset{t}{1}$ by $\overset{c}{1}$ to get $\overset{mm}{1}$ $(x \times x^3 \Rightarrow x^4)$
227.7	Multiply 6 by $\overset{m}{4}$ to get $\overset{m}{24}$ $(6 \times 4x^2 \Rightarrow 24x^2)$

Passage	Example
227.10	Multiply 7 by $\overset{mc}{3}$ to get $\overset{mc}{21}$ $(7 \times 3x^5 \Rightarrow 21x^5)$
227.17	$\overset{m}{10}\ \overset{c}{4} = \overset{mm}{3}$ simplifies to $10\ \overset{t}{4} = \overset{m}{3}$ $(3x^4 = 4x^3 + 10x^2 \Rightarrow 3x^2 = 4x + 10)$
228.1	$\overset{t}{20}\ \overset{m}{10} = \overset{c}{3}$ simplifies to $20\ \overset{t}{10} = \overset{m}{3}$ $(3x^3 = 10x^2 + 20x \Rightarrow 3x^2 = 10x + 20)$
228.4	$\overset{t}{39} = \overset{m}{10}\ \overset{c}{1}$ simplifies to $39 = \overset{t}{10}\ \overset{m}{1}$ $(x^3 + 10x^2 = 39x \Rightarrow x^2 + 10x = 39)$
228.11	Multiply $\overset{t}{5}$ by $\overset{t}{4}$ ϑ 13 to get $\overset{m}{20}$ ϑ $\overset{t}{65}$ $(5x \times (13 - 4x) \Rightarrow 65x - 20x^2)$
228.15	Multiply $\overset{t}{2}$ ϑ 8 by $\overset{m}{4}$ ϑ 7 to get $\overset{m}{32}$ ϑ $\overset{t}{14}$ ϑ 56 $\overset{c}{8}$ $((8 - 2x) \times (7 - 4x^2) \Rightarrow 8x^3 + 56 - 14x - 32x^2)$
229.4	Divide $\overset{m}{10}$ by $\overset{t}{2}$ to get $\overset{t}{5}$ $(10x^2 \div 2x \Rightarrow 5x)$
229.8	Divide $\overset{c}{15}$ by $\overset{t}{3}$ to get $\overset{m}{5}$ $(15x^3 \div 3x \Rightarrow 5x^2)$
229.13	Divide $\overset{m}{12}$ by $\overset{m}{3}$ to get 4 $(12x^2 \div 3x^2 \Rightarrow 4)$
229.17	Divide $\overset{t}{12}$ by 4 to get $\overset{t}{3}$ $(12x \div 4 \Rightarrow 3x)$
230.4	Divide $\overset{m}{3}$ ϑ $\overset{c}{12}$ by $\overset{t}{2}$ to get $\overset{t}{\frac{1}{2}1}$ ϑ $\overset{m}{6}$ $((12x^3 - 3x^2) \div 2x \Rightarrow 6x^2 - 1\frac{1}{2}x)$
230.8	Divide $\overset{t}{3}$ ϑ $\overset{m}{10}$ by 2 to get $\overset{t}{\frac{1}{2}1}$ ϑ $\overset{m}{5}$ $((10x^2 - 3x) \div 2 \Rightarrow 5x^2 - 1\frac{1}{2}x)$
230.13	Divide $\overset{m}{6}$ by $\overset{c}{3}$ to get $\frac{2}{\overset{t}{1}}$ $(6x^2 \div 3x^3 \Rightarrow \frac{2}{x})$
230.17	Divide $\overset{m}{10}$ by $\overset{t}{1}$ ϑ 3 to get $\frac{\overset{m}{10}}{\overset{t}{1}\ ϑ\ 3}$ $(10x^2 \div (3 - x) \Rightarrow \frac{10x^2}{3-x})$

B. Some sample problems solved in other books

[1] A problem of Ibn al-Bannāʾ's solved by "the four proportional numbers".

In some problems, the method of "the four proportional numbers" is applied directly. This example is from the chapter on business transactions (*muʿāmalāt*) in Ibn al-Bannāʾ's *Essays on Arithmetic*. It asks for the price of a certain amount of a commodity given the price of another amount of the same commodity.[1]

> If someone said to you, "Five *amdāʾ*[2] for ten dinars.[3] How much do fourteen *mudy* cost?"
>
> These are four proportional amounts: the first is ten dinars, the second is five *amdāʾ*, the third is the unknown dinars, and the fourth is fourteen *mudy*. The third in this problem is the unknown. You multiply the first by the fourth, and you divide the outcome by the second, giving you the third, which is the cost of fourteen *mudy*, the unknown, and that is twenty-eight dinars.

[2] A problem of Ibn al-Bannāʾ's solved by single false position.

This problem, also from Ibn al-Bannāʾ's *Essays on Arithmetic*, shows how the method of "the four proportional numbers" is used in single false position. In this method a convenient but probably incorrect value is posited for the solution, here 60, and the calculations are performed on this value. The answer is then found from a proportion.[4]

> If someone said to you, "A quantity (*māl*): you add its half and a third of what remains, and a fourth of what remains, and a fifth of what remains, to get twenty dirhams. How much is the quantity?"
>
> You can take the third and the fourth and the fifth of sixty. Then take its half, giving thirty. And take a third of what remains, giving ten, which leaves twenty. Take its fourth, giving five. There remains fifteen. Take its fifth, giving three. Then gather the thirty with the ten and the five and the three, and you add them all, so the sum is forty-eight.

---

[1] (Ibn al-Bannāʾ 1984, 212.7).
[2] The *mudy*, plural *amdāʾ*, was a unit of capacity measure (Rebstock 2008). The plural form of Arabic nouns are used for numbers from 3 to 10 only, so fourteen of them are *mudy*.
[3] The dinar (*dīnār*) was the standard gold coin in Islamicate countries.
[4] (Ibn al-Bannāʾ 1984, 206.17).

> The ratio of this forty-eight with respect to the sixty is as the ratio of the twenty with respect to the unknown quantity. So you work it out as described above to get the quantity, which is fifteen.

Like al-Hawārī's problem at 200.1, this enunciation requires some explanation. If we use modern notation and call the unknown A, and we make the remainders $B = A - \frac{1}{2}A$, $C = B - \frac{1}{3}B$, and $D = C - \frac{1}{4}C$, then the problem asks to add together $\frac{1}{2}A + \frac{1}{3}B + \frac{1}{4}C + \frac{1}{5}D$. It would not be efficient to solve this problem by algebra, either premodern or modern. If we let the unknown be x, then the equation would be set up as $\frac{1}{2}x + \frac{1}{3}(x - \frac{1}{2}x) + \frac{1}{4}(x - \frac{1}{2}x - \frac{1}{3}(x - \frac{1}{2}x)) + \frac{1}{5}(x - \frac{1}{2}x - \frac{1}{3}(x - \frac{1}{2}x) - \frac{1}{4}(x - \frac{1}{2}x - \frac{1}{3}(x - \frac{1}{2}x))) = 20$. Even if you manage to simplify this to $\frac{1}{2}x + \frac{1}{6}x + \frac{1}{12}x + \frac{1}{20}x = 20$, it is much easier to pick a value like 60 and work through the operations with it.

For another problem solved by single false position, see problem [7] below.

[3] Ibn Qunfudh's problem (14), solved by algebra.

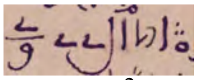
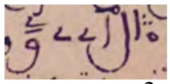
Problems [3], [4], and [5] are simple examples that show how the rules illustrated by al-Hawārī in the chapter on algebra are applied in the algebraic solutions to problems. For each problem, we give the breakdown according to the three stages. These stages are not indicated in the texts.

Ibn Qunfudh completed his commentary on Ibn al-Bannāʾ's *Condensed Book* in 1370. It is titled *Lowering the Veil from the Faces of Arithmetical Operations*. This book shows the algebraic notation.[5]

> [Enunciation]
> **Problem.** Ten: you divide it into two parts, and you multiply one of them by the other, resulting in twenty-two and two ninths.
>
> [*Stage 1*]
> Make one of the parts a thing and the other ten less a thing, and multiply one of them by the other, resulting in ten things less a *māl*, and it is equal to twenty-two and two ninths, and it is this: $\overset{t}{10}\ \ell\ \overset{m}{1} = 22\frac{2}{9}$.
>
> [*Stage 2*]
> So you restore by adding the excluded [amount] to each [side], so the problem becomes: ten things equal a *māl* and twenty-two and two ninths, and it is this: $\overset{t}{10} = \overset{m}{1}\ 22\frac{2}{9}$.
>
> [*Stage 3*]
> This results in the fifth type [of equation]. So work it out, which is that you halve the roots and you square the half to get twenty-five. Subtract the number from it, leaving two and seven ninths. Take its root, to get one and two-thirds. And if you add it to the half, which is five, it results in the thing by addition,

[5] (Ibn Qunfudh manuscript, 297.13).

> and if you subtract it, it results [in the thing] by subtraction. One of them is [one] part, and the remainder from the ten is the other part. So you find one of the parts to be six and two thirds, and the other three and a third, and that's it.

The enunciation of the problem is an arithmetic question which is solved using algebra. Ibn Qunfudh applies the rule for multiplying polynomials in stage 1, a restoration (*al-jabr*) to simplify the equation in stage 2, and the rule for solving the type 5 equation in stage 3. The type 5 equation here has two solutions, and in this problem they lead to the two parts of ten. The two equations in notation are shown as  and , in the manuscript. They correspond to the modern $10x - x^2 = 22\frac{2}{9}$ and $10x = x^2 + 22\frac{2}{9}$, respectively. In this problem Ibn Qunfudh shows "things" in notation with only the three dots from the letter ش, though he shows the whole letter in other examples.

[4] A geometry problem of al-Karajī's solved by algebra.

This next problem is from al-Karajī's *Sufficient Book on Arithmetic*, written in the early eleventh century. There is no need for stage 2 because the equation that is set up at the end of stage 1 is already simplified. Its solution in stage 3 is trivial.[6]

> [*Enunciation*]
> If someone said, "A rectangle: its length is twice its width, and its area is equal to its perimeter. How much are its area and its sides?"
>
> [*Stage 1*]
> Make its length two things and its width a thing, and multiply the length by the width to get two *māl*s. This is the area, and it equals the perimeter, which is six things [$\genfrac{}{}{0pt}{}{m}{2} = \genfrac{}{}{0pt}{}{t}{6}$].
>
> [*Stage 3*]
> So the one thing equals three, which is the width.

[5] ʿAlī al-Sulamī's problem (I.36), solved by algebra.

This author's tenth-century book *Sufficient Introduction on Calculation by Algebra and What One Can Learn from Its Examples* contains many problems, among which is the following[7]:

> [*Enunciation*]
> Two men encounter a horse. The first says to the second, if you offer a fourth and a fifth of what you have in addition to what I have, then I will have the price of the horse. And the second says to the first, if you offer a sixth and a

[6] (al-Karajī 1986, 204.8).
[7] (ʿAlī al-Sulamī manuscript, fol. 54b.1).

seventh of what you have in addition to what I have, then I will have the price of the horse.

[*Stage 1*]
So make what the first has a thing, and what the second has a number that has a fourth and a fifth, and that is twenty.[8] If the first takes from the second a fourth and a fifth of what he has, he will have a thing and nine dirhams, which is the price of the horse.

Then the second takes from the first a sixth and a seventh of a thing, so he will have twenty and thirteen parts of forty-two of a thing, which is also the price of the horse. Then confront what they have, [i.e., establish the equation $\overset{t}{1}\ 9 = 20\ \overset{t}{\frac{13}{42}}$]

[*Stage 2*]
and cast away like terms, leaving twenty-nine parts of forty-two equals eleven dirhams [$\overset{t}{\frac{29}{42}} = 11$].

[*Stage 3*]
Complete the thing, by which you multiply everything you have by forty-two and divide it by twenty-nine, to get: the thing is fifteen and twenty-seven[9] parts of twenty-nine [$15\frac{27}{29}$], which is what the first has, and the second has twenty dirhams.

If you want the results to be whole numbers, multiply what you have by twenty-nine, so the first has four hundred sixty-two and the second has five hundred eighty.

[6] Ibn al-Bannāʾ's problem (I.10), solved by algebra.

Many authors, Ibn al-Bannāʾ included, condensed stages 2 and 3 of their solutions. Here is the enunciation and the first of five different algebraic solutions to problem (I.10) from his *Book on the Fundamentals and Preliminaries in Algebra*. Note how he manipulates the operations to avoid unresolved divisions in his equation, and his justification for this manipulation after the solution.[10]

[*Enunciation*]
Ten: you divide it into two parts. You divide each of them by the other, so it results in a total of four and a fourth.

[*Stage 1*]
You make one of the parts a thing, so the other is ten less a thing. Then you square each of them and you add the squares to get a hundred and two *māl*s less twenty things [$100\ \overset{m}{2}\ \ell\ \overset{t}{20}$]. Keep this in mind. Then you multiply the

[8] The problem is indeterminate, so the author is allowed to pick one of the numbers as he pleases.
[9] The MS mistakenly has "nineteen".
[10] (Saidan 1986, 562.4).

surface of the parts by the result of the divisions, which is four and a fourth, to get forty-two things and half a thing less four *māl*s and a fourth of a *māl*. This equals the sum of the remembered squares [$\overset{t}{42\frac{1}{2}}\ \ell\ \overset{m}{4\frac{1}{4}} = 100\ \overset{m}{2}\ \ell\ \overset{t}{20}$].

[*Stage 2*]
Then you restore and confront, to get the fifth type [of equation: $\overset{t}{62\frac{1}{2}} = 100\ \overset{m}{6\frac{1}{4}}$].

[*Stage 3*]
So you look for the thing, finding two by subtraction, which is one of the parts, and eight by addition, which is the other part.

You saw the reason for this way above in the chapter on division, in the first part, which is: for any two numbers, if each one is divided by the other, then the sum of their squares is equal to the product of their surface by the sum of the two quotients.

We can represent the enunciation with modern algebraic notation by naming the two parts of ten a and b so that $a + b = 10$, with the condition that $\frac{a}{b} + \frac{b}{a} = 4\frac{1}{4}$. Ibn al-Bannāʾ names one of the parts "a thing", so the other is necessarily "ten less a thing". The modern versions would be x and $10 - x$, and we would set up the equation as

$$\frac{x}{10-x} + \frac{10-x}{x} = 4\frac{1}{4},$$

and maybe combine the terms on the left to get

$$\frac{x^2 + (10-x)^2}{x(10-x)} = 4\frac{1}{4},$$

after which we would multiply both sides by $x(10 - x)$.

Like other Arabic algebraists, Ibn al-Bannāʾ avoids operations in equations. He instead reformulates the problem according to an arithmetical rule he had given earlier in the book, and which was stated rhetorically: for any two numbers a and b, $a^2 + b^2 = ab\left(\frac{a}{b} + \frac{b}{a}\right)$. This way, the equation is set up without operations: "forty-two things and half a thing less four *māl*s and a fourth of a *māl*…equals" "a hundred and two *māl*s less twenty things". There is no addition, subtraction, or scalar multiplication in this equation. Our modern version $42\frac{1}{2}x - 4\frac{1}{4}x^2 = 100 + 2x^2 - 20x$, however, is built from addition, subtraction, scalar multiplication, and exponentiation. See our commentary above at 219.1, 219.4, and 229.8.

Because stages 2 and 3 follow prescribed rules, he does not need to spell out the steps. Like some other algebraists, Ibn al-Bannāʾ writes "restore and confront" to mean restoration and/or confrontation in stage 2. For this equation two restorations are performed.

[7] A problem of al-Ḥaṣṣār's solved by single false position, algebra, and double false position.

Al-Ḥaṣṣār wrote his *Book of Demonstration and Recollection in the Art of Dust-Board Reckoning* in the late twelfth century. He is one of several authors to solve the same arithmetic problem by different methods.[11]

> If someone said to you, "A quantity (*māl*): you add its third and its fourth so it brings back twenty-one dirhams".
>
> [*Solution by single false position*]
> In this problem you can take the third and the fourth from twelve. Then you take from the twelve its third and its fourth, which is seven. The ratio of this seven with respect to twelve is as the ratio of the twenty-one with respect to the required quantity. So multiply the twelve by the twenty-one, and you divide the result by the seven. This gives you thirty-six, which is the quantity.
>
> [*Solution by algebra*]
> And if you wish, work it out by algebra (*al-jabr*), which is that you make the quantity a thing. You take from it its third and its fourth, and that is three sixths of a thing and half a sixth of a thing. This equals the twenty-one [$\overset{t}{\frac{1\ \ 3}{2\ \ 6}} = 21$].
>
> So we say to you: restore the three sixths and a half so that it yields a thing. What determines that is one and five sevenths, and we will explain it, God willing, in the chapter on the restoration of fractions.[12] So you multiply one and five sevenths by twenty-one. This gives you thirty-six, which is the quantity.
>
> [*Solution by double false position*]
> And if you wish, work it out by the [method of] scales, which is that you take any number you wish for the front [scale], which you are allowed to do. So you make the front scale three. You take its third and its fourth: it is one and three fourths. Confront it with the twenty-one. You find it is smaller. So you made an error in the "three" scale of nineteen and a fourth falling short, and keep this error in mind.
>
> Then you take the other scale and let it be, for example, four. You take its third and its fourth, and that is two and a third. Confront it with the twenty-one. So you made an error in the "four" scale of eighteen and two thirds falling short.
>
> Multiply the error of the first scale, which is nineteen and a fourth, by the second scale, which is four. That gives seventy-seven. Keep it in mind. Then multiply the error of the second scale, which is eighteen and two thirds, by the first scale, which is three. It gives you fifty-six. Subtract it from the seventy-seven, leaving twenty-one. Then you subtract the smaller of the two errors from the greater. The remainder is three sixths and half of a sixth. Divide the

[11] (al-Ḥaṣṣār manuscript, fol. 72r.6).
[12] Al-Hawārī explains the restoration of fractions at 154.1.

twenty-one by it, giving you thirty-six, which is the quantity itself, and we will clarify this ratio when we get to it.

C. Chronological list of mathematicians and other scholars

Below is a list of exceptionally brief notices on some of the different mathematicians and philosophers mentioned in this book. References are given to (Rosenfeld and Ihsanoğlu 2003) and (Lamrabet 2014) where applicable. For example, for al-Ḥaṣṣār we write (#532, M55), which means that he is scholar #532 in Rosenfeld & Ihsanoğlu, and scholar M55 in Lamrabet. Rosenfeld and Ihsanoğlu give numbered lists of individual titles by each author. For example, "[M2]" after an Arabic title indicates book M2 in their list. After Aristotle and Euclid all dates are CE, and with a few AH.

- Aristotle (384-322 BC). Three aspects of Aristotle's thought impacted al-Hawārī's book, most likely through the works of Ibn Sīnā. One is the distinction between quality and quantity at 92.17, another is the distinction between sensible and intelligible objects at 133.3, and the third is the division of the genus of "quantity" into discrete and continuous, discussed in our commentary at 117.2 and 133.3. For a brief overview of Aristotle's influence on Arabic mathematics, see Section 4 in our introduction.
- Euclid (ca. 300 BC). Euclid wrote his *Elements* in Greek around 300 BC in Alexandria. The work consists of thirteen "Books", or what we might call long chapters. Books I through VI cover plane geometry, Books VII to IX discuss number theory, the long Book X deals with the theory of quadratic irrationals in a geometric context, and Books XI to XIII deal with three-dimensional geometry.[1]

  Euclid's *Elements* is reported to have been first translated into Arabic by al-Ḥajjāj ibn Yūsuf ibn Maṭar during the reign of the caliph Hārūn al-Rashīd (786-809). Al-Ḥajjāj later produced a second and improved translation during the reign of al-Maʾmūn (813-833). Later in the century Isḥāq ibn Ḥunayn (830-910) translated the *Elements*, and this was subsequently revised by Thābit ibn Qurra (836-901). While the surviving manuscripts apparently all derive from Thābit's version, they also contain passages from al-Ḥajjāj. In addition, quotations from other books preserve parts of the earlier translations. The textual history of Euclid's *Elements* in Arabic is complex and at present not well understood.

  Euclid's *Data*, another work on geometry, was also translated into Arabic and influenced the ways some Arabic mathematicians presented proofs.

  The influence of Euclid's *Elements* on al-Hawārī's book is seen in the definitions of "number" (see our commentary at 65.2), "oddly-odd" (66.7), "multiplication" (95.2), "ratio", "part", "parts" (133.1), "medial" (163.4), and "ex-aequali" (195.2). The terms "side", "surface", and "square" (66.17) also ultimately come from Euclid, as do the various manipulations of ratios covered at 196.16. The entire theory of quadratic irrationals beginning at 173.4 derives from an arithmetical reading of

[1] The standard English translation of the *Elements* is (Euclid 1956). The Greek text with the English translation by Richard Fitzpatrick is available online: http://farside.ph.utexas.edu/Books/Euclid/Elements.pdf (accessed May 1, 2019). The extant Arabic translation has not been published.

Elements Book X. Some fragments of al-Ḥajjāj's translation of this book may have found their way into al-Hawārī's book. See our comments at 173.10.

- Nicomachus of Gerasa (fl. second century). Nicomachus was a Neopythagorean philosopher who wrote his *Arithmetical Introduction* more as a prolegomenon (introduction) to philosophy than as a treatise on mathematics. He devotes the first six chapters to a philosophical argument for the primacy of arithmetic in the mathematical sciences, and he covers aspects of what we would call elementary number theory in the rest of the book. Ḥabīb ibn Bihrīz translated the *Arithmetical Introduction* into Arabic from a Syriac version shortly before 822. Thābit ibn Qurra translated Nicomachus's work into Arabic again some decades later, this time directly from the Greek.[2]

  Nicomachus's influence is evident in the definitions of "evenly-even", etc. (65.10), the distinction between disparities of quantity and quality in sequences (73.7, 73.17), the sieve of Eratosthenes (127.10), and in the list of the kinds of proportion that al-Hawārī copied from Ibn al-Bannāʾ's *Lifting the Veil* (195.2).

- Diophantus (ca. 300). This Alexandrian mathematician wrote his *Arithmetica* in thirteen "books". Six survive in Greek and another four in the Arabic translation of Qusṭā ibn Lūqā. In this work Diophantus solves determinate and indeterminate problems by algebra.[3]

- al-Khwārazmī, early ninth century (#41). Muḥammad ibn Mūsā al-Khwārazmī worked in Baghdad and authored works on Indian arithmetic, algebra, astronomy, astrology, geography, and the Jewish calendar. The earliest known books on algebra are al-Khwārazmī's *Book of Algebra* (*Kitāb al-jabr wa l-muqābala*, [M3]) and Ibn Turk's book by the same title. Only a fragment of the latter is extant, while al-Khwārazmī's book survives in its entirety. Al-Khwārazmī's *Book on Indian Reckoning* (*Kitāb al-ḥisāb al-hindī*, [M1]) covers the rules of calculating with Arabic numerals, and survives only in a medieval Latin reworking of a Latin translation.[4]

- Ibn Turk, early ninth century (#59). The only part of ʿAbd al-Ḥamīd Ibn Turk's *Book of Algebra* (*Kitāb al-jabr wa l-muqābala*, [M1]) that is extant is the part giving proofs for the rules for solving simplified equations.[5]

- Thābit ibn Qurra, 836-901 (#103). Abū l-Ḥasan Thābit ibn Qurra al-Ḥarrānī was a prolific translator and scholar who worked in all branches of the mathematical sciences. He revised Isḥāq ibn Ḥunayn's Arabic translation of Euclid's *Elements*, and he translated Nicomachus's *Arithmetical Introduction* as well as some works of Archimedes into Arabic.

[2] Thābit's version is published in (Nicomachus 1959). Ḥabīb ibn Bihrīz's translation survives only in a Hebrew translation from 1317 of a redaction by al-Kindī (Freudenthal and Zonta 2007). The Greek text is published in (Nicomachus 1866), and an English translation from the Greek is published in (Nicomachus 1938).

[3] The extant Greek books are published in (Tannery 1893–1895), and the Arabic books in (Sesiano 1982); (Diophantus 1984). See (Christianidis and Oaks 2013) for a comparison between the method of Diophantus and Arabic algebra.

[4] The *Book of Algebra* is published with an English translation in (al-Khwārizmī 2009). This is translated from the French edition of 2007. The *Book on Indian Reckoning* is published in (al-Khwārizmī 1997). See also (al-Khwārizmī 1992).

[5] The Arabic text is published with English and Turkish translations in (Sayili 1962).

Among Thābit's original contributions to mathematics are his calculations of area and volume in the tradition of Archimedes, including the volume of a section of a paraboloid. These are not numerical calculations. Instead, they equate the areas and volumes to other known areas and volumes. Thābit also wrote a short treatise on algebra titled *Establishing the Correctness of Algebra Problems by Geometric Proofs* (*Qawl fī taṣḥīḥ masāʾil al-jabr bi l-barāhīn al-handasiyya*, [M19]) in which he proved the rules for solving the three composite quadratic equations in the style of Euclid's *Data*.[6]

- Qusṭā ibn Lūqā, died ca. 910 (#118). Among the many contributions Qusṭā made to mathematics, the two of interest to the current study are his translation of Diophantus's *Arithmetica* and his short treatise in which he proved the rules of double false position via geometry in the style of Euclid's *Data*.[7]
- Abū Kāmil, late ninth century (#124). Abū Kāmil, whose full name is Shujāʿ ibn Aslam ibn Muḥammad ibn Shujāʿ, is best known for his comprehensive *Book on Algebra* (*Kitāb fī l-jabr wa l-muqābala*, [M1]), written as a kind of commentary on al-Khwārazmī's algebra book. The rules in algebra for finding the *māl* directly, first mentioned at 214.2 ultimately come from this book.[8]
- ʿAlī al-Sulamī, tenth century (#267). This Syrian mathematician is known only for his *Sufficient Introduction on Calculation by Algebra and What One Can Learn from its Examples* (*al-Muqaddima al-kāfiyya fī ḥisāb al-jabr wa l-muqābala wa mā yuʿrafu bihi qiyāsuhū min al-amthila*, [M1]), which survives in a single manuscript. The book exhibits borrowings from Abū Kāmil and al-Khwārazmī.[9] His full name is Abū l-Ḥasan ʿAlī ibn al-Muslim ibn Muḥammad ʿAlī al-Fatḥ al-Sulamī.
- Ikhwān al-Ṣafāʾ (Brethren of Purity), tenth century (#226). The Brethren of Purity were a group of anonymous mystical scholars centered in the city of Basra. Their highly influential *Epistles of the Brethren of Purity* (*Rasāʾil Ikhwān al-Ṣafāʾ*, [E1]) is a kind of encyclopedia of all knowledge situated in a universalist philosophical/theological setting. The first of its four parts is on mathematics, where the main influences for arithmetic are the Greek Neopythagoreans, especially Nicomachus, and for geometry Euclid's *Elements*.[10]
- al-Uqlīdisī, tenth century (#232). Abū l-Ḥasan Aḥmad ibn Ibrāhīm al-Uqlīdisī wrote his *Chapters on Indian Arithmetic* (*al-Fuṣūl fī l-ḥisāb al-Hindī*, [M1]) in 952-3. It is the oldest extant Arabic treatise on calculating with Arabic numerals. The nickname "al-Uqlīdisī" can be translated as "the Euclidean", presumably because he worked on Euclid's *Elements*. His book on arithmetic is unrelated to Euclid's work.[11]

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[6] Thābit's translation of Nicomachus is published in (Nicomachus 1959). The treatise on algebra is published in (Luckey 1941); (Thābit ibn Qurra 2009, 159-169), the former with a German translation and the latter with a French translation. An English translation appears in (al-Khwārizmī 2009, 34-35, 38, 41-42).
[7] The four extant books of Qusṭā's translation of Diophantus have been edited and translated twice: (Sesiano 1982), into English, and (Diophantus 1984), into French. His treatise on double false position is edited and translated into German in (Suter 1908–1909).
[8] A facsimile of the Istanbul manuscript of the *Book on Algebra* is published in (Abū Kāmil 1986). An edition with a German translation is published in (Abū Kāmil 2004), and an edition with a French translation appears in (Abū Kāmil 2012).
[9] (ʿAlī al-Sulamī manuscript).
[10] English translations of the mathematical portions of the *Epistles* have been published in (Goldstein 1964) and (El-Bizri 2012), the latter also containing the Arabic text.
[11] The Arabic text is published in (al-Uqlīdisī 1984), and Saidan's English translation is published in (al-

- Abū l-Wafāʾ, 940-998 (#256). The earliest extant book dedicated primarily to finger-reckoning is Abū l-Wafāʾ's *Book of What is Necessary for Scribes, Businessmen, and Others in the Science of Arithmetic* (*Kitāb fīmā yaḥtāju ilayhi al-kuttāb wa l-ʿummāl wa ghayruhum min ʿilm al-ḥisāb*, [M2]), written between 961 and 976.[12] Full name: Abū l-Wafā Muḥammad ibn Muḥammad ibn Yaḥyā ibn Ismāʿīl ibn al-ʿAbbās al-Būzjānī.
- al-Karajī, died ca. 1025 (#309). Fakhr al-Dīn Abū Bakr Muḥammad ibn al-Ḥasan (or al-Ḥusayn) al-Karajī was a Persian mathematician and engineer who wrote his works on algebra and arithmetic in Baghdad. His three main works, in chronological order, are:

  [Book of] al-Fakhrī on the Art of Algebra (*al-Fakhrī fī ṣināʿat al-jabr wa l-muqābala*, [M2]). This is al-Karajī's book on algebra, modeled on Abū Kāmil's *Book on Algebra* and Diophantus's *Arithmetica*. It was probably completed in 401H/1010-1 CE.

  The Sufficient [Book] on Arithmetic (*al-Kāfī fī l-ḥisāb*, [M1]). A work on calculation by finger-reckoning with a section on algebra. It was probably completed in 402H.

  Marvelous [Book] of Arithmetic (*al-Badīʿ fī l-ḥisāb*, [M3]). This book covers various techniques of calculation, beginning with an introduction to arithmetic based in Euclid and Nicomachus, then a section on algebra, and ending with an exposition on techniques of solving indeterminate problems by algebra.[13]
- Kūshyār ibn Labbān, ca. 970-1030 (#308). Kūshyār was a Persian mathematician and astronomer who wrote a book on calculating with Arabic numerals, titled *Principles of Indian Reckoning* (*Kitāb fī uṣūl ḥisāb al-Hindī*, [M1]).[14] Full name: Abū l-Ḥasan Kūshyār ibn Labbān ibn Bāshahrī al-Jīlī.
- Ibn Sīnā, ca. 970-1037 (#317). Abū ʿAlī al-Ḥusayn ibn ʿAbdallāh ibn Sīnā, known in Latin as Avicenna, was the most important philosopher of medieval Islam. He was second only to Aristotle in influence in the Islamic world and in the late medieval and early modern West. For epistemology and ontology he built on the works of Aristotle, and it is most likely through him that Aristotelian ideas are present in al-Hawārī's book.[15]
- al-Baghdādī, died 1038 (#320). Abū Manṣūr ʿAbd al-Qāhir ibn Ṭāhir al-Baghdādī was a Persian mathematician and judge best known for his *Completion of Arithmetic* (*al-Takmila fī l-ḥisāb*, [M1]). This book covers calculation with Indian numerals on the dust-board, sexagesimal arithmetic, number theory, and business arithmetic.[16]
- Ibn al-Samḥ, early eleventh century (A93). Abū l-Qāsim Aṣbagh ibn Muḥammad ibn Aṣbagh ibn al-Samḥ was a mathematician, astronomer, and physician who

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Uqlīdisī 1978).

[12] The Arabic text is edited in (Saidan 1971). See (Saidan 1974) for a description of the contents in English.

[13] The Arabic texts of these three books are published, in order, in (Saidan 1986), (al-Karajī 1986), and (al-Karajī 1964).

[14] A facsimile of the Istanbul manuscript with an English translation is published in (Kūshyār ibn Labbān 1965).

[15] For more on Ibn Sīnā, see his entry in the online *Stanford Encyclopedia of Philosophy*, https://plato.stanford.edu/entries/ibn-sina/ (accessed July 29, 2018).

[16] The Arabic text is published in (al-Baghdādī 1985).

taught in Grenada. His *Sufficient Book on Mental Reckoning* (*Risāla kāfiya fī ʿilm al-ḥisāb*) covers techniques of finger-reckoning and double false position.[17]

- Ibn al-Haytham, 965-1041 (#328). Known also by his Latinized name Alhazen, Ibn ʿAlī al-Ḥasan ibn al-Ḥasan ibn al-Haytham is best known for his groundbreaking work in optics. What interests us is his short work *The Arithmetic of Transactions* (*Ḥisāb al-muʿāmalāt*, [M24]).[18] It was common for scientists conducting advanced work to produce elementary texts like this. Naṣīr al-Dīn al-Ṭūsī (below) is another.

- al-Bīrūnī, 973-1048 (#348). Abū l-Rayḥān Muḥammad ibn Aḥmad al-Bīrūnī wrote extensively on a wide range of topics including anthropology, linguistics, natural science, mathematics, and astronomy. Book II of his *Book of Instruction in the Elements of the Art of Astrology* (*Kitāb al-tafhīm li-awāʾil ṣināʿat al-tanjīm*, [A2]) covers arithmetic.[19] The first part, on number theory (pp. 72-95), is largely taken from Nicomachus, with some influence from Indian sources. The second part (pp. 96-119) covers calculation: multiplication, division, roots, sexagesimal and base ten arithmetic, algebra, double false position, and *abjad* calculation.

- al-Qurashī, 1030-1067. Abū l-Ḥasan ʿAlī ibn al-Khiḍr al-Qurashī hailed from Syria. His *Note on the Elements of Calculation and Inheritance* (*al-Tadhkira bi-uṣūl al-ḥisāb wa l-farāʾiḍ*) is a summary of his more extensive, and lost, *Book of Sustenance* (*Kitāb al-Maʿūna*). It covers practical arithmetic, with a focus on the division of estates.[20]

- al-Khayyām, 1048-1131 (#420). Better known in the West as Omar Khayyam, Ghiyāth al-Dīn Abū l-Fatḥ ʿUmar ibn Ibrāhīm al-Khayyāmī (Khayyām) was a poet, mathematician, and astronomer best known to historians of mathematics for his *Treatise on the Proofs of Algebra Problems* (*Risāla fī l-barāhīn ʿalā masāʾil al-jabr wa l-muqābala*, [M2]), in which he classifies and solves the twenty-five equations of degree three and less. Al-Khayyām saw algebra as a tool for geometric problem-solving, so he gives constructions for solving each equation type. He also gave numerical solutions where he could, but rules for solving irreducible cubic equations would not be found until the sixteenth century. Al-Khayyām was able to justify the appropriation of algebra for solving problems in geometry by regarding the numbers of the algebraists as the dimensionless measures of continuous magnitudes.[21]

- al-Samawʾal, died ca. 1175 (#487). Al-Samawʾal ibn Yaḥyā ibn ʿAbbās al-Maghribī was a native of Baghdad who later worked in Iran. He built on the works of al-Karajī and others to develop computational rules for polynomials ("composite numbers") in his *The Dazzling [Book] on the Science of Calculation* (*al-Bāhir fī ʿilm al-ḥisāb*, [M1]).[22]

[17] A facsimile of the Escurial manuscript with a Spanish translation is published in (Ibn al-Samḥ 2006).
[18] Published in (Rebstock 1998).
[19] A facsimile with an English translation is published in (al-Bīrūnī 1934).
[20] A facsimile of the Medina manuscript, with a German translation and commentary, is published in (al-Qurashī 2001). For a description of the book in English, see (Rebstock 2002).
[21] For the Arabic text with a French translation of al-Khayyām's mathematical works see (Rashed and Vahabzadeh 1999). An English translation of this book, but without the Arabic text, is published in (Rashed and Vahabzadeh 2000). Earlier editions and translations have also been published. See (Oaks 2011a) for the ontological foundation of his algebra.
[22] This book is published in (al-Samawʾal 1972).

- al-Ḥaṣṣār, died before 1194 (#532, M55). Abū Bakr Muḥammad ibn ʿAbdallah ibn ʿAyyāsh al-Ḥaṣṣār's *Book of Demonstration and Recollection in the Art of Dust-Board Reckoning* (*Kitāb al-bayān wa l-tadhkār fī ṣanʿat ʿamal al-ghubār*, [M1]) is the earliest extant Arabic book showing the notation of fractions with the division bar, among other innovations. He also wrote a more comprehensive book on arithmetic, the *Complete [Book] on the Art of Number* (*al-Kāmil fī ṣināʿa al-ʿadad*, [M2]). Al-Ḥaṣṣār worked in al-Andalus and Morocco.[23]

- Ibn al-Yāsamīn, died 1204 (#521, M83). Originally from Fez in Morocco, Abū Muḥammad ʿAbdallah ibn Muḥammad ibn Hajjāj ibn al-Yāsamīn al-Adrīnī composed his *Grafting of Opinions of the Work on Dust Figures* (*Talqīḥ al-afkār fī l-ʿilm bi-rushūm al-ghubār*, [M3]) in Seville. This book is a hybrid between a textbook on Indian arithmetic and one on finger reckoning. It is the earliest book we know to show the Arabic algebraic notation, though it is clear by the way he presents it that it was not his invention. Ibn al-Yāsamīn is also well known for his 54-line *Poem on Algebra* (*Urjūza fī l-jabr*, [M1]), which inspired as many commentaries as Ibn al-Bannāʾ's *Condensed Book*.[24]

  Ibn al-Bannāʾ copied many parts of Ibn al-Yāsamīn's *Grafting of Opinions* nearly word-for-word into his *Condensed Book*. Much of the chapter on the addition of whole numbers (73.2 through 79.13), the paragraph on multiplication at 98.4, the rules (mostly) from finger-reckoning from 109.7 through 114.4 (except 113.6), and the first couple of sentences on division at 117.2-5 were taken from it.

- Ibn Munʿim, twelfth-thirteenth centuries (#556, M89). Abū Jaʿfar Aḥmad ibn Ibrāhīm ibn Munʿim al-ʿAbdarī was a Moroccan mathematician who wrote his book *Understanding Calculation* (*Fiqh al-ḥisāb*, [M1]) at a more theoretical level than the textbooks of people like al-Ḥaṣṣār or Ibn al-Yāsamīn.[25]

- Ibn Fallūs, 1194-1239 (#584). Shams al-Dīn Abū l-Ṭāhir Ismāʿīl ibn Ibrāhīm ibn Ghāzī al-Māridīnī was known by the name Ibn Fallūs. It was during a pilgrimage to Mecca that he wrote his treatise on algebra with a rhyming title: *Kitāb niṣāb al-ḥabr fī ḥisāb al-jabr* (*Preparation for writing on calculation in algebra*, [M3]).[26] He also wrote a textbook on finger-reckoning, *Direction to Reckoners Showing the Right Path in Revealing the Science of Arithmetic* (*Irshād al-ḥussāb fī al-maftūḥ min ʿilm al-ḥisāb*, [M2]).

- Ibn Badr, ca. thirteenth century (#587). Abū ʿAbdallah Muḥammad ibn ʿUmar Ibn Badr al-Balansī wrote his *Brief Book on Algebra* (*Kitāb fīhi ikhtiṣār al-jabr wa l-muqābala*, [M1])[27] with borrowings from Abū Kāmil and al-Khwārazmī.

- Naṣīr al-Dīn al-Ṭūsī, 1201-1274 (#606). One of the most prolific scholars of medieval Islam, Naṣīr al-Dīn Abū Jaʿfar Muḥammad ibn Muḥammad al-Ṭūsī wrote on philosophy, theology, and a variety of scientific topics. In the sciences he is best known for his work on planetary astronomy, but we are more interested in his rela-

[23] A facsimile of a late twelfth-century manuscript is available online: (al-Ḥaṣṣār manuscript).

[24] T. Zemouli's edition of the Arabic text of *Grafting of Opinions* appears in (Zemouli 1993). We lack pages 103-116 of this book, but we have a typeset version covering these pages from the edition that Zemouli will publish at some time in the future, (Zemouli n.d.). The *Poem on Algebra* is published with an English translation in (Abdeljaouad 2005a).

[25] The Arabic text is published in (Ibn Munʿim 2005).

[26] (Ibn Fallūs manuscript).

[27] The Arabic text is published in (Saidan 1986).

tively insignificant textbook *Gathering of Arithmetic by Means of Board and Dust* (*Jāmiʿ al-ḥisāb bi l-takht wa l-turāb*, [M17]).[28]

- al-Fārisī, died 1319 (#674). Kamāl al-Dīn al-Fārisī was a Persian mathematician who wrote his scientific examination of the rules of finger reckoning, *Foundation of the Rules on Elements of Benefits* (*Asās al-qawāʿid fī uṣūl al-fawāʾid*, [M2]), in the form of a commentary on a textbook of his teacher Ibn al-Khawwām.[29]
- Ibn al-Bannāʾ, 1256-1321 (#696, M134). See section 1.2 in the Introduction for bio-bibliographic details.
- al-Hawārī, early fourteenth century (#747, M148, M150). See section 1.2 in the Introduction for bio-bibliographic details.
- Ibn Qunfudh, 1339-1407 (#780, M193). Abū l-ʿAbbās Aḥmad ibn Ḥasan ibn ʿAlī ibn al-Khaṭīb ibn Qunfudh al-Qustantīnī completed his commentary on Ibn al-Bannāʾ's *Condensed Book*, titled *Lowering the Veil from the Faces of Arithmetical Operations* (*Ḥaṭṭ al-niqāb ʿan wujūh aʿmāl al-ḥisāb*, [M1]), in 1370. He also wrote *Foundations for Beginning the Commentary on Ibn al-Yāsamīn's Poem* (on algebra) (*Mabādīʾ al-sālikīn fī sharḥ urjūzat Ibn al-Yāsamīn*). Both books show the Arabic algebraic notation.[30]
- al-Mawāḥidī, second half of the fourteenth century (M176). Abū ʿAbd al-Raḥmān Yaʿqūb ibn Ayyūb al-Mawāḥidī al-Jazūlī wrote the popular commentary titled *Achieving the Desire of Commenting on Ibn al-Bannāʾ's Condensed [Book]* (*Taḥṣīl al-munā fi sharḥ Talkhīṣ Ibn al-Bannāʾ*), ca. 1382.[31]
- al-ʿUqbānī, died 1408 (#781, M195). Abū ʿUthmān Saʿīd ibn Muḥammad al-Tujībī al-Tilimsānī al-ʿUqbānī wrote his *Commentary on the Condensed [Book]* (*Sharḥ al-Talkhīṣ*)[32] with a focus on providing Euclidean-style proofs to the rules in Ibn al-Bannāʾ's book. He was a native of Tlemcen, and spend his career between cities now located in Algeria and Morocco.
- al-Ḥanbalī, died 1409 (#782). Taqī al-Dīn ibn ʿIzz al-Dīn al-Ḥanbalī copied numerical examples from al-Hawārī's book into his own *Commentary on the Condensed [Book]* (*Sharḥ al-Talkhīṣ*).[33]
- Ibn al-Hāʾim, ca. 1355-1412 (#783). Abū l-ʿAbbās Shihāb al-Dīn Aḥmad ibn Muḥammad ibn Imād al-Dīn ibn ʿAlī was known by the name Ibn al-Hāʾim. Among his many books on arithmetic and algebra are his 1387 *Commentary on the Poem of al-Yāsamīn* (*Sharḥ al-Urjūza al-Yāsmīnīyya*, [M13]), his 1389 *Guidebook for the Science of Aerial Calculation* (*al-Maʿūnah fī ʿilm al-ḥisāb al-hawāʾī*, [M1]), and his *Abridgement [of the] Condensed [Book] of Ibn al-Bannāʾ* (*Mukhtaṣar Talkhīṣ Ibn al-Bannāʾ*, [M17]).[34]

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[28] The Arabic text is published in (Saidan 1967).
[29] The Arabic text is published in (al-Fārisī 1994).
[30] A facsimile of a fifteenth-century manuscript of the first work is available online: (Ibn Qunfudh manuscript).
[31] We consulted (al-Mawāḥidī manuscript), which is available online.
[32] The Arabic text is published in (Harbili 1997).
[33] MSS Paris 2463/1; Tunis 16448/1.
[34] The first is edited with a partial French translation in (Ibn al-Hāʾim 2003), the second is edited in (Ibn al-

- al-Kāshī, died 1429 (#802). Giyāth al-Dīn Jamshīd ibn Masʿūd al-Kāshī (or al-Kāshānī) was a mathematician and astronomer in the court of Ulugh Beg in Samarqand. He is best known for his work in planetary astronomy and for his calculations of 2π and the sine of one degree, each accurate to the equivalent of 16 decimal places. What interests us is his 1427 book *Key to Calculation* (*Miftāḥ al-ḥisāb*, [M1]).[35]
- Ibn al-Maghribī, fifteenth century (#910). We mention his *Poem on Reckoning with Finger-Joints* (*Manẓūma fī l-ḥisāb al-ʿuqūd*, [M1]) in our Introduction.[36] His full name is Abū l-Ḥasan ʿAlī ibn al-Maghribī.
- al-Qalaṣādī, died 1486 (#865, M229). Abū l-Ḥasan ʿAlī ibn Muḥammad al-Qurashī al-Basṭī al-Qalaṣādī was a native of Baza in al-Andalus. His *Commentary on the Condensed [Book] on the Operations of Arithmetic* (*Sharḥ Talkhīs aʿmāl al-Ḥisāb*, [M7]) shows the Arabic algebraic notation.[37]
- Sibṭ al-Māridīnī, 1423-1506 (#873). Muḥammad ibn Muḥammad ibn Aḥmad Abū ʿAbdallah Badr [Shams] al-Dīn al-Miṣrī al-Dimashqī was the timekeeper at the al-Azhar mosque in Cairo, and authored dozens of works on arithmetic and astronomy. We cite two works in particular: his *Student's Guide to the Way of Arithmetic* (*Irshād al-ṭullāb ilā wasīlat al-ḥisāb*, [M7]) and *The Light of al-Māridīnī on Commentary on Ibn al-Yāsamīn* (*al-Lamʿah al-Māridīnīyah fī sharḥ al-Yāsamīnīyah*, [M10]).[38]
- Ibn Ghāzī, 1437-1513 (#913, M246). Abū ʿAbdallah Muḥammad ibn Aḥmad ibn Muḥammad Ibn Ghāzī al-ʿUthmānī al-Miknāsī al-Fāsī further condensed Ibn al-Bannāʾ's *Condensed Book* into a 461-line poem titled *Desire of Reckoners* (*Munyat al-ḥussāb*, [M1]). Later, in 1483, he wrote a commentary on his poem titled *Aim of the Students in Commentary on Desire of Reckoners* (*Bughyat al-ṭullāb fī sharḥ munyat al-ḥussāb*, [M2]). This book shows the Arabic algebraic notation, including an entire worked-out problem.[39]
- Luca Pacioli, ca. 1446-1517. Italian Renaissance author whose massive, printed, and highly influential 1494 *Summa de Arithmetica Geometria Proportioni & Proportionalita* presents most of the mathematical knowledge of his time. Much of it ultimately comes from Arabic sources, often via Fibonacci.[40]

Hāʾim 1988), and the third is unpublished.

[35] The Arabic text is published in (al-Kāshī 1969).

[36] Partially translated into English in (Saidan 1968).

[37] Published with a French translation in (al-Qalaṣādī 1999).

[38] The Arabic texts are published in (Sibṭ al-Māridīnī 2004) and (Sibṭ al-Māridīnī 1983), respectively.

[39] For a facsimile of the Library of Congress manuscript see (Ibn Ghāzī manuscript). The Arabic text is published in (Ibn Ghāzī 1983).

[40] (Pacioli 1494).

D. Glossary

For each Arabic word, we give references to the text, and thus also to the translation. References to explanations of the word or concept in our commentary are preceded by a “C”. For words that were used in both technical and non-technical ways, we include references to both. The listing follows the alphabetical order of the roots of the words.

ا

آحاد *āḥād*: units, 65.2, 65.8, 68.19, 69.3,16. C65.2.
اس *uss*: 1. index, 70.15, 71.12, 101.19, 107.12. C68.18.
2. power, 225.1,8, 226.3, 227.15, 229.19. C68.18.
الا *illā*: less, 76.19, 86.11, 140.2, 173.18, 219.5. C141.7, C173.10, C219.4.
امام *imām*: denominator, 94.12, 120.1, 135.12, 138.1, 170.10, 206.6. C93.6.
اول *awwal*: prime, 66.7. C66.7.

ب

بسط *basaṭa*: to numerate, 93.3, 134.1, 138.5, 145.9. C93.10.
بسط *basṭ*: numerator, 93.17, 137.13, 139.10, 142.8, 170.10. C93.10.
مبسوط *mabsūṭ*: numerator, 144.6, 145.3, 151.2, 158.7. C93.10.
بسيط *basīṭ* (and related forms): simple, 66.9, 67.17, 69.15, 134.2, 217.8. C66.7.
مبعض *mub‘‘iḍ*: portioned, 137.1, 139.10.
بقي *baqiya*: to leave, remain, 72.2, 85.5, 88.10, 113.4, 165.1, 229.19. C83.1.
باقي *bāqī*: remainder, 72.7, 84.12, 90.9, 141.6, 176.2, 201.4, 216.2, 220.2, 231.6. C83.1.
بقية *baqiya*: residue, 89.2, 90.5, 91.13, 92.6. C83.1.
باقية *bāqiya*: residual, 92.1,12, 93.12. C83.1.
متباينة *mutabāyna*: 1. different (i.e., relatively prime), 122.13, 123.11. C122.2.
2. incommensurable, 180.11, 182.5. C173.10.

ث

مستثنى *mustathnā* (and related forms): excluded, 137.12, 140.1, 142.11, 183.17, 220.5, 230.1. C140.1, C173.10, C219.4.

محطوط اليه *maḥṭūṭ ilayhi*: reduced number, 129.12, 154.2, 155.14.
حل *ḥalla*: to decompose, 68.15, 117.16, 120.1, 123.4. C68.11.
حل *ḥall*: decomposition, 68.15, 117.2, 122.9.
حمل *ḥamala*: to add, 71.18, 84.17, 108.10, 167.12, 200.4, 214.7, 216.11. C73.2.

خ

مختلف *mukhtalif*: distinct, 136.8, 137.12, 139.1, 142.7, 145.14. C93.15, C136.8, C139.1.
مختلفة *mukhtalifa*: difference, 73.20.

د

درهم *dirham*: 1. dirham (unit of currency), 95.10, 117.16, 158.2, 207.7. C211.8.
2. dirham (unit in arithmetic/algebra), 219.3, 220.3, 221.13, 223.7, 228.3. C211.8.
ادق *adaqq*: finer, 94.11, 135.8, 136.7, 158.10.

ذ

ذهب *dhahaba*: to take away, 169.12, 199.1. C83.1.

ر

ربع *rabaʿa*: to square, 80.5, 112.17, 113.14, 214.7, 215.8, 232.3.
مربع *murabbaʿ*: square, 66.17, 112.17, 164.17, 173.4, 211.10. C66.17.
تربيع *tarbīʿ*: squaring, 79.18, 80.10, 112.16.
مرتبة *martaba*: rank, 65.1, 68.18, 74.22, 174.2, 180.6, 230.14. C68.18, C174.1.
رد *radda*: transform, 143.1, 180.1,6, 184.9, 188.2, 228.8. C179.20.
ركب *rakiba*: 1. to compose (numbers), 122.13, 123.5. C66.7, C68.11.
2. to combine (a proportion), 121.23, 196.17, 204.2. C196.16.
مركبة *murakkaba* (and related forms): 1. composed; composite; composition (numbers), 66.12,16, 68.1, 120.1, 127.4,12, 128.2. C66.7.
2. composite (equations), 211.14, 212.6, 217.1. C211.13, C219.4.

ز

زوج *zawj*: even, 65.6.

زوج الزوج *zawj al-zawj*: evenly-even, 65.12. C65.10.

زوج الفرد *zawj al-fard*: evenly-odd, 65.17. C65.10.

زوج الزوج والفرد *zawj al-zawj wa l-fard*: evenly-evenly-odd, 66.1. C65.10.

زاد *zāda*: 1. to add, 71.13, 110.2, 113.17, 175.7, 204.19, 220.10. C73.2.
2. to exceed, 77.6, 108.9, 169.12. C73.2.

زائد *zāʾid*: 1. appended, 87.11, 183.19, 189.3, 211.5, 228.9. C219.4.
2. exceeds, 198.15, 200.6, 201.9, 203.5. C219.4.

زيادة *ziyāda*: added; additional; excess, 77.7, 108.13, 115.5, 200.6.

س

مسطح *musaṭṭaḥ*: 1. surface (in arithmetic), 67.3, 80.10, 182.5, 225.6, 231.14. C66.17.
2. surface (in geometry), 133.7. C66.17.

سقط *saqaṭa*: to drop, 76.13, 109.17, 123.4, 140.11, 166.17, 175.13, 199.4, 216.18, 231.4. C83.1.

اسم *ism*: 1. name, 59.1, 66.13, 67.16, 69.15, 71.6, 134.1, 157.9, 229.5. C68.18.
2. term, 173.5, 174.5, 176.1, 211.12, 217.2, 225.1, 229.15.

ذو الاسمين *dhū l-ismīn*: binomial (lit., “two unified names”), 173.13, 175.11, 180.13, 188.18. C173.10.

سمى *sammā*: to denominate, 94.1, 108.9, 121.15, 166.9, 181.16, 187.1. C93.6.

تسمية *tasmiya*: denomination, 93.6, 109.7, 118.15, 151.1. C118.14.

مسمى *musammā*: denominated number, 94.3, 124.8, 153.5. C93.6.

مسمى منه *musammā minhu*: denominating number, 93.6, 123.18, 153.5, 181.18. C93.6.

ش

اشتراك *ishtirāk*: common divisor, 122.4, 146.3, 153.10, 230.11. C122.2.

شيء *shayʾ*: thing, 211.8, 213.7, 219.10, 225.2, 227.20. C211.8.

ص

صحيح *ṣaḥīḥ*: 1. whole; whole number, 61.10, 65.2, 92.15, 149.2, 154.10, 163.1, 207.10.
2. posited number, 198.14, 199.14, 200.10, 201.10. See footnote in translation at 198.14.

صرف *ṣarafa*: to convert, 157.10, 158.9. C157.1.
صرف *ṣarf*: conversion, 95.8.
مصروف *masrūf*: fraction to be converted, 158.7,12, 159.1.
مصروف اليه *masrūf ilayhi*: converted fraction, 158.1,8,12.
تصفير *taṣfīr*: voiding, 114.7. C74.17.
صفر *ṣifr*: zero, 69.9, 75.5, 84.1, 98.22, 112.8, 114.4, 119.13, 170.8. C74.17.
اصم *aṣamm*: deaf (i.e., prime), 127.5,14, 128.12. C66.7, C134.2.
اجزاء صم *ajzāʾ ṣumm*: deaf parts (i.e., prime), 66.9, 126.2, 127.9, 128.12. C134.2.
صنف *ṣanf*: kind, 207.8, 208.8. C68.18.

ض

ضرب *ḍaraba*: to multiply, 71.13, 95.10, 96.3, 129.3, 183.2, 211.11, 226.9.
ضرب *ḍarb*: 1. multiplication, 92.8, 95.1, 98.4, 103.18, 157.11, 225.1, 227.6. C95.2, C149.2, C183.4, C225.1.
2. variety, 65.6, 66.15, 67.17; type, 73.3,10, 83.2, 170.14, 211.1,14, 214.3, 223.6; kind, 95.15, 98.4. C68.18.
مضروب *maḍrūb*: 1. multiplicand, 92.12, 95.4, 96.4, 99.17, 102.4, 149.4, 226.9.
2. multiplication, 93.3, 98.9.
مضروب فيه *maḍrūb fīhi*: multiplier, 92.12, 95.4, 96.4, 101.1, 103.4, 149.4, 226.9.
ضعف *ḍaʿafa*: to duplicate, 149.5, 186.1.
تضعيف *taḍʿīf*: duplication; duplicating, 95.2,5, 114.7, 149.2, 184.10, 186.1. C74.17, C95.2, C149.2, C179.1, C186.1.
ضلع *ḍilʿ*: side, 66.18, 67.2,13, 68.9,12. C66.17.
ضاف *ḍāfa*: 1. to add, 97.15, 102.6, 105.3, 110.10, 215.18. C73.2.
2. to attach, 124.5, 135.1, 144.4.
مضاف *muḍāf*: combined, 137.14. C137.13.

ط

طرح *ṭaraḥa*: 1. to subtract, 76.4, 83.7, 84.2, 147.1, 179.1, 211.3, 220.1, 221.1. C83.1.
2. to cast out, 87.18, 88.1, 91.2, 125.16.
طرح *ṭarḥ*: 1. subtraction, 83.1, 91.12, 147.3, 176.11, 211.7, 219.1. C83.1,2, C219.4.
2. modulus, 91.13, 125.6, 126.3, 127.1.
مطروح *maṭrūḥ*: subtrahend, 83.6, 86.15, 148.1, 182.1, 220.13.
مطروح منه *maṭrūḥ minhu*: minuend, 83.6, 86.4, 91.12, 148.1, 220.10.
منطرحة *munṭariḥa* (and related forms): cast out entirely, 87.22, 88.6, 125.8,9, 126.1, 127.1.
مطلوب *maṭlūb*: required number, 76.14, 101.14, 147.12, 171.3, 198.18, 220.17, 228.13, 231.12.

ع

عد *ʿadda*: to count, 66.8, 127.11, 128.1,12, 166.1. C127.9.

عدد *ʿadad*: number, 61.9, 65.2, 68.18, 69.15, 71.13, 73.2, 133.1, 163.3, 170.7, 173.13, 211.8. C65.2, C219.1.

عدة *ʿidda*: number, 79.1,3,10.

عدل *ʿadala*: to equal, 211.13,15, 216.13, 223.2, 227.14, 227.16, 228.1. C211.13, C213.7.

معادلة *muʿādala*: 1. equation, 217.2,4, 227.16,18, 228.2. C213.7.
2. equalization, 211.5, 220.11, 221.8. C213.7.

متعادلان *mutaʿādilān*: two sides of [the] equation, 220.8, 223.1.

عقد *ʿaqd*: power of ten (lit., “bent finger-joint”), 87.17, 109.19, 110.7, 111.3. C110.5.

معقول *maʿqūl*: intelligible, 133.6. C133.3.

علة *ʿilla*: cause, 87.11.

على *ʿalā*: to surpass, 121.6. C73.2.

معنى *maʿnā*: meaning, 68.12, 94.12, 95.5,9,11,13, 113.2, 117.9, 211.1, 213.8. C95.3.

غ

غربال *ghirbāl*: sieve, 127.10. C127.9.

ف

فرد *fard*: odd, 65.6.

مفرد *mufrad* (and related forms): simple, 109.20, 137.13, 211.14, 213.2. C137.13, C211.13.

منفصلة *munfaṣila*: detached, 141.9, 142.8,13. C141.7.

منفصل *munfaṣil*: 1. apotome, 173.4, 173.16, 176.4,18, 182.9, 188.18. C173.10.
2. discrete, 117.4, 133.8. C117.2, C133.3.

فضل *faḍala*: to exceed, 78.1,7. C73.2.

فضل ما بين *faḍl mā bayna*: difference between (sometimes just *mā bayna* or *faḍl bayna*), 78.2, 111.21, 173.9, 197.1, 201.1, 204.14, 229.5.

تفاضل *tafāḍul* (and related forms): disparity, 73.7,15, 76.7, 79.1.

ق

قبل *qabila*: 1. to confront, 198.10, 199.5, 207.3, 217.7, 221.19, 223.9. C223.1.
2. to match, 99.18.

مقابلة *muqābala*: confrontation, 199.6,12, 211.3, 221.8. C211.1, C220.5.

قدر *qadr*: size (of a ratio), 124.6.

تقريب *taqrīb*: approximation, 166.12, 168.6, 169.1, 173.2.

قسم *qasama*: to divide, 61.10, 65.2, 72.1, 93.10, 117.16, 118.1, 119.18, 151.4, 187.2, 189.1, 229.2,12,16, 230.13. C117.2.

قسم *qism*: 1. kind, 65.2, 73.6; type, 82.1, 95.3, 157.11; variety, 163.2. C68.18.
2. part, 118.8, 139.1, 144.8, 202.4,10.

قسمة *qisma*: division, 72.1, 117.2, 118.14,15, 120.20, 151.4, 188.18, 229.6. C68.18, C95.2, C117.2,9, C118.14, C230.11.

مقسوم *maqsūm*: dividend, 93.14, 117.8, 119.2, 151.5, 152.14, 187.3, 229.2,14. C93.6.

مقسوم عليه *maqsūm ʿalayhi*: divisor, 93.6, 117.2, 118.4,18, 119.7, 151.5, 181.15, 188.7, 229.3,18. C93.6.

منقطع *munqaṭiʿ*: disconnected, 140.1,18, 143.1. C140.1.

قانون *qānūn*: basic rule, 170.7, 191.1.

قوة *quwwa*: area (lit., "power"), 177.8,14. C163.4, C177.5. See also *munṭaq fī l-quwwa* below.

مقام *maqām*: denominator, 122.15, 171.2. C93.6.

قياس *qiyās*: rule, 175.18, 183.10, 184.7, 187.18.

ك

كسر *kasr*: fraction, 65.2, 93.15, 133.4, 134.1, 135.10, 143.5, 154.6, 157.4, 170.10. C65.2, C131.1, C133.3, C134.2.

مكعب *mukaʿʿab*: cube, 67.12,15, 68.14,15, 74.1, 80.2. C67.12, C211.8.

كعب *kaʿb*: 1. cube root, 67.13,16, 68.9,12. C67.12.
2. cube (the third power in algebra), 221.1, 225.2,13, 226.7, 227.10, 229.8. C67.12, C211.8.

كم *kamm* (and *kammīya*): quantity, 73.17, 93.5. C73.17, C92.17.

كيف *kayf* (and *kaifīya*): quality, 73.7, 93.5. C73.17, C92.17.

ل

لفظ *lafẓ*: 1. term, 95.5,8,11,13, 184.9, 188.1. C95.3.
2. word, 70.3, 73.24, 117.11.

م

مال *māl*: 1. quantity, 199.1,19, 200.1, 201.1, 203.13, 213.8,14, 215.7. C211.8.
2. *māl* (the second power in algebra), 211.8,15, 213.7, 214.9, 215.12, 225.4,8, 226.1. C211.8.

ن

نزل *nazala*: to remove, 114.8, 123.1, 142.2, 146.3, 221.5. C83.1.

منزلة *manzila*: place, 69.1,17, 71.8,19, 76.18, 83.10, 90.4, 112.6, 164.1. C68.18.

نسبة *nisba*: 1. ratio, 73.8, 78.4, 108.10, 113.10, 117.5, 123.21, 124.6, 133.3, 163.3, 195.7. C131.1, C133.3, C193.1.
2. proportion, 121.23, 191.4, 195.2. C193.1, C195.2.

نسبة عددية *nisba ʿadadiyya*: arithmetic progression, 73.17, 74.5.

نسبة هندسية *nisba handasiyya*: geometric progression, 73.7, 78.13. C73.17.

منتسب *muntasib*: related, 135.10, 138.1, 139.4. C135.1,10.

منطق *munṭaq*: rational (lit., "expressible" in speech), 163.2, 167.9, 172.10, 174.2, 177.8, 183.10. C134.2.

غير منطق *ghayr munṭaq*: surd (lit., "inexpressible"), 163.2,4,6. C163.2.

منطق في القوة *munṭaq fī l-quwwa*: rational in square (lit., "expressible in power"), 163.7, 180.7, 188.7. C163.4.

نقص *naqaṣa*: to subtract, 112.9, 168.14, 173.6, 183.16, 204.10, 215.4, 232.8. C83.1.

ناقص *nāqiṣ*: 1. deleted, 87.11, 189.3, 211.5, 220.10, 228.9. C219.4.
2. falls short, 198.12, 199.7, 202.7, 205.19. C219.4.

تنقيل *tanqīl* (and related forms): shifting, 95.16, 97.3, 98.8, 99.14, 157.7.

منتهى اليه *muntahā ilayhi*: upper extreme, 79.13, 80.5,20, 81.2, 82.2.

نيف *nāfa*: excess, 108.9.

نوع *nawʿ*: species, 65.10, 66.7, 71.13, 211.3,8, 220.5, 226.9; kind, 66.16, 67.18, 157.2, 158.1, 195.2; type, 92.15, 97.1, 99.14, 118.14, 207.9. C68.18.

و

و *wa*: and, C219.1.

موسط *muwassaṭ* (and *mutawassit*): medial, 163.8, 176.14, 180.8, 187.10, 188.7. C163.4.

موسطين *muwassaṭayn*: bimedial, 176.13,18, 177.14. C176.10.

وصل *waṣala*: to join (i.e., form a binomial), 174.15, 175.1,3,5,7,9. C173.10.

متصل *muttaṣil*: 1. continuous, 117.6. C117.2, C133.3.
2. connected, 140.14,16, 141.9, 143.1. C140.14.
3. joining, 177.9,15. C173.10.
4. binomial (lit., "joining"), 188.19,20, 189.12,14. C173.10.

متصلة *muttaṣila*: connected, 142.2,8,11. C140.14.

موضع *mawdhiʿ*: place, 108.12, 109.5. C68.18.

وفق *wafiqa*: to reconcile (i.e., cancel common factors), 120.3,17, 123.1,5. C119.18.

على توالي *ʿalā tawālī*: consecutive; consecutively, 79.13,15, 80.2,5, 81.16, 89.6.

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اللّباب في شرح تلخيص أعمال الحساب

لعبد العزيز بن علي بن داود الهواري المصراتي
كان حيّا سنة 705 هـ / 1304 م

تقديم ودراسة وتحقيق
مهدي عبد الجوّاد (جامعة تونس)
جفري أوكس (جامعة اندِيانابُلِس)

منشورات الجمعية التونسية لديداكتيك الرياضيات

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المؤلفان:

مهدي عبد الجواد (جامعة تونس)

وجفري أوكس (جامعة انديانابلس)

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بسم الله الرحمن الرحيم

قال العبد الخاضع الذليل لربه المرتجي ثوابه وستر ذنبه عبد العزيز بن

علي بن داود الهواري المصراتي عفا الله عنه

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وبعد اطال الله بقاء سيدنا الوزير المعظم السني السري المبارك

المحترم الفقيه العالم العلامة المكرم ذي المناقب العلية والهمم الفاضلة

الابية السنية سراج الدولة الميمونة المرينية ابي محمد عبد الله بن

عمادنا الشيخ الفقيه الصالح الناسك الزاهد الحسن السيرة والمسالك

المرحوم المقدس الاطهر ابي مدين جعله الله من المشفعين في المحشر

خلد الله في المتقين ذكرهم وحفظ من عور الايام وردهم وصدرهم

فاني استخرت الله عز وجل في ان اشرح كتاب شيخنا الفقيه

الاوحد السري الامجد المشارك الاوحد ابي العباس احمد بن محمد بن

عثمان الازدي ابقى الله بركتهم وادام بمنه حفظهم حظوتهم ورفعتهم

المسمى بتلخيص اعمال الحساب اطرزه باسمكم واحليه بحليتكم ليكون

وسيلة

نسخة المدينة المنورة (الورقة 1 ظ)

الفهـــــــــــرس العــــــــام

اللباب في شرح تلخيص أعمال الحساب

تمهيد

تعرض البحث حول "مسائل خاصة بتدريس الرياضيات في القرون الوسطى داخل البلاد العربية والإسلامية"[1] إلى قرار الوزير الأول خير الدين باشا التونسي تحديث التدريس بالجامع الأعظم وإقحام كتاب **اللباب في شرح تلخيص أعمال الحساب** ضمن قائمة الكتب المبرمجة للدراسة في المرحلة الثالثة – وهي العليا – في التعليم الزيتوني. وعندما اطلع الاستاذ بجامعة مينيابوليس جَفْرَايْ أُوكْسْ والمختص بتاريخ الجبر العربي، على هذا المقال استفسرني عن المصراتي، مؤلف **اللباب**، وعن آثاره العلمية. فاتضح لي أن المرحوم الأستاذ محمد سويسي قد وصف **اللباب** بالعمل " القيم والنفيس " واستند إلى نسخة مصورة للمخطوط في درسه وتحقيقه **تلخيص أعمال الحساب** لابن البنا[2]. كما أن أحمد جبار ومحمد أبلاغ تعرضا إلى **اللباب** وأشارا إلى أنه أوّل شرح **للتلخيص** وأن نسخا عديدة موجودة حاليا في مكتبات العالم ومنها نسخة بالمكتبة الوطنية بتونس[3]. فعندما تحصلنا على بعض الصور الشمسية أو الرقمية من المخطوط أيقينا أن هذا الكتاب جدير بالدراسة والتحقيق والترجمة كمساهمة منا للتعريف بالتراث العلمي العربي.

وها هنا الجزء الأول من عملنا المشترك وهو يحتوي على القسم المؤلف باللغة العربية، علما أننا قد أنهينا الترجمة الانكليزية لنص الهواري المصراتي وسننشرها لاحقا في مجلد آخر.

نظمنا هذا الكتاب إلى قسمين وملاحق.

يحتوي القسم الأول على تقديم للمؤلف وللكتاب، وخصصنا القسم الثاني لتحقيق علمي لكتاب **اللباب**. وتحتوي الملاحق على جداول نعرض فيها جميع مسائل كتاب **اللباب**، وعلى الفهارس.

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[1] Abdeljaouad, Mahdi (2003). Issues in the History of Mathematics Teaching in Arab Countries, *ICME-10 : TSG29 : The history of the teaching and learning of mathematics*.

[2] ابن البنا المراكشي، **تلخيص أعمال الحساب** ، تحقيق محمد سويسي. تونس 1969.

[3] أحمد جبار ومحمد أبلاغ، **حياة وأعمال ابن البنا المراكشي**، 2001. (ص. 46 وص. 99-109).

في الجزء الأول من القسم الأول، أثبتنا القليل مما يُعرف عن المؤلف، عبد العزيز الهواري المصراتي، وحاولنا التدقيق في اسمه ونسبه ومهنته. وبعد ما أثبتنا اسم الكتاب أُحصينا نسخه المخطوطة المعروفة حاليا ووصفنا خمس نسخ تمكنا من اقتنائها[1]: أقدمُها وأجملها نسخة مذهبة، تمت كتابتها سنة 746هـ/1345م (مكتبة عارف حكمت بالمدينة المنورة).

في الجزء الثاني، اولينا أهمية خاصة للمحتويات العلمية لكتاب **اللباب** ومنهجه والمستجدات فيه. فقارنا نص **التلخيص** الذي ورد بأكمله في **اللباب** بالنص الذي حققه محمد سويسي واستفدنا من الفوارق بينهما. ثم قابلنا فقرات **رفع الحجاب** التي أحال إليها المؤلف مع نص الكتاب الذي حققه محمد ابلاغ. ثم قدمنا تحليلا لمادة الكتاب : طبيعة الأمثلة العددية، استعمال الرموز في بابي الصحيح والكسور وغيابهم من بابي الجذور والجبر، التناسب والعمل بالكفات، وأخيرا حل المسائل الجبرية.

وفي كتاب **اللباب**، نجد فقرات أملاها ابن البنا على تلميذه، الهواري، ولم تنشر من قبل: الأولى تخص ثلاثة أوجه للعمل بالكفات، وهي طرق لحل المسائل التطبيقية، والثانية هي ثلاث مسائل في الأعداد المضمرة قدمناها كما جاءت في صيغتها اللفضية ثم في صيغتها العصرية.

في الجزء الثالث، قدمنا مستجدات كتاب **اللباب**، وهي أساسا النضريات والمسائل التي أملاها ابن البنا على تلميذه الهواري المصراتي، ولم تنشر من قبل.

في الجزء الرالبع ، ابرزنا مكانة **اللباب** في تاريخ تعليم الرياضيات في المغرب والمشرق ، وتأثيره على من خلفه. فبَيَّنا أن عددا كبيرا من مسائل هذا الكتاب

[1] شكرنا الحار للأستاذة منيرة القنوني التي مكنتنا من التحصل على صورة شمسية من هذا المخطوط.

نقلها ابن غازي المراكشي سنة 890هـ/1485م في كتاب **بغية الطلاب في شرح منية الحساب.**

وكتاب **اللباب** صَاحَبَ كتاب **التلخيص** في عديد من حلقات تدريس الحساب والجبر، حتى في أواخر القرن الثامن عشر الميلادي. ونجد مثالا من ذلك في الكتاب المسمى **أمثلة من التلخيص لابن البنا والحاوي لابن الهائم** والذي ألفه الرياضي العثماني شكر زاده (المتوفى سنة 1202هـ/1787م) وقدم فيه ما لا يقل عن 111 قاعدة وأمثلة ومسائل منقولة من **اللباب**. فقابلنا بين النصين وتأكدنا من تطابقهما في جميع الحالات.

القسم الثاني من الكتاب، خصصناه لتحقيق علمي لكتاب **اللباب**.

حافظ الهواري المصراتي على ترتيب أجزاء كتاب **التلخيص** وتجزئتهم إلى أقسام وأبواب. فيحتوي **اللباب** على ثلاثة أجزاء : الأول في العدد المعلوم وفيه قسم يختص بالعدد الصحيح (أقسام العدد ومراتبه – الجمع (وفيه الجمع على توالي الأعداد) – الطرح (وفيه الاختبار بالطروح الثلاثة) – الضرب (وفيه الانواع المختلفة من طرق الضرب) – القسمة (وفيه القسمة بالمحاصة والقسمة بحل الأعداد وفصل في بحث الأعداد الأوّلية بالغربال). أما القسم الثاني فهو للكسور، والقسم الثالث للجذور.
الجزء الثاني من **اللباب** ينقسم إلى قسمين: الأول في العمل بالنسبة ويحتوي على فصل في العمل بالكفات. والقسم الثاني خاص بالجبر والمقابلة (تعريف الجبر والمقابلة واصطلاحاتهما – الضروب الستة – الجمع والطرح - الضرب ومعرفة الأس والاسم – القسمة).

ملاحظة: بما أننا استعملنا خمس نسخ في هذا التحقيق وجدنا في المقابلة متغيرات عديدة. فاعتمدنا كمرجع أصلي نسخة المدينة المنورة (النسخة "م")، وهي الأقدم والأكمل، وأشرنا إلى المتغيرات بأرقام متتالية ترجع القارئ إلى الحاشية في أسفل كل ورقة. وتتضمن نفس الحاشية السفلى بعض التعليقات أو التصحيحات.

تحتوي خاتمة الكتاب على الملحقات والفهارس.

في الملحق الأول، أحصينا جميع الحالات التي نقلها الهواري المصراتي من كتاب **رفع الحجاب** لابن البنا.

وفي الملحق الثاني، نقدم جداول تحتوي على جميع المسائل العددية والجبرية التي وردت في كتاب الهواري المصراتي مُرتبة حسب أبوابه، ونستعمل في بعضها الرموز العصرية لتسهيل الفهم. وأحلنا كل مسألة إلى ورقتها في مخطوط المدينة المنورة لكتاب **اللباب** (مخطوط م)، وإلى صفحتها في كل من كتابي ابن غازي وشكر زاده كلما نقلها أحدهما.
وأختمنا الكتاب بمعجم للمصطلحات الواردة في كتاب **اللباب في شرح أعمال الحساب** ولفهرس للأعلام ولقائمة الكتب المستعملة في هذا البحث.

نتوجه في النهاية بالشكر الجزيل لمن ساعدنا في بلورة هذا العمل، لا سيما الأستاذ طيب عشاش والأستاذ منصف الحاجي والأستاذة سامية عاشور، وعلى اسهامهم الثمين في مراجعة هذا البحث.

المهدي عبد الجواد
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جَفْرَايْ أُوكْسْ
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تقديم المؤلف و الكتاب

يحتوي هذا القسم على أربعة أجزاء:

- الجزء الأول في التعريف بالمؤلف
- والجزء الثاني في محتويات كتاب **اللباب في شرح تلخيص أعمال الحساب**
- والجزء الثالث في مستجدات كتاب **اللباب**
- والجزء الرابع في مكانة كتاب **اللباب** في تاريخ تعليم الحساب.

الجزء الأول : التعريف بالمؤلف

نستعرض في هذا الجزء الصيغ المختلفة لاسم المؤلف ونسبه وماهيته.

1.1 تقديم الهوّاري المصراتي

يذكر محمد سويسي أنّه يوجد بمكتبة المخطوطات التونسية شرح لكتاب **تلخيص أعمال الحساب** لابن البنا المراكشي (تـ. 721هـ/1321م)، ألفه تلميذ من تلاميذ ابن البنا، وهو عبد العزيز الهواري وينعت محمد سويسي هذا الشرح بالقيّم والنفيس "إذ هو أصل مباشر يمكن أن تعتبر شروحه هي الشروح الشفوية التي حلل بها ابن البناء نفسه تلخيصه، فنقلها عنه تلميذه الهوّاري مسجّلا على الخصوص الصورة التطبيقية لأوضاع العمليات التي يصفها صاحب التلخيص وصفا موجزا مقتضبا." (سويسي 1969: 7).

وبما أننا تمكّنا من اقتناء خمس نسخ مختلفة من كتاب الهوّاري قرّرنا تحقيق معانيه وضبط مصادره وتقييم طرافته والبحث عن آثاره في التأليف والتعليم. وقمنا بترجمته إلى الانقليزية لكي يتسنى للباحث الأجنبي في تاريخ الرياضيات التعرّف المتميز على هذا العمل[1].

2.1 اسم المؤلف و نسبه

اختلف المؤرخون حول اسم المؤلف واسم كتابه والسبب في ذلك عدم تطابق الأسماء الموجودة في النّسخ المعروفة للكتاب.

هو عبد العزيز بن علي بن داود الهوّاري المِصْرَاتِي (بحرف الصاد).

نجد هذه النسبة في مخطوط مكتبة عارف حكمت بالمدينة المنورة، وهي أقدم نسخة معروفة حاليا من هذا الكتاب وكذلك في جل نسخ المخطوط التي تحصلنا عليها ودرسناها في هذا البحث. ويقول مؤرخ الرياضيات العربية أحمد لمْرَابِط إن هذه النسبة تجعل الشارح أصيل **مصراته** الليبية ونزيل المغرب الأقصى[2].

ويقر عبد السلام الشدادي بأنّ **هوّارة** كانت في الأصل مجموعة قبلية بربرية تسكن في ناحية طرابلس بليبيا قبل أن تنتقل إلى إفريقيا والمغرب في القرنين الأوّل والثاني بعد الفتوحات الإسلامية.[3]

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[1] ننوي نشر ترجمة كتاب الهواري إلى الانقليزية بالولايات المتحدة في آخر هذه السنة.

[2] Lamrabet Driss (1994), Introduction. n° 392, p. 97.

[3] عبد السلام الشدادي، "مقدمة ابن خلدون"، الجزء الثالث، ص. 460.

وفي **بغية الطلاب في شرح منية الحساب** يشير ابن غازي المكناسي (تـ. 1513/919) إلى "المصراتي" مرتين، الأولى صفحة 19 قائلا : "... وإياه اعتمد المصراتي"، والثانية صفحة 210 حيث جاء: "قال ابو محمد عبد العزيز المصراتي لما ذكر ما في **رفع الحجاب** هنا ...".[1]

أمّا النسبة الثانية أي "المسراتي" (بحرف السين) فهي ترجع أصلا إلى محمد المنّوني الذي يذكر أنّ عبد العزيز بن علي بن داوود الهوّاري المسراتي هو من قبيلة هوارة البربرية وأنّه أصيل بني مسريت المجاورة لمدينة فاس.[2] ويثبت ذلك محمد سويسي في مقالته: **الأرقام والرموز في الرياضيات العربية**[3] وكذلك أحمد جبار ومحمد أبلاغ في كتابهما: **حياة ومؤلفات ابن البنا المراكشي**[4]. ويقتصر جلّ المؤرخين كحاجي خليفة وخير الدين الزركلي ورضا كحالة وجواد إزقي بنعته بالهوّاري وكذلك بروكلمان.
وحرّف ناقل نسخة أكسفورد اسم الشارح تحريفا كاملا إذ يسمّيه في مقدمة الكتاب: "الهَدادي المعداني" ويكتب في هامش المخطوط: "نقل السيوطي أن الهدادي بفتح الهاء وبالدالين المهملين والتخفيف في الدال الأولى نسبة إلى هداد من الأزد والمَعداني بفتح الميم والعين المهملة نسبة إلى معدان جده".[5]
وكتابة "الهوازي" بحرف "الزاي" في بعض النسخ (خاصة في آخر نسخة أكسفورد وفي أوّل وآخر نسخة تونس) يمكن أن تكون تحريفا أدخله ناسخ أول ثم تواصل من بعده.

3.1 من هو االهوّاري المصراتي ؟

لا يعرف شيء كثير عن عبد العزيز بن علي بن داود الهوّاري المصراتي: اسمه غائب من الكتب المتخصصة في بيو- بيبلوغرافية العلماء ككتاب **الاحاطة في أخبار غرناطة** لابن الخطيب (تـ. 1374/776) وكتب التراجم والطبقات القديمة ككتب ابن حجر العسقلاني (تـ. 1449/853) وابن القاضي (تـ.

[1] ابن غازي المكناسي، "**بغية الطلاب**"، حلب 1983.
[2] محمد المنوني، "ورقات"، (ص. 331-332).
[3] محمد سويسي، **نماذج** ، باب "الأرقام والرموز في الرياضيات العربية"، (ص. 237-259). أنظر خاصة ص. 250.
[4] أحمد جبار ومحمد أبلاغ : **حياة ومؤلفات ابن البنا المراكشي**، 2001. أنظر خاصة ص. 90.
[5] في تمهيد التحقيق لكتاب **تلخيص أعمال الحساب** يخص محمد سويسي فصلا هاما (صفحات 5-7) يذكر فيه ملاحظات فوبك (Woepcke) وحاجي خليفة وما جاء في فهرس كازري (Casiri) وفي فهرس درنبورغ (Derenbourg) ورينو (Renaud) حول الهوّاري وكتابه.

1616/1025). وأوّل اشارة إلى الهواري توجد في **كشف الظنون** لحاجي خليفة (القرن 11هـ/17م).

كان عبد العزيز الهوّاري تلميذا لابن البنا حوالي سنة 700هـ/1300م، إذ كما أورده أحمد جبار ومحمد أبلاغ (2001: 46) هو "الوحيد من بين طلبة ابن البنا الذي تصدى لشرح كتاب **التلخيص** بتشجيع من ابن البنا نفسه. ويكون بذلك أول من قام بشرح هذا الكتاب وهي المهمة التي سيقوم بها أساسا رياضيون تتلمذوا على طلبة ابن البنا". وحصر جبار وأبلاغ تاريخ كتابة هذا الشرح "ما بين سنة 702 و 706 هجريا". وتمكنّنا من اثبات تاريخ انهاء تأليف **اللباب** في 18 ذي القاعدة سنة 12/704 جوان 1305، كما نصّ عليه أقدم مخطوط معروف حاليا لكتاب الهوّاري وهو مخطوط المدينة المنورة (الورقة 64 و).
ولا نعرف هل تابع عبد العزيز الهوّاري دروس أستاذه ابن البنا التي كان يلقيها بالجامع الكبير بمراكش أو هو "من بين الذين كانوا يتلقون دروسا ومذكرات خاصة من أستاذهم في منزله" (أبلاغ وجبار 2001: 46). والأرجح أنه كان من بين الصنف الثاني إذ شجعه ابن البنا على شرح كتاب **التلخيص**.

وللهواري أخ يُدعى محمد بن علي بن داود الهواري المصراتي، الذي درس الفرائض على "شيخه الطانجي علي بن عبد الرحمن اليَفْرَنِي المكناسي"، وأنهى محمد الهواري تأليف **نزهة الرائض في علم الفرائض**[1] سنة 1347/748.

لا يمكننا قبول اشارة أحمد لمْرَابِط (2008: 34) بأن الهوّاري انتقل بعد تكوينه في مراكش إلى مدينة سبته حيث اشتغل عدلا وأجاز للقاضي محمد بن يحيى العشقري، ولابن الجياب استاذ ابن الخطيب، وللقاضي ابن سلمون الكناني. لقد اتّضح لنا بعد التثبت في كتاب **الإحَاطة في أخبار غرناطة** لابن الخطيب وفي كتاب **درة الحجال في أسماء الرجال** لابن القاضي أنّ هناك شيخا عدلا مشهورا عاش في سبتة بين سنة 1220/617 وسنة 1301/701، وحدد ابن القاضي بدقة اسمه: "عبد العزيز بن ابراهيم بن عبد العزيز بن أحمد بن نبيه أبو فارس الهوّاري الجزيري السبتي" وأشار إلى أنه أجاز لابن جابر[2]، في حين اكتفى ابن الخطيب بالكنايات: "أبو فارس عبد العزيز الهوّاري الجزيري" وذكر أنه أجاز

[1] محمد المنوني، **نشاط الدراسات الرياضية في مغرب العصر الوسيط الرابع**، (ص. 97-98). يقتبس المنوني هذه المعلومات من المخطوط عدد س 8389 بالخزانة العامة في الرباط.
[2] ابن القاضي، **درة الحجال**، المجلد 3، عدد 1082، (ص. 133-134).

العديد من طلبته[1] وهم القاضي أبو عبد الله بن أبي بكر (تـ. 1340/741)، والقاضي أبو بكر محمد بن أحمد (تـ. 1346/747)، والإمام علي بن علي القرشي (تـ. 1343/744)، ولابن الجياب (تـ. 1348/749)، والقاضي ابن سلمون الكناني (حي سنة 1284/683).
وذكر كذلك كل من ابن جابر[2] والتجيبي[3] وابن سليمان[4] في برامجهم الشيخَ العدلَ أبا فارس عبد العزيز الهواري كأستاذهم ورئيس جماعتهم.
ولا يمكن أن يكون هذا الشيخ المتوفى سنة 1301/701 قد ألف كتاب **اللباب في شرح التلخيص** سنة 1305/705، إضافة إلى أنّ جميع هؤلاء المؤلفين والشيوخ لم يذكروا لا عليا ولا داودا، اسمي أب وجد الهواري المصراتي.

وفاة الهواري المصراتي

لم يحدّد حاجي خليفة تاريخ وفاة الهواري. واسماعيل باشا البغدادي هو أول من إقترح 745هـ/1344م كحد أقصى لسنة وفاة الهواري، لكنه لم يعلل هذا الاقتراح بأي قرينة[5]. وأصبح هذا التاريخ هو المتداول في كتب التراجم[6] وكتب تاريخ الحساب العربي.

[1] ابن الخطيب، **الاحاطة**، المجلد 2، ص. 176-180 وص. 239-249، والمجلد 4، (ص. 197-200 وص. 125-128 وص. 309-310).
[2] ابن جابر الوادي آشي التونسي (تـ. 1338/749). **برنامج ابن جابر الوادي آشي التونسي**، 1979. الجزء 1، عدد 196. (ص. 142-143).
[3] القاسم بن يوسف التجيبي (تـ. 1329/730) : **برنامج التجيبي**، 1981. (ص. 74-75).
[4] أبو الحسن ابن سليمان القرطبي (تـ. 1329/730).
[5] اسماعيل باشا البغدادي، **هدية العارفين** 1951، المجلد الاول، (ص. 582).
[6] عُمر رضا كحالة، **معجم المؤلفين** (ص. 787).

الجزء الثاني : محتويات "كتاب اللباب في شرح تلخيص أعمال الحساب"

في المشغل الأول لهذا الجزء، نستعرض جميع مخطوطات كتاب **اللباب** المعروفة حاليا، ونصفها كلما أمكن ذلك.
أما محتوى كتاب **اللباب،** فهو موضوع مشغلنا الثاني، وقد بدأنا بتقديم أوّلي للكتاب، تليه مقابلة بين نصّ كتاب **تلخيص أعمال الحساب** لابن البنا كما ورد في كتاب **اللباب** ونصّه الذي حقّقه محمد سويسي ونشره سنة 1969.
وأما المشغل الثالث لهذا الجزء، فهو تحليل دقيق لأمثلة الكتاب، خاصة العددية والجبرية، والنظر المعمّق في أنواع الرموز والعلامات والصور المستعملة في حساب الأعداد الصحيحة والكسور وفي حساب الكفات.

1.2 عنوان مخطوط الهوّاري ونسخه المعروفة حاليا

الاختلاف الثاني حول الهوّاري يخصّ عنوان كتابه. فبعضهم يقول إنّه : "اللباب في شرح تلخيص أعمال الحساب لأبي العباس بن أحمد الأزدي" أو "الباب لشرح تلخيص أعمال الحساب"[1] وكذلك "التلخيص في الحساب للهواري"، ويقول البعض الآخر: "كتاب شرح التلخيص" والباقي: "كتاب اللباب". والأرجح أن العنوان الأصح هو: "اللباب في شرح تلخيص أعمال الحساب"، كما جاء ذلك في نسخة المدينة المنورة، أقدم نسخة معروفة حاليا، كما ذكرنا آنفًا.

2.2 مخطوطات "اللباب في شرح تلخيص أعمال الحساب"

المخطوطات المُعتمدة في هذا التحقيق

أ. مخطوط رقم "21 حساب" مكتبة عارف حكمت بمكتبة المدينة المنورة. عنوانه: "اللباب في شرح تلخيص أعمال الحساب لأبي العباس بن أحمد الأزدي". سنرمز لهذا المخطوط في التحقيق بحرف (م). يحتوي هذا المخطوط على 62 ورقة (1و – 62ظ). مقياس كل واحدة منها هو 21×16 صم ، وفي كل صفحة 19 سطرا بها 11 كلمة تقريبا. النسخة مذهبة والخط نسخي. ويقول المؤلف في آخر الكتاب: "وكان الفراغ منه يوم السبت ثامن عشر ذا القعدة عام أربعة وسبعمائة". أما تاريخ هذه النسخة واسم ناسخها فهو بائن في خاتمة المخطوط: "كمل يوم الثلاثاء ثامن عشر، شهر ربيع

[1]" حسب **هدية العارفين** لابراهيم باشا البغدادي. ص. 582 .

الأول، عام ستة وأربعين وسبعمائة. وكتب لنفسه بخط يده الفانية راجيا عفو ربه، عبد الرحمن بن أبي بكر بن أحمد بن علي بن أحمد بن علي بن سبع بن مالك النفزي عفا الله عنه". فيكون هذا المخطوط أقدم نسخة من الكتاب معروفة حاليا (1345/746).

ب. مخطوط مجموعة شهيد علي باشا بالمكتبة السليمانية باسطنبول، رقم 2/1977. عنوانه: "شرح التلخيص في الحساب للهواري". سنرمز له في التحقيق بحرف (أ). يحتوي هذا المخطوط على 52 ورقة (54و – 103ظ). وفي كل صفحة 21 سطرا بها 12-13 كلمة تقريبا. تاريخ النسخة واسم ناسخها: "علّقه بنفسه عن نسخة سقيمة، للضرورة، عمر بن عثمان بن عمر الحُسيني بمدينة القسطنطينية المحروسة،غفر الله لمن ترحّم على كاتبه وعلى جميع المُسلمين بتاريخ عشرين رمضان المُكرّم سنة ثمانين وثمانمائة هجرة".(1475/880).

ت. مخطوط مكتبة أكسفورد: رقم Marsh 378. عنوانه : "كتاب اللباب". سنرمز له في التحقيق بحرف (ب). يحتوي هذا المخطوط على 63 ورقة (109و – 162ظ). وفي كل صفحة 25 سطرا بها 9-10 كلمة تقريبا. تاريخ النسخة واسم الناسخ غائبان.

ث. مخطوط المكتبة الوطنية بتونس. رقم 9940. عنوانه : "كتاب شرح التلخيص". سنرمز له في التحقيق بحرف (ت). يحتوي هذا المخطوط على 32 ورقة (54و – 103ظ). مقياس كل واحدة منها هو 21,5×18 صم ، وفي كل صفحة 29 سطرا بها 14-15 كلمة تقريبا. اسم الناسخ وتاريخ النسخة: " وقد نُجز بعون الله القدير ليلة الجمعة رابع ليلة خلت من جمادى الآخرة من شهور سنة إثنتين وثمانين وألف، على يد العُبيد الفقير المُعترف بالعجز والتقصير تحويل مُصنّف العلم الشريف بدمشق الشام يسين بن مصطفي الإمام بتكية المرحوم السلطان سليمان خان عليه الرحمةُ والرضوانُ والرضى الحنفي المآثر برضى عنهما وعن المسلمين آمين." (1671/1082).

ج. مخطوط مكتبة مجلس الشورى الاسلامي بتهران. رقم 2/239. عنوانه : "شرح القانون يا شرح التلخيص". سنرمز له في التحقيق بحرف (ط). اسم المؤلف في الصفحة الأولى (10و): عبد العزيز بن علي بن داود الهواري المعراقى وفي الصحة الأخيرة (56ظ): "عبد العزيز بن علي بن داوود الهواري المصراتي". يحتوي هذا المخطوط على 46 ورقة (10و – 56ظ) مقياس كل واحدة منها هو 15×20 صم، وفي كل صفحة 23 سطرا. قرأ هذا المخطوط علي بن ناصر الدين بن الطرابلسي الحنفي، سنة

1564/972. وامتلكه صالح مصطفى (حي سنة 1709/1121) وأحمد الطيبي. ويحمل المجموع خاتم الحاج مصطفى صدقي (تـ. 1769/1183).

المخطوطات غير المستعملة في هذا التحقيق

مخطوطان ذكرهما Brockelmann[1]

الأول: مكتبة أكسفورد : مخطوط رقم "Bodlean I, 76/3".

الثاني: مكتبة المتحف الهندي: مخطوط رقم "Loth 3/770". عنوانه : "اللباب. شرح تلخيص أعمال الحساب " لعبد العزيز بن على بن داوود الهواري المصراتى". يحتوي على 51 ورقة (19ظ – 69ظ). مقياس كل واحدة منها هو 17,78×13,335 صم ، وفي كل صفحة 23 سطرا. اسم الناسخ وتاريخ النسخة: "محمد بن عبد الله الطراني الأزهري الشافعي"، فى آخر جمادى الاولى سنة 17/856 جوان 1452.

مخطوطان من مكتبة الاسكوريال بمدريد، ذكرهما Derenbourg [2]

الأول: مخطوط رقم 2/948. النص الثاني بعد كتاب **التلخيص** لابن البنا من مجموعة سبعة نصوص. يحتوي على 49 ورقة (11ظ – 59ظ). مقياس كل واحدة منها 18×13,5 صم ، وفي كل صفحة 17 سطرا. كتاب الهوّاري ناقص في أوله وتاريخ النسخة: 28 صفر 867/ 22 نوفمبر 1462.

الثاني: مخطوط رقم 953. عنوانه: "كتاب غاية الكُتّاب: شرح تلخيص أعمال الحساب". تأليف: "الشيخ الامام العالم العلامة الحبر البحر الفهامة المحقق المدقق عبد العزيز بن على بن داود الهواري المصراتي، عفا الله عنه وعن جميع المسلمين. آمين، آمين، آمين". تحتوي هذه النسخة على 79 ورقة، مقياس كل واحدة منها 18×13,5 صم ، وفي كل صفحة 17 سطرا. تاريخ النسخة غائب، لكن هناك علامة امتلاكها سنة 1567/975.

مخطوطان من المكتبة السليمانية باسطنبول ذكرهما رمضان ششن[3] .

[1] Brockelmann : Geschichte II, page 331.

[2] Derenbourg H. (1941), *Les Manuscrits arabes de l'Escurial*, tome II, fascicule 3: Sciences exactes et occultes, Paris: Langues Orientales (p.79 & p. 95-94).

[3] رمضان ششن: **نوادر المخطوطات العربية في مكتبات تركيا** 1975. (ص. 571-572)

الأول: مجموعة لا له لي: مخطوط رقم 2740. عنوانه: "اللباب في شرح تلخيص أعمال الحساب لابن البنا". وهو من القرن العاشر الهجري (أي السادس عشر ميلادي) وله 59 ورقة.
الثاني: مجموعة قليج علي باشا: مخطوط رقم 683. عنوانه: "اللباب في شرح تلخيص أعمال الحساب لابن البنا". وتاريخ النسخة: 1659/1070. تحتوي هذه النسخة على 61 ورقة (13ظ – 73ظ).

مخطوطات المغرب الأقصى:
الأول[1]: مخطوط الخزينة العامة بالرباط، رقم ق 846. وأصله من مكتبة الزاوية الناصرية بتمكروت. عنوان المخطوط على الغلاف: "شرح تلخيص ابن البنا للهواري فى الحساب". وفي المقدمة اسم المؤلف: "عبد العزيز بن على بن داوود الهواري المصراتي"، وعنوان الكتاب: "الكتاب فى شرح تلخيص أعمال الحساب".
الثاني[2]: مخطوط الزاوية الحمزية، رقم 2/145.
الثالث[3]: مخطوط مكتبة تمكروت بالمغرب الأقصى، رقم 3080.
الرابع[4]: مخطوط المكتبة الحسينية بالرباط، رقم 2/2186. يحتوي المخطوط أولا على كتاب **رفع الحجاب** لابن البنا (44 ورقة) ثم على كتاب الهوّاري المصراتي (64 ورقة). الخط مغربي لكن حالة النص رديئة ولم نتمكن من قراءة العناوين والكلمات المكتوبة باللون الأحمر في الصورة الشمسية التي تحصلنا عليها.

مخطوط دار الكتب المصرية[5]، رقم 6829 كـ عربي. عنوانه على الغلاف واسم المؤلف: "الباب فى شرح تلخيص أعمال الحساب لعبد العزيز بن على بن داود". يحتوي على 53 ورقة (1و – 53ظ). مقياس كل واحدة منها هو 21×14,5 صم. وتاريخ النسخة: 1640/1050.

مخطوط مكتبة وزارة الأوقاف بالقاهرة. رقم 1077. عنوانه على الغلاف: "شرح تلخيص ابن البنا فى علم الحساب". والعنوان واسم المؤلف في

[1] محمد المنوني، **نشاط الدراسات الرياضية في مغرب العصر الوسيط الرابع** 1985. (ص. 83-84)
[2] نفس المرجع.
[3] محمد المنوني، **دليل المخطوطات دار الكتب الناصرية بتمكروت**، 1985.
[4] **كشاف الكتب المخطوطة بالخزانة الحسينية بالرباط.**
[5] http://www.manuscripts.idsc.gov.eg/Manuscript/Manuscript_OPAC/SimpleResult.aspx#

مقدمة الكتاب: "اللباب فى شرح تلخيص أعمال الحساب لعبد العزيز بن على بن داود الهدادى المعدانى". واسم المؤلف كما جاء في أخر المخطوط: "عبد العزيز بن على بن داود الهوارى المصراتي". يحتوي على 80 ورقة. مقياس كل واحدة منها هو 23×17 صم ، وفي كل صفحة 19 سطرا. اسم الناسخ وتاريخ النسخة: "يوسف رزين، الشافعى مذهباً، الأحمدى طريقة، البناوى بلدا، فى يوم الثلاثة المبارك خمس عشر يوما مضت من صفر سنة 1270" (أي 1853م).

مخطوط دارالكتاب الظاهرية بدمشق[1]، رقم 1/6666 عام. عنوانه: "اللباب في شرح تلخيص أعمال الحساب". تأليف "عبد العزيز بن على بن داود الهواري". تحتوي هذه النسخة على 112 ورقة (1ظ – 112ظ). مقياس كل واحدة منها 21,5×15,5 صم ، وفي كل صفحة 19 سطرا. ولها هامش عرضه 4 صم عليه شروح وتوصيات وتعليقات. اسم الناسخ: يحيى بن تقي الدين بن اسماعيل بن عبادة. وتاريخ النسخة: 1 صفر 25/1002 أكتوبر 1593.

مخطوط المكتبة الوطنية بتونس. رقم 7/9783. عنوانه: "كتاب الباب لشرح التلخيص في علم الحساب". وهو رسالة ضمن مجموع به 8 رسائل. يحتوي المخطوط على 20 ورقة (63ظ – 82ظ)، وكأنه من مذكرات طالب إذ تختلط فيه نصوص قصيرة من كتب مختلفة. الخط مشرقي وتاريخ النسخة: 1611/1020. حبّس هذا الكتاب الوزير خير الدين باشا التونسي على مكتبة جامع الزيتونة، سنة 1875/1292.

[1] **فهرس مخطوطات دار الكتاب الظاهرية**، عدد 48، (صفحة 33-34).

وهذه قائمة مخطوطات كتاب **اللباب** مرتبة حسب تاريخ النسخ:

المخطوط	العنوان	المؤلف	تاريخ النسخة
المدينة: 21 حساب	اللباب في شرح تلخيص أعمال الحساب		1345/746
المتحف الهندي: Loth 3/770	اللباب في شرح تلخيص أعمال الحساب	داوود	1452 /856
اسكوريال: 2/948	ناقص في أوله		1462/867
أكسفورد: Marsh 378	اللباب في شرح تلخيص أعمال الحساب	الهدادي المعداني	بعد 1467/872
شهيد علي باشا: 2/1977	شرح التلخيص في الحساب للهواري		1476/880
تهران: 2/239	شرح القانون يا شرح التلخيص	المعراقى	1564/972
اسكوريال: 953	كتاب غاية الكُتّاب: شرح تلخيص		قبل 1567/975
الظاهرية، دمشق: 1/6666	اللباب في شرح تلخيص أعمال الحساب		1593/1002
تونس: 7/9783	كتاب الباب لشرح التلخيص في علم الحساب		1611/1020
دار الكتب بمصر: 6829 كـ	الباب فى شرح تلخيص أعمال الحساب	بن داود	1640/1050
قليج علي باشا: 683	اللباب في شرح تلخيص أعمال الحساب لابن البنا		1660/1070
تونس: 9940	كتاب شرح التلخيص		1671/1082
وزارة الأوقاف القاهرة: 1077	الباب فى شرح تلخيص أعمال الحساب	الهدادي المعداني	1853/1270

3.2 محتويات كتاب الهوّاري المصراتي

ان كتاب **اللباب في شرح تلخيص أعمال الحساب** هو أول شرح **لتلخيص** ابن البنا، شرع الهوّاري في انجازه وهو لا يزال طالبا من بين طلاب ابن البنا، وشجعه أستاذه على ذلك العمل.

يتناول الهوّاري المصراتي في كتابه نفس المواضيع الموجودة في **التلخيص** إذ يتجزأ الكتاب إلى جزئين: الأول "في العدد المعلوم" والثاني "في القوانين التي يمكن بها الوصول إلى معرفة المجهول المطلوب من المعلوم المفروض". وينقسم الجزء الأول إلى ثلاثة أقسام: قسم "في العدد الصحيح" وقسم "في الكسور" وقسم "في الجذور"، وينقسم الجزء الثاني إلى قسمين: القسم الأول في العمل بالنسبة مشتملا على فصل حول العمل بالكفات، والقسم الثاني في العمل بالجبر والمقابلة.

يقول محمد سويسي : "إن التلخيص جملة أعمال ألقاها ابن البنا على تلاميذه، وعلّق عليها الهوّاري بما حفظه من شروح شيخه" (سويسي 1969: هامش صفحة 35). ويتبع الهوّاري الطريقة التقليدية في الشرح إذ يشتمل كتاب **اللباب** على النص الكامل لكتاب **التلخيص** مبوبا إلى فقرات يتبع كل منها شرح المفاهيم والقواعد معتمدة على أمثلة عددية سهلة الفهم.

وبعد أن نبّه الشارح إلى أنه شرع في شرحه كتاب **التلخيص** بعد انهاء ابن البنا من كتابته **رفع الحجاب**، الذي هو أيضا شرح وتكملة لكتاب **التلخيص**، فلم يتردّد عن نقل الكثير من التحاليل والأمثلة العددية الواردة في كتاب **رفع الحجاب** دون أي إشارة إلى مصدرها. ولم يذكر الهوّاري كتاب **المقالات الأربع في العدد** وكتاب **الأصول والمقدمات في الجبر والمقابلة** التي هي من أولى مؤلفات ابن البنا في الحساب والجبر[1] ، ولم يستشهد كذلك بأي رياضي آخر ما عدا إشارة قصيرة إلى ابن الياسمين[2] (مخطوط م، الورقة 23ظ: 12-15) .

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[1] أحمد جبار ومحمد أبلاغ ، **حياة ومؤلفات ابن البنا المراكشي**. الرباط 2001.

[2] ابن الياسمين هو أبو محمد عبد الله بن حجاج الأدريني، وهو شاعر ورياضي من خادمي الخليفة المُوحّدي يعقوب المنصور (1184/580 – 1198/595)، ثمّ ابنه الخليفة الناصر لدين الله (1198/595 – 1215/612). ولد ابن الياسمين في فاس وتكوّن في المغرب ثمّ في الأندلس. ودرّس في مرّاكش حيث مات ذبيحًا سنة 1204م. ووصلت إلينا أرجوزته في الجبر والمقابلة. وله **كتاب تلقيح الأفكار برشوم حروف الغبار** وهو من أقدم كتب الحساب والجبر والهندسة الذين وصلوا إلينا من الغرب الإسلامي.وحققه تهامي الزمولي في أطروحة ماجيستار سنة 1993.

مقارنة نص "التلخيص" المنقول في "اللباب" والنص المحقق

يميز الهوّاري المصراتي فقرات كتاب **التلخيص** بكتابة علامة في صدر كل فقرة، وهذه العلامة تختلف من ناسخ إلى الآخر: فهي حرف "ص" في نسخة اسطنبول وهو الحرف الأخير من لفظ "تلخيص" وربما حرف دال على كلمة "المصدر". وحرف "م" في نسخة أكسفورد وهو الحرف الأول من لفظ "مؤلف". أما ناسخ نسخة الرباط المختلة الأوراق فهو يبتدأ فقرات كتاب **التلخيص** بكلمة "قوله".
وكذلك شأن فقرات الشرح إذ يشير إليها الناسخون بحرف "ش"، وهو الحرف الأول من لفظ "شرح". ويبدأ ناسخ نسخة المدينة المكرّمة برسم سطر أحمر فوق نص جمل كتاب **التلخيص**، لكنه سرعان ما يعدل عن ذلك، فتختلط نصوص المؤلف مع نصوص الشارح. وليس في نسخ أكسفورد وتونس وتهران أي علامة تميز بين الأصل والشرح.

ويمكّننا التمييز بين ما لابن البنا وما للهواري بالإستناد إلى نسخة شهيد علي باشا (مخطوط أ)، فنضبط نصا جديدا لكتاب **التلخيص** قابلا للمقارنة مع النص المعروف الذي حققه محمد سويسي ونشره. فيلاحظ قارئ نسخة استانبول أن نص **التلخيص** عند الهوّاري المصراتي أطول وأثرى من النص المحقق، ولكن يعتبر ناسخ نسخة تونس كل الزيادات من إضافات الهوّاري المصراتي.

لاحظنا ثلاثة أنواع من الإضافات:
جمل أو فقرات، البعض منها تكملة لازمة لما سبقها من نص كتاب **التلخيص**، ويجعل غيابها المفهوم ناقصا.
محتويات منقولة حرفيا من كتاب **رفع الحجاب**، وقد استمع لها الهوّاري المصراتي أثناء دروس أستاذه واعتبرها التلميذ لازمة.
البعض الآخر شرح لما سبق، واعتباره من نص كتاب **التلخيص** صعب إذ لا يتماشى مع غرضه المبني على الإجاز.

وتفسير الاختلافات بين النسخ يرجع حسب محمد أبلاغ (2008: 53) إلى فرضيتين:
الفرضية الأولى هي أن النساخ الأوّل – الذين ربما كانوا تلاميذ ابن البنا – هم الذين أدخلوا هذه التعديلات وذلك حسب فهمهم للغرض الذي توخاه ابن البنا.
والفرضية الثانية، وهي الأرجح، تتلخص في كون ابن البنا قدم صياغات مختلفة من كتابه هذا.

فقرة سقطت من تحقيق محمد سويسي:

حين نقارن نصّ **التلخيص** الذي حقّقه محمد سويسي مع ما ورد في شرح الهوّاري المصراتي نجد ما يلي:

جدول مقارنة بين نصّ محمد سويسي و نصّ الهواري المصراتي:

الفقرات	سويسي[1] (صفحة 42)	الهوّاري : مخطوط المدينة (م)
الأولى	"وإن اختلف الوضع فاضرب الباقي في الأول يكن المطلوب." (سطر 9)	"وإن اختلف الوضع فاضرب الباقي في الأول يكن المطلوب. واختلاف الوضع هو أن يكون في البيت الأول غير الواحد." (ورقة 8 و: 1-2)
الثانية	وإن كانت أعداد على تفاضل آخر، فاضرب أصغرها في فضل الأكبر عليه واقسم على الفضل بين الأصغر والعدد الذي يليه وزد الخارج على الأكبر. يكن الجواب." (سطر 11-12)	"وإن كانت أعداد على تفاضل آخر، فاضرب أصغرها في فضل الأكبر عليه واقسم على الفضل بين الأصغر والعدد الذي يليه وزد الخارج على الأكبر. يكن الجواب". (ورقة 8 و: 7-9)
الثالثة	(فقرة غائبة)	"وإن تفاضلت الأعداد بعدّة معلومة دون التّضعيف فاضرب التفاضل في عدّة الأعداد إلّا واحد، فما خرج فاحمل عليه العدد الأوّل، فما بلغ فهو آخر الأعداد، فاجمعه مع الأوّل واضربه في نصف عدّة الأعداد، يكون الجواب." (ورقة 8 ظ: 3-6)

ففي الفقرة الأولى، نلاحظ في الشرح زيادة الجملة: " واختلاف الوضع هو أن يكون في البيت الأول غير الواحد"، التي توجد أيضًا في كتاب **رفع الحجاب**[2]. ويمكن اعتبارها إمّا من الشروح الشفوية لابن البنا عند إلقاء درسه وإما من زيادات الناسخ تفسيرا للجملة الأولى.

وفي الفقرة الثانية، النصان متطابقان.

[1] أبن البنا، **تلخيص أعمال الحساب**، تحقيق محمد سويسي، تونس 1969.

[2] أبن البنا، **رفع الحجاب عن وجوه أعمال الحساب**، تحقيق محمد أبلاغ، الرباط 1994. (ص. 217)

وفي الفقرة الثالثة، النص غائب تمامًا من كتاب **التلخيص**. والراجح أنه سقط من بعض نسخ هذا كتاب لأنه لازم منطقيا لفهم الحالات المختلفة المدروسة هنا. ومما يؤيّد هذا أنّ نفس الفقرة منسوبة في شرح **التلخيص** للقلصادي[1] إلى كتاب **التلخيص**. ولا بدّ هنا من الرّجوع إلى ملاحظة هامة لأحمد جبار (2002: 221) حيث اكتشف أن فصولا عديدة من كتاب **التلخيص** منقولة من كتاب **تلقيح الأفكار في العمل برشوم الغبار** لابن الياسمين[2]، وهي التي تخص فصول جمع الأعداد الصحيحة في كتاب **التلخيص** (1969: 41-42) وضربها وقسمتها (1969: 47-53). فإن مراجعتنا لهذه الفصول أثبتت أن الفقرة: "وإن تفاضلت الأعداد ... يكون الجواب المطلوب" منقولة حرفيا من كتاب **التلقيح** فيصح اعتبارها من النص الأصلي لكتاب **التلخيص**.

فقرات سقطت من تحقيق محمد سويسي للتلخيص وموجودة في رفع الحجاب:

رقم الورقة في **اللباب** [3]	رقم الصفحة في **رفع الحجاب** [4] 4	الموضوع
10 و: 1-4	228: 13-16	الجمع على توالي الأعداد مع الابتداء من غير الواحد
11 ظ: 1-6	245: 8-15	طرح سلسلة من الأعداد
26 ظ: 15 إلى 27 و: 2	263: 4-9	معنايان للقسمة
30 و: 9-12	267: 11-14	العمل غير المشهور في التسمية
35 و: 10-12	275: 8-9	الاستثناءات في الكسور (ذكرها ابن البنا كذلك في كتاب **المقالات الأربع**)
39 ظ: 15	279: 18	تصريف الكسور
41 و: 5-6	283: 3-4	المُنطق وغير المُنطق في الجذور

[1] القلصادي، **شرح تلخيص أعمال الحساب**، تحقيق فارس بنطالب، بيروت 1999. (ص. 52: 15-17)
[2] ابن الياسمين، **تلقيح الأفكار**، تحقيق الزمولي، الجزائر 1993. (ص. 114-116، 119، 131-132، 136).
[3] نسخة المدينة المنورة (مخطوط م).
[4] **رفع الحجاب**، تحقيق محمد أبلاغ.

مفاهيم موجودة في كتاب "اللباب"، ينسبها بعض ناسخيه إلى "تلخيص ابن البنا" لكنها سقطت من تحقيق محمد سويسي لهذا الكتاب ولا توجد في كتاب "رفع الحجاب"، ولا نعرف مصدرها[1]:

- في النوع الثاني من أنواع الضرب بالتسمية: "والعقد المفرد هو ما في أوّل كلّ مرتبة، مثل العشرة والمائة وشبه ذلك". (مخطوط م، 24 و: 2-3).
- في أسماء الكسور: "والسمّي هو الأكبر من العددين المنسوب أحدُهُما إلى الآخر". (مخطوط م، 32ظ: 19 إلى 33و: 1).
- في بسط الكسور التي في وسطها عدد صحيح:

"ومعنى الإضافة إلى ما قبله أن يكون الكسر الأوّل مأخوذا من الصّحيح وحده ويكون معه قسما والكسر الباقي قسما مثل المُختلف، فتضرب بسط كلّ قسم في إمام غيره وتجمع الجميع." (مخطوط م، 36و: 5-8)

"والإضافة إلى ما بعده أن يكون الكسر الأوّل مأخوذا من الصّحيح والكسر الذي بعده. فالصّحيح مُضاف إلى ما بعده وهو مُقدّم. فتبسطه معه وتضرب ذلك في مبسوط الباقي، وهو الكسر الأوّل، لأنّه مُبعّض منه." (مخطوط م، 36و: 13-16)

- في تجذير الكسور: "وهي بالنسبة إلى التجذير، أعني الكسور، على أربعة أضرب، أحدها أن يكون للبسط جذر منطق وللإمام مثله. فالعمل فيه ما ذكر." (مخطوط م، 43 ظ: 7-8).
- في طرح الجذور: "وفيه وجه آخر: وهو أن يقسم أحد المطروحين على الآخر، ثم يؤخذ فضل ما بين الخارج والواحد ويضرب في المقسوم عليه، منهما يخرج المطلوب" (مخطوط م، 47 و: 18-19).

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[1] ينقل ابن غازي المكناسي في **بغية الطلاب** (حلب 1983) عن الهواري بعض هذه المفاهيم والأمثلة المصاحبة لها (أنظر النوع السابع ص. 90: 14-15 ، والكسور المؤلفة ص. 131: 15-18 وص. 132: 7-10 ، وتجذير الكسور ص. 152: 7-8 إلى 154: 4),

4.2 الأمثلة العددية في كتاب الهوّاري المصراتي

يشير الهوّاري المصراتي أنّه شرح كتاب **التلخيص** بعد أن إنتهى ابن البنا من تأليف كتاب **رفع الحجاب**، أي بعد سنة 1301/701. ويُعتبر شرح الهواري المصراتي، قبل كل شيء، شرحا للمفاهيم وليس شرحا أدبيا للمفاهيم أو نحويا لتركيب الألفاظ والعبارات كما هو شأن كثير من شروح كتاب **التلخيص** الأخرى. وليس هو ملخص من كتاب **رفع الحجاب**، الذي ينعتُهُ جبار وأبلاغ بأنه: "شرح رياضي وفلسفي لتلخيص أعمال الحساب كما أنه مكمل نظري لهذا الأخير"[1]. إن التفحّص الدقيق لمحتوى نصّ الهوّاري يُثبت أنّ الشارح التزم باتباع تسلسل المواضيع كما وضعها ابن البنا في كتابه (العدد المعلوم ، حساب الكسور، حساب الجذور، مسائل المجهول: التناسب، حساب الخطأين، الجبر والمقابلة).

واكتفى الهوّاري بتفاسير قليلة للمفاهيم مع تقديم أمثلة عددية ومسائل متنوعة ليست موجودة في كتاب **التلخيص**. فكلما نقل الشارح فقرات من كتاب **رفع الحجاب** وكانت فيها أمثلة عددية أوردها الهوّاري في كتابه.

لكن فائدة هذا الشرح تكمن في كثرة الأمثلة العددية الأخرى التي استنبطها الشارح كما تعهد به في مقدمة كتاب **اللباب** حيث يكتب[2]:

> "فجعلته ممثلا، إذ الكتاب الموضوع عليه ذله المُترجم برفع الحجاب قد استوفى فيه، رضي الله عنه، جميع ما يحتاج إليه ما عدا الأمثلة إلا اليسير منها. وسنورده في موضعه مع ما لا بدّ منه، إن شاء الله تعالى". (مخطوط م، 2و: 2-5)

تنقسم أمثلة الهوّاري المصراتي إلى ثلاثة أنواع:

الأمثلة المنقولة من كتاب **رفع الحجاب** وهي، كما قال الشارح، قليلة (أنظر إلى الملحق الأول): فهي 32 مثال ومسألة موزّعة حسب الأبواب.

- الاستثناءات المكررة في الصحيح (1).
- ضرب الكسر في الكسر (4)
- مسألتان في تفسير معنى الضرب
- مسألتان في تفسير معنى القسمة
- أنواع الكسور (3)
- تصريف الكسور (2)
- تجذير الصحيح (2)

[1] أحمد جبار ومحمد أبلاغ، **حياة ومؤلفات ابن البنا المراكشي**، الرباط 2001. (ص. 100).

[2] أرقام الأوراق من نسخة المدينة المنورة (مخطوط م).

- أنواع الجذور (4)
- ذوات الأسماء (4)
- مسائل الكفات التي لا تستعمل النسبة (3)
- جمع الأجناس الجبرية المختلفة (5)

الثمانية الأمثلة الخاصة بجمع الأعداد المتتالية (مخطوط م: 6 و–7 ظ) هي أيضًا موجودة في كتاب **تلقيح الأفكار في العمل برشوم الغبار** لابن الياسمين[1]. ولا غرابة في ذلك إذ لاحظ أحمد جبار أن العديد من رياضيي الأندلس والمغرب تدارسوا مسائل جمع الأعداد المتتالية (طبيعية أو أفرادا فقط أو أزواجا فقط) ومربعاتها ومكعباتها، ودرّسوها كما يشهد على ذلك فهرس كتاب **البيان والتذكار** للحصار[2] حيث الباب الخامس منه يخص الجمع على توالي الأعداد ومربعاتها ومكعباتها، والباب السادس تضعيف بيوت الشطرنج[3]. وطوّر ابن المنعم العبدري (ت. 1228/626) النظريات الخاصة بالأعداد المتتالية وبرهن على قوانينها وعمم دراستها للأشكال العددية[4]. ويفترض أحمد جبار أن هذه النظريات كانت لا تزال تدرّس في عهد ابن البنا.

أمثلة ومسائل أخرى غير منقولة من **رفع الحجاب**، وعددها:
- في الصحيح (96)
- في الكسور (32)
- في الجذور (54)
- في النسبة والكفات (6)
- في الجبر (64).

وقابلنا أمثلة الهوّاري بما كتبه ابن البنا في كتاب **المقالات الأربع في العدد** وفي كتاب **الأصول والمقدمات في الجبر والمقابلة**، وبالأمثلة العددية والمسائل الموجودة في كتب الحساب والجبر المعروفة حاليا والمؤلفة قبل سنة

[1] ابن الياسمين، **تلقيح الأفكار**، تحقيق الزمولي، الجزائر 1993. (ص. 136-144).
[2] أبو بكر محمد بن عبد الله بن عيّاش الحصّار هو رياضي أندلسي قد زار مرّاكش حوالي سنة 1150 وتوفي قبل آخر القرن 12. وألّف الحصّار كتابين في الحساب: أوّلهما كتاب **البيان والتذكار** يسمّى أيضا بالكتاب الصّغير، وثانيهما **الكتاب الكامل في صناعة العدد**، وهو الذي اكتشف منه أخيرًا محمد أبلاغ وأحمد جبار السفر الاول. أنظر أبلاغ وجبار، **أكتشاف السفر الاول**، 1989. (ص. 189-203).
[3] المصدر السابق: (الملحق ص.201-203)
[4] ابن منعم، **فقه الحساب**، (ص.81-150)

1300/700، مثل كتب الحصار وابن الياسمين وابن المنعم. فلم نجد أي تطابق يذكر. وهو ما يجعلنا نفترض أن الهوّاري اخترع جل الأمثلة الواردة في كتابه.

5.2 الرموز في كتاب الهوّاري المصراتي

ظهرت الرّموز بالأندلس وبشمال إفريقيا في عهد الموحّدين (القرن 6هـ/12م)، وهي رموز خاصة للكسور والجذور وللجمل والمعادلات الجبرية صاحبت انتشار كتب حساب الغبار المنشورة في المغرب الإسلامي أي الحساب الهندي الذي تستعمل فيه لوحة يرشّ عليها غبار ويكتب على الغبار بعود. ولذا سمى ابن الياسمين كتابه: كتاب **تلقيح الأفكار في العمل برشوم الغبار**.

أرقام هندية عربية مشرقية و أرقام هندية عربية مغربية

الأرقام هي صور وأشكال يصطلح عليها جماعة من الحُسّاب لإجراء العمليات الحسابية بأدقّ وسيلة وأسرعها. والنّظام العشري يستخدم عشرة أشكال: الأعداد من واحد إلى تسعة والصّفر. والتّرقيم العشري المنازلي هو الذي تكون قيمة الرّقم فيه تابعة لمنزلته ومنازله عشرية: فقيمة الأربعة في المنزلة الأولى أربعة، وفي المنزلة الثانية أربعون، وفي الثالثة أربعمائة، وهكذا تصاعديًا. فالأرقام العشرة كافية للدلالة على كلّ الأعداد مهما كبرت أو صغرت، وهي تُوظّف لإجراء عمليات حسابية مختلفة، كالجمع والطرح والضرب والقسمة، وكذلك للدلالة على الكسور والجذور وحتّى العبارات الجبرية.

واكتشف العرب في الهند التّرقيم العشري المنازلي. فيقول العالم المؤرّخ الفقيه والقاضي صاعد الأندلسي في كتابه **طبقات الأمم**:

> "ومما وصل إلينا من علومهم (الهند) في العدد حساب الغبار الذي بسّطه أبو جعفر محمد بن موسى الخوارزمي، وهو أوجز حساب وأصغره وأقربه تناولا وأسهله مأخذًا وأبدعه تركيبًا، يشهد للهـــند بذكاء الخواطر وحسن التوليد وبراعة الاختراع." (صاعد الأندلسي 1985: 58)

وتحتوي مُقدّمات كُتُب الحصّار وابن الياسمين مثلا وصفًا للأرقام المُختلفة المُستعملة عند المُسلمين.

فينبّه الحصّار في **الكتاب الكامل في صناعة العدد** أنّ حُسّاب عصره يستخدمون ثلاثة أنظمة حسابية: حساب الغبار والحساب الرّومي[1] وحساب الجمّل[2] ويبدأ بوصف النّظام الأوّل، ثمّ النّظامين الآخرين. ويضع حساب الغبار في مناخه الاجتماعي ويقول:

"الباب الثالث في وضع الرّشوم للأعداد وتصريفها إلى مراتبها.
في هذا البلاد جرت عادة أهل صناعة الحساب وأهل الأعمال وخاصة الدّواوين في بلادنا باستعمال رشوم جعلوها خطًا مُتعارفًا بينهم يصلون به إلى معرفة الأعداد وتمييز بعضها من بعض، فصار خطًا كسائر الخطوط، العبرانية واللاتينية والحميرية وغير ذلك من الرّشوم التي جُعلت خطًا."
(ورقة 6و من مخطوط رقم 313 ، خزانة ابن يوسف، مرّاكش)

ثمّ يعلّل الحصار الاسم الذي خُصّص لهذا النّوع من الحساب:

"وهي عندهم على ضربين: ضرب منه يسمّونه الغبار ويُسمّى أيضًا هذا الضرب بالهندي. وإنّما سمّوه كذلك لأنّ أصل عمل الحساب بها إنّما هو أنّهم يعمدون إلى لوح من خشب ويبسطون عليه غبارًا دقيقًا، ثمّ يأخذ الذي يريد تعلّم الحساب عودًا صغيرًا على هيئة القلم يرشم به تلك الخطوط في ذلك الغبار ويعمل به مسألته الذي يريد عملها من الحساب. فإذا انقضى عملها مسح على الغبار وضمّه".
(نفس المرجع)

ونفهم من الفقرة التالية أنّ نوعين من الألواح كانا يستخدمان، الأوّل يستعمل الرّمل والمحو والثاني المدد والمحو كذلك، لكن الأوّل أسهل الاستعمال للمتعلّم.

"وإنّما فعلوا ذلك تقريبا على المتعلّم وتسهيلا عليه حتّى لا يحتاج إلى مداد ولوح ومحو في كلّ وقت، فأقاموا الغبار مقام المدد ووجدوه أسهل للعمل. فلذلك سمّوه بالغبار". (نفس المرجع)

ومن جهته يؤكّد ابن الياسمين[3]:

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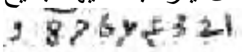
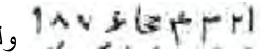
[1] الحساب الرّومي أو الزّمامي أو القلم الفاسي: هو ضرب خاص من الحساب أصله يوناني استُعمل في الدّواوين السلطانية في اسبانيا وواصل حسّاب الدّواوين الأندلسية والمغربية استخدامه كصناعة خاصة بهم.
[2] حساب الجُمّل: هو أقدم نظام استعمله العرب وهو يستخدم حروف الأبجد كأرقام للدّلالة على الأعداد والذاكرة والعقود الأصابع في العمليات الحسابية.
[3] ابن الياسمين، **تلقيح الأفكار**، تحقيق الزمولي، الجزائر 1993. (الورقة 8 من المخطوط)

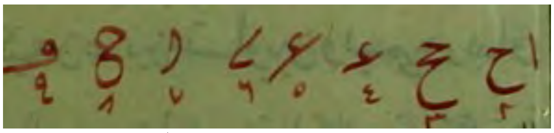
"أن الرّشوم التي وضعت للعدد تسعة أشكال يتركب عليها جميع العدد، وهي التي تُسمّى أشكال الغبار، وهي هذه:  وقد تكون:  ولكن الناس عندنا على الوضع الأول، ولو اصطلحت مع نفسك على تبديلها أو عكسها لجاز، ووجه العمل على حاله لا يتبدل. وقد صنعها قوم من جواهر الأرض مثل الحديد والنحاس من كل شيء منها أعداد كثيرة ويضرب بها ما شاء من غير نقش ولا محو. وأما أهل الهند فيتخذون لوحا أسود يمدون عليه الغبار وينقشون فيه ما شاءوا، ولذلك سُمّي حساب الغبار. ...".

ان نسخ كتاب الهوّاري التي درسناها تستعمل الأرقام حسب الجهة التي أنجز فيها. غير أن ناسخ نسخة المدينة المنورة، وهي أقدم نسخة وصلتنا والتي تمت كتابتها سنة 1347/747، يستعمل أرقاما مغربية غريبة نوعًا ما، تدل على أنه شرقي وأنه اجتهد في رسم الحروف حسب ما جاء في أبيات أوردها الهوّاري

(نسخة المدينة : 3 ظ – 4و):

ألف وحاء ثمّ حجّ بعده عـو * وبعد العو عينٌ تُـــرسم
هـاء وبعد الهاء شكل ظاهر * يبدو كمخطاف كذلك تنظـمّ
صفـران ثامنها وألف بينــها * والواو تاسعها كذلك تـفهم

وهذه هي سلسلة صور الأشكال التي استعملت في ذلك المخطوط، فنلاحظ أنّ الناسخ وضع تحت صورة كل رقم مغربي صورة الرقم المشرقي المطابق له وواصل رسم الصورتين معا في الصفحات الموالية إلى أن اكتسب مهارة كافية في كتابة الحروف المغربية فاكتفى بصورها.


(نسخة المدينة : 3 ظ)

الكسور في "كتاب اللباب في شرح التلخيص" للهوّاري المصراتي

اهتمّ عربُ الغرب الإسلامي بالأبواب الخاصّة بالكسور في كتب حساب الغبار المنشورة في القرن الثاني عشر وبعده ويمكن القول إنّهم بالغوا في بحثهم عن خواصّ الكسور. وأبدعوا كذلك في تصنيف الكسور وتعريفها وتنظيمها.

ومن جهة أخرى فإنّهم أدخلوا رموزًا جديدة باستعمالهم الخطّ الفاصل بين البسط والمقام وقد مكّنهم ذلك من تسهيل استعمالها في العمليات. و لم تكن هذه الرّموز متماثلة وموحّدة في أوّل الأمر، ثمّ استقرّت في أوائل القرن الرابع عشر مع نشر كتاب **المقالات الأربع في العدد** لابن البنا، وهي نفس الأشكال الموجودة عند الهوّاري المصراتي.

أنواع الكسور وصورها:

الكسر البسيط: $\frac{1}{2}$ ، $\frac{1}{3}$ ،... ، $\frac{1}{9}$ ، $\frac{1}{10}$ وكذلك: $\frac{1}{11}$ و $\frac{1}{17}$.

الكسر المنتسب: $\frac{2\quad 4\quad 5}{3\quad 5\quad 6}$. وقراءته: خمسة أسداس وأربعة أخماس سدس وثلثي خمس سدس.

وكتابته في عصرنا هكذا: $\frac{5}{6}+\frac{4}{5\times6}+\frac{2}{3\times5\times6}$.

الكسر المختلف: $\frac{5}{6}\ \frac{4}{5}$. وقراءته: خمسة أسداس وأربعة أخماس.

وكتابته في عصرنا هكذا: $\frac{5}{6}+\frac{4}{5}$.

الكسر المبعّض: $\frac{5\ |\ 3}{6\ |\ 4}$. وقراءته: ثلاثة أ رباع خمسة أسداس.

وكتابته في عصرنا هكذا: $\frac{5\times3}{6\times3}$.

وبذل الهوّاري كابن البنا والرّياضيين المغاربة السابقين جهدًا كبيرًا لحساب بسط كلّ نوع من الكسور مهما تشعّبت قراءتها واختلفت صورها. وكلّ عملية على الكسور تبتدئ ببسط كلّ كسر قبل إجراء جمعها أو طرحها أو ضربها أو غير ذلك من العمليات. فهذه أمثلة بسط المستثنى والعدد المتكوّن من عدد صحيح وكسر، أو من كسر وصحيح ، أو من صحيح وكسر وصحيح:

قراءة الكسر المستثنى: $\frac{6}{8}$ إلا $\frac{1}{9}$: ستّة أثمان إلّا تسع واحد ، وبسطه: 46،

وكتابته في عصرنا هكذا: $\frac{6}{8}-\frac{1}{9}$ وهو $\frac{46}{72}$.

قراءة عدد صحيح وكسر: $5\frac{5\quad 3}{6\quad 4}$: خمسة وخمسة أسداس وثلاثة أرباع سُدس، وبسطه: 143.

وكتابته في عصرنا هكذا: $5+\frac{5}{6}+\frac{3}{4\times6}+$ وهو $\frac{143}{24}$.

قراءة كسر وصحيح: $10\frac{6}{8}\frac{4}{7}$: أربعة أسباع وستّة أثمان عشرة، وبسطه: 740 .

وكتابته في عصرنا هكذا: $(\frac{4}{7}+\frac{6}{8})\times10$ ، وهو $\frac{740}{56}$.

قراءة صحيح وكسر وصحيح $\frac{3}{6}5\frac{4}{9}$ ، لدينا حالتين:

القراءة الأولى: أربعة أتساع خمسة، وثلاثة أسداس، وبسطه: 147.

وكتابته في عصرنا هكذا: $(\frac{4}{9}\times5)+\frac{3}{6}$ ، وهو $\frac{147}{54}$.

القراءة الثانية: أربعة أتساع الخمسة والثلاثة الأسداس، وبسطه: 132.

وكتابته في عصرنا هكذا: $\frac{4}{9}\times(5+\frac{3}{6})$ ، وهو $\frac{132}{54}$.

إنّ كثرة القراءات المُمكنة للكسور (في صيغتها المكتوبة أو المُصوّرة) هي نتيجة تشعّب التصنيفات النظرية التاريخية للكسور العربية ولحرص حسّاب المغرب على التدقيق والتعميم، وغاية هذا الحرص هو النزعة الموسوعية الشمولية.

العمليات على الأعداد والكسور في كتاب الهوّاري المصراتي

يقدّم الهوّاري المصراتي العمليات على الأعداد والكسور بنفس الطريقة المتبعة في كتب الرّياضيين المغاربة كالحصار وابن الياسمين: فهي صور طبق الأصل منقولة من صورتها على اللوح، العدد أو الكسر الثاني تحت العدد أو الكسر الأوّل وفوقهما أو يليهما الحاصل من الجمع أو الطرح أو الخارج من الضرب أو القسمة:

مثال طرح تسعة وستين وأربعمائة وثلاثة آلاف من ثلاثة وأربعين وخمسمائة وستة آلاف:

$$\begin{array}{c}3074\\ \hline 6543\\ 3469\end{array}.$$

مثال ضرب ثلاثة أرباع وثلث فى ثلاثة أتساع وأربعة أسداس التسع وخُمس سُدس التّسع. الصورة:

$$\frac{1\ 3}{3\ 4} ، \frac{1\ 4\ 3}{5\ 6\ 9}$$

تكون صورة الخارج:

$$\frac{1\ 0\ 0\ 4}{4\ 5\ 6\ 9} .$$

وكذلك اخترع الحُسّاب الأندلسيون والمغاربة رموزا تتماشى مع النظام العشري التنازلي وتُسهّل العمل الجبري وتجعل حلّ المسائل أسرع. فنجد في كتاب ابن الياسمين رموزا للدلالة على الجذر وعلى الشيء والمال والمعادلة الجبرية. لكن رموز الجذور والعبارات الجبرية غائبة في كتب ابن البنا وكذلك في شرح الهوّاري. ولم ينتشر استعمالها إلا في المغرب الاسلامي[1] في النصف الثاني من القرن 7هـ/14م. فنجدها في مؤلفات الغُربي (كان حيا 1350/751) والمواحدي (كان حيا 1382/784) وابن قنفذ القسنتيني (تـ. 1408/811) والعقباني (تـ. 1408/811) وابن هيدور التادلي (تـ.1413/816) والقلصادي (تـ.1486/891) وابن غازي المكناسي (تـ. 1513/919).

6.2 العمل بالكفات في كتاب "اللباب في شرح أعمال الحساب"

على غرار ما جاء في كتاب **التلخيص** يقدّم الهوّاري العمل بالكفات كوجه خاص من العمل بالنسبة الهندسية ويعلّل قول ابن البنا "إن الكفات فهي من صناعة الهندسة" بالإستشهاد إلى ما جاء في كتاب **رفع الحجاب** من "أنّ نسبة خطأ كل كفة إلى فضل ما بين كفة والعدد المجهول كنسبة العدد المفروض إلى المجهول." (297: 17-18)

ويختم الهوّاري باب العمل بالكفات بنقل فقرات من **رفع الحجاب**[2] يبيّن فيها ابن البنا أنه يمكن استخراج المجهول بالكفات في مسائل ليس فيها تناسب.

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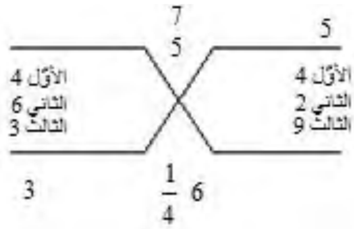
[1] وقلّ استعمال الرّموز المغربية في الشرق الإسلامي حيث لا نجدها أساسا الا في نسخ كتب مغربية انتشرت خارج شمال إفريقيا ككتب القلصادي وابن غازي وفي شروحها وشروح كتاب **تلخيص أعمال الحساب** لابن البنا. فمعلوم أن ابن مجدي (ت. 1446/850) يستعمل الكتابة المغربية للكسور في كتاب **حاوي اللباب وشرح تلخيص أعمال الحساب** وكذلك عبد القادر السّخاوي (ت. 1506/910) في الكتاب **المختصر في علم الحساب** وعثمان ابن مالك الدمشقي (حي 1593/1002) في كتاب **شمس النهار في صناعة الغُبار**. وفي تركيا ظهرت الرّموز المغربية في كتاب **تحفة الأعداد لذوي الرشد والسداد** لابن حمزة الجزائري (ت. 1614/1022) وهذا الكتاب كُتب باللغة التركية واستعمل في المدارس العثمانية.

[2] **رفع الحجاب**، صفحة 299: 1-14 وصفحة 299: 25 إلى 300: 20.

المسألة الأولى هي مسألة رجال يتبايعون دابّة. وهي من المسائل "السيالة" أي التي قد تؤدي إلى أجوبة عديدة. وهذا النوع انتشر في كتب الحساب القديمة قبل حلول العصر الإسلامي. ونجد نفس هذه المسألة في **كتاب الأصول والمقدمات في الجبر والمقابلة** لابن البنا[1]، ومسائل تشبهها في **كتاب الفخري** للكراجي[2]، لكن حلّ المسائل في هذين الكتابين يقع باستعمال الطرق الجبرية.

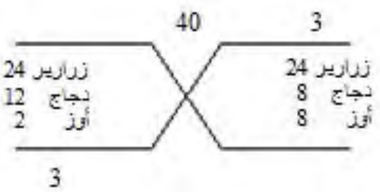
"ثلاثة رجال تبايعوا دابّة: فقال الأوّل للثاني: اعطني نصف ما معك، إلى ما معي يكون معي ثمنُ الدّابة. وقال الثاني للثالث: اعطني ثلث ما معك، إلى ما معي يكون معي ثمنُ الدّابة. وقال الثالث للأوّل: اعطني رُبع ما معك، إلى ما معي يكون معي ثمنُ الدّابة"... وصورة ذلك هذه:



(مخطوط م، 54ظ)

والمسألة الثانية هي مسألة من مسائل الطيور، وهي أيضًا سيالة وقديمة. نشر أبو كامل (تـ 930/318) أوّل رسالة عربية حول هذا الموضوع حيث يلتجئ إلى طريقة جبرية بحتة لحل هذه المسألة.[3]

"أربعون طائرا، ما بين أوزّ ودجاج وزرازير بأربعين درهمًا. الزّرازير ثمانية بدرهم، والدّجاج واحدة بدرهمين، والأوزّ واحدة بثلاثة دراهم. كم أخذ من كلّ صنف من الطير؟"... وصورة ذلك هذه:



(مخطوط م، 55ظ)

[1] ابن البنا، **كتاب الأصول والمقدمات في صناعة الجبر** ، تحقيق سعيدان 1986.(ص. 570-571).
[2] الكراجي، **كتاب الفخري**، تحقيق سعيدان 1986.(ص. 224-228).
[3] أبو كامل، **كتاب طرائف الحساب** ، سعيدان (1986: 61-80).

في قراءاتنا كتب الحساب العربية المؤلفة قبل القرن الرابع عشر لم نتعرض فيها إلى حلّ ابن البنا لهذا النوع من المسائل السيالة بطريقة الكفات. أما بعد القرن 7هـ/15م، فنجدها في **بغية الطلاب** بابن غازي المكناسي (ص. 218-222)، وفي كتاب **رشف الرضاب من ثغور اعمال الحساب** للقطرواني[1].

[1] مخطوط الخزينة العامة بالرباط، رقم 416. منسوخة سنة 1501/907. (ص. 182-186 و 188).

الجزء الثالث : مستجدات كتاب الهواري المصراتي

نقدّم في هذا الجزء فقرات أملاها ابن البنا على تلميذه الهواري المصراتي ولم تنشر من قبل، الأولى تخصّ ثلاثة أوجه للعمل بالكفات والثانية تتكون من ثلاث مسائل في الأعداد المضمرة.

1.3 ثلاثة أوجه للعمل بالكفات أملاها ابن البنا على الهواري

بعد نقل الخورزميات المنصوصة في كتاب **التلخيص** لحل المسائل بطريقة الكفات يشرع الهوّاري بإضاحها معتمدًا على ستة أمثلة عددية، ويذكر أن هذا الوجه لا يُعمل به إلّا في ما فيه تناسب. ثمّ ينقل ما أملاه عليه ابن البنا، وهي ثلاثة أوجه أخرى للعمل بالكفات:

" ثلاثة أوجه،

أحدها : تضرب فضل ما بين الكفّتين في أحد الخطأين، فإن كان الخطآن زائدين أو ناقصين، قسمت الضرب على ما بينهما. وإن كان أحدُهما زائدًا والآخر ناقصًا قسمت الضرب على مجمُوعهما، فما خرج من ذلك تزيده على الكفّة التي ضربت في خطئها إن كان ناقصًا وتنقصُه منها إن كان زائدًا. يحصل المطلوب.

والوجه الثاني : تضرب ما بين الكفّتين في مجموع الخطأين إن كانا زائدين أو ناقصين، وتقسم على ما بينهما. وإن كان أحدُهُما زائدًا والآخر ناقصًا، تضرب ما بين الكفّتين في فضل ما بين الخطأين وتقسم على مجموع الخطأين، فما خرج من ذلك تحفظه. فإن شئت زدت المحفوظ على ما بين الكفتين وأخذت نصف المجتمع، تزيده على الكفّة التي خطأها أكبر إن كان ناقصًا وتنقصه منها إن كان زائدًا، يحصل المطلوب.

وإن شئت فخذ المحفوظ وما بين الكفّتين وانقص أقلّهما من أكثرهما وخذ نصف الباقي، تزيده على الكفّة التي خطأها أقلّ إن كان ناقصًا وتنقصه منها إن كان زائدًا، يحصل المطلوب.

والوجه الثالث : تضرب ما بين الكفّتين في العدد المفروض لك، فإن كان الخطأن زائدين أو ناقصين، قسّمت الضرب على ما بينهما، وإن كان أحدُهما زائدًا والآخر ناقصًا قسّمت الضرب على مجمُوعهما، يحصل المطلوب. فافهمه" (مخطوط المدينة، 53 ظ – 54 و).

2.3 مسائل يختم بها الكتاب

ينهي الهوّاري كتابه بفصل يحتوي على ثلاث مسائل ينسبها المؤلف لأستاذه ابن البنا:

> "ولنختم هنا هذا التّأليف بثلاث مسائل من ملح الحساب، إذ لا يزال الحُسّاب يختمون بها مُصنّفاتهم" (مخطوط المدينة، ورقة 62 ظ: 6-7) ...
> "وهذه المسائل الثلاثة أيضًا ممّا أملى عليّ شيخُنا الفقيه **ابو العبّاس**."
> (ورقة 63 و: 5-6).

هل لهذا الفصل أهمية تذكر وما هو نوع الرياضيات الواردة فيه ومدى طرافتها؟
جرت العادة عند مؤلفي كتب الحساب والجبر انهاء كتبهم بطرح سلسلة من المسائل، إمّا مصحوبة بحل واحد أو بحلول مختلفة وإمّا خالية من أي حلّ.
كمثال عن الحالة الاولى ما جاء من تفسير في آخر كتاب **الأصول والمقدمات في الجبر والمقابلة** لابن البنا:

> "وهذا الجزء مسائله لا تتناهى كثرة ولكن أذكر منها ما أرى أنه يتنبه بها على استعمال الحيلة في ايجاد الجواب في كل مسألة يمكن الجواب عنها، ويظهر للطالب فيها أيضا كيف تصريف تلك الأصول التي قدمناها في الجزء الأول من هذا الكتاب، مع أنها لا تخلو من رياضة و تدبّر." (سعيدان 1986: 556).

وكمثال عن الحالة الثانية ما جاء في آخر فصل من كتاب **الفوائد البهائية في القواعد الحسابية** لابن الخوام (تـ. 1324/724)، وهي سلسلة مكونة من 33 مسألة :

> "التي لا يمكن أن يوتي بجواب لواحد منها ... ولسنا ندّعي فيها أننا نقيم البرهان على امتناعها ونقول إنما لا يمكننا عملها، فمن كان في قوته الوصول إليها ففي قوته ما ليس في قوتنا". (ابن الخوام، مخطوط تونس عدد 8607، ورقة 28 ظ).

مسائل الأعداد المضمرة

أمّا الأمثلة الثلاثة التي اختتم بها الهوّاري كتابه: **اللباب في شرح تلخيص أعمال الحساب**، تحت إملاء أستاذه ابن البنا، فهي من نوع ثالث: نوع المسائل المُضمرة التي يمكن أن تُعتبر من مجال الترفيه والتسلية واللغز ولكنها في الحقيقة مسائل تهدف امتحان قدرات الطالب الحسابية إذ هي مبنية على القوانين

التي دُرست في الأبواب الأولى من الكتاب وصحة الجواب فيها يتطلّب تحليلا دقيقا للمعطيات وبراهين رياضية تفسّر الخطوات المُتّبعة للوصول إلى الحلّ.
وكل مسألة من مسائل العدد المُضمر يمكن أن تفهم بعد اللجوء إلى الحساب أو الجبر، وصيغة العمل فيها هو حوار بين شخصين: سائل و مسؤول. فالسائل يطلب من رفيقه أن يفكر في عدد وأن يخفيه عنه، فهذا هو العدد المُضمر. ثمّ يأمره بالقيام بسلسلة من العمليات الحسابية محورها العدد المُضمر، ويطلبه أن يصرّح له بالنتيجة النهائية للعمليات، فهذا هو العدد المُصرّح به. فيحوّل السائل العدد المُصرّح به عن طريق بعض الأعمال الحسابية و يستخرج إثرها العدد المُضمر.
فيمكن أن نعتبر العدد المُضمر عددا مجهولا والعدد المُصرّح به عددا معلوما. فالمسألة ترجع إلى مسألة حسابية يُستخرج منها العدد المجهول من العدد المعلوم.
ويمكن لهذه المسائل أن تحتوي أكثر من عدد مُضمر وأكثر من عدد مُصرّح به.

وتداول الحُسّاب مسائل الأعداد المُضمرة قبل الإسلام وبعده. فالكندي (تـ. 870/257) هو أول عربي ألّف **"رسالة في الحيل العددية وعلم أضمارها"**. وألّف الأنطاكي (تـ. 987/377) أيضا **"رسالة في استخراج الأعداد المُضمرة"**.[1] وخصّ عبد القادرالبغدادي (تـ. 1038/430) الباب الثاني عشر من كتابه **"التكملة في الحساب"** للبحث "في استخراج الخبيء والأعداد المُضمرة" (صفحات 291-296). ويختم ابن الياسمين (تـ. 1204/601) كتابه **"تلقيح الأفكار في العمل برشوم الغبار"** بفصل يحتوي على 11 مسألة من الأعداد المُضمرة. فحللها مالك بوزاري ولاحظ امتناع المؤلف عن تقديم أي برهان للخوارزميات المتّبعة للوصول إلى العدد المُضمر[2].

المسائل الثلاثة في الأعداد المُضمرة لابن البنا

إن المسائل الثلاثة التي أملاها ابن البنا على طالبه الهوّاري المصراتي هي من نوع استخراج الأعداد المُضمرة وهي تشابه مسائل ابن الياسمين ولكنها تختلف عنها وتختلف عن كل المسائل الواردة في المراجع المشار إليها أعلاه. فيمكن أن نعتبر أن ابن البنا استنبطها خصيصا ليختم بها الهوّاري كتابه وامتحانا لقرّائه.

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[1] الرسالتان موجودتان ضمن مخطوط عدد 4830 بمجموعة أيا صوفيا (اسطنبول): الاول في الورقات 81أ-86أ، والثاني في الورقات 35ب-36ب.
[2] أنظر (Djebbar : 1999) و (Bouzari : 2000).

ولم يذكر النص البراهين اللازمة لإثبات مراحل العمل، بل اكتفى ابن البنا بسرد التعليمات الواحدة تلو الأخرى، في صيغة أوامر غير معللة من السائل إلى المسؤول.

نقدم في ما يلي تحليلا رياضيا لهذه المسائل. ونحاول اتّباع أوامر السائل واستعمال القوانين الجبرية المعهودة والطرق الحسابية المقترحة. فاتضحت لنا صحّة الخوارزميات في المسائل الثلاثة.

ويختم ابن البنا كل مسألة بتعميمها إذ يعوّض فيها العشرة المفروضة في مدخل المسألة بأي عدد آخر مفروض.

المسألة الأولى:

"أن نأمره أن يُسقط عدده من عشرة، ثمّ يسقط مُربّع الباقي من مُربّع عدده.
إن كان الباقي أقلّ، يُخبرنا بالباقي، فنقسمه نحن على عشرة. وما خرج نزيد عليه نصف بقيته إلى العشرة، يكون المُضمر.
وإن كان مُربّع الباقي أكثر، فيسقط منه مُربّع المُضمر ويُخبرُنا بالباقي، فنقسمه على العشرة ونسقط الخارج من عشرة. ونصف الباقي هو العدد المُضمر.
وإن شئنا أن نأمره بإسقاط عدده المُضمر من غير العشرة، ونتّبع العمل، يحصل المطلوب." (مخطوط م، 62 ظ: 8-13)

التحليل

- العدد المجهول (المُضمر): س.
- العدد المعلوم (المُصرّح به): ب.

مراحل العمل:

$10 - \text{س}$ $\quad\quad$ س

الفرضية الأولى: $10 - \text{س} < \text{س}$

$$\text{ب} = \text{س}^2 - (10 - \text{س})^2 \longrightarrow \frac{\text{ب}}{10} + \frac{1}{2}\left(10 - \frac{\text{ب}}{10}\right) = \text{س}$$

البرهان:

$$\text{ب} = \text{س}^2 - (10 - \text{س})^2 = 20\text{س} - 100$$

$$\frac{\text{ب}}{10} + \frac{1}{2}\left(10 - \frac{\text{ب}}{10}\right) = 5 + \frac{\text{ب}}{20} = 5 + \frac{20\text{س} - 100}{20} = 5 + (\text{س} - 5) = \text{س}$$

الفرضية الثانية: $10 - \text{س} > \text{س}$

$$\text{ب} = (10 - \text{س})^2 - \text{س}^2 \longrightarrow \frac{1}{2}\left(10 - \frac{\text{ب}}{10}\right) = \text{س}$$

البرهان

$$\text{ب} = (10 - \text{س})^2 - \text{س}^2 = 100 - 20\text{س}$$

$$\frac{1}{2}\left(10 - \frac{\text{ب}}{10}\right) = 5 - \frac{\text{ب}}{20} = 5 - \frac{100 - 20\text{س}}{20} = 5 - (5 - \text{س}) = \text{س}$$

الوضعية العامة: تعويض 10 بأي عدد شئت.

المسألة الثانية :

"نأمره أن يقسم العشرة بقسمين، يُضمرها. ثمّ نأمره أن يقسم مُربّع أحدهما على مُسطّحهما ويُخبرُنا بالحاصل. فإذا علمناه، فهو نسبة أحد القسمين إلى الآخر، فنقسم العشرة على تلك النّسبة. وكذلك نفعل في أيّ عدد شئنا غير العشرة. فيخرج القسمان المُضمران." (مخطوط م، 62 ظ: 13-17)

التحليل

- عددان مجهولان (مُضمران): س و ش حيث $10 = \text{س} + \text{ش}$.
- العدد المعلوم (المُصرّح به): النسبة ن.

مراحل العمل

إقسم ش^2 على س ش $\longrightarrow \frac{\text{ش}^2}{\text{س ش}} = \text{ن}$

البرهان

$$\text{ن} = \frac{\text{ش}^2}{\text{س ش}} = \frac{\text{ش}}{\text{س}}.$$

$$\frac{10}{\text{ن}} = \frac{\text{س} + \text{ش}}{\text{ن}} = \frac{\text{س} \cdot \frac{\text{س} + \text{ش}}{\text{س}}}{\text{ن}} = \frac{(\text{ن} + 1)\,\text{س}}{\text{ن}} \longrightarrow \text{س} = \frac{10\,\text{ن}}{\text{ن}(\text{ن} + 1)} = \frac{10}{\text{ن} + 1}$$

$\text{ش} = 10 - \text{س}$.

الوضعية العامة: تعويض 10 بأي عدد شئت.

المسألة الثالثة :

"عددٌ مُضمر وقسمان مُضمران، كم هو؟ وكم كلّ واحد من قسميه؟
نأمره أن يضرب أحد القسمين في الآخر وأن يُربّع كلّ واحد منهما، ثمّ يسقط مُربّع الأصغر من مُربّع المُسطّح ويخبرنا بالباقي ويسقط المُسطّح من مُربّع الأكبر ويخبرنا بالباقي. فنأخذ جذر فضل ما بين المُخبر عنهما يكون ما بين القسمين. فنقسم

عليه مجموع ما أخبرنا به، يكون العدد المُضمر وهو مجموع القسمين. فإن زدنا عليه جذر فضل ما بينهما كان ضعف أكبرهما، وإن نقّصناه من مجموعهما بقي ضعف أصغرهما". (مخطوط م، 62 ظ: 17 إلى 63 و: 5)

التحليل

- ثلاثة أعداد مجهولة (مُضمرة): س و ش و ج حيث ش>س و ج = س+ش.
- عددان معلومان (المُصرّح بهما): ب و ت.

مراحل العمل

إضرب س في ش ⟵ س ش

ربّع س و ش ⟵ ش^2 و س^2

إسقط س ش من ش^2 ⟵ ش^2 − س ش = ب

إسقط س^2 من س ش ⟵ س ش − س^2 = ت

$$\text{ج} = \frac{\text{ب} + \text{ت}}{\sqrt{\text{ب} - \text{ت}}} \quad \text{،} \quad \text{ش} = \text{ج} + \sqrt{\text{ب} - \text{ت}} \quad \text{،} \quad \text{س} = \text{ج} - \sqrt{\text{ب} - \text{ت}}$$

البرهان

ب + ت = (ش^2 − س ش) + (س ش − س^2) = ش^2 − س^2

ب − ت = (ش^2 − س ش) − (س ش − س^2)

= ش^2 − 2 س ش + س^2 = (ش − س)^2.

$$\sqrt{\text{ب} - \text{ت}} = \sqrt{(\text{ش} - \text{س})^2} = \text{ش} - \text{س}.$$

$$\frac{\text{ب} + \text{ت}}{\sqrt{\text{ب} - \text{ت}}} = \frac{\text{ش}^2 - \text{س}^2}{\sqrt{(\text{ش} - \text{س})^2}} = \frac{(\text{ش} - \text{س})(\text{ش} + \text{س})}{\text{ش} - \text{س}} = \text{ش} + \text{س} = \text{ج}.$$

$$\text{ج} + \sqrt{\text{ب} - \text{ت}} = (\text{ش} + \text{س}) + (\text{ش} - \text{س}) = 2\ \text{ش}.$$

$$\text{ج} - \sqrt{\text{ب} - \text{ت}} = (\text{ش} + \text{س}) - (\text{ش} - \text{س}) = 2\ \text{س}.$$

الجزء الرابع : مكانة كتاب اللباب في تاريخ تعليم الحساب

في هذا الجزء، نستعرض مسائل كتاب **اللباب** التي نقلها ابن غازي المكناسي في كتاب **بغية الطلاب في شرح منية الحساب** في القرن الخامس عشر الميلادي، وكذلك مسائل الهواري المصراتي التي نقلها الرياضي العثماني شكر زاده في كتابه **أمثلة من التلخيص لابن البنا والحاوي لابن الهائم**، المؤلف في القرن الثامن عشر الميلادي.

ونختم المقدمة بذكر ادماج كتاب **اللباب** ضمن برنامج التعليم بجامع الزيتونة بتونس وذلك سنة 1875/1292. وهذا آخر دليل على أهمية كتاب الهواري المصراتي في تاريخ تعليم الحساب.

1.4 الهوّاري المصراتي و ابن غازي المكناسي

جاء في تقديم محمد سويسي لابن غازي المكناسي (تـ 919 هـ / 1513 م):

> "اعتمد < ابن غازي > في شروحه العلمية في مادة الحساب على **تلخيص الحساب** لابن البنا وعلى **رفع الحجاب** و **مختصر** الإمام أبي بكر الحصار وعلى رسائل شيخ شيوخه أبي الحسن ابن هيدور التادلي. ويرجع أيضا إلى ابن قنفذ أحمد بن حسين بن علي القسنطيني في شرحه على **التلخيص** وهو **حط النقاب عن وجوه أعمال الحساب** (...)، كما استدل بأقوال القاضي أبي عثمان العقباني في شرح التلخيص وتعاليق تلميذه أبي عبد الله المكناسي".
> (**بغية الطلاب في شرح منية الحساب**، ص: يب و يج).

لا شك أن ابن غازي كان يعرف كتاب الهوّاري المصراتي إذ هناك إحالتان له في **بغية الطلاب** :

الأولى خاصة بأسوس الأعداد، يقول ابن غازي عند تفسيره تعريف الأس: "والأسّ عبارة عن مرتبة العدد. فأسّ الآحاد واحد، يعني أنها في المرتبة الأولى، وأسّ العشرات اثنان، يعني أنها في المرتبة الثانية، وأسّ المئين ثلاثة، يعني أنها في المرتبة الثالثة. بهذا فسره [ابن البنا] في **المقالات** وهو المعروف، وإياها اعتمد المصراتي وغيره". (ابن غازي 1983، ص. 19: 8-11)

الثانية خاصة بالعمل بالكفات، يقول ابن غازي: "قال أبو محمد عبد العزيز المصراتي لما ذكر ما في **رفع الحجاب** هنا: وهذا تبديل تفصيل، تبديل نسبة

الجزء إلى كفته كنسبة العدد المعلوم إلى المجهول، على ما تبين في **رفع الحجاب**." (ابن غازي 1983، ص. 210: 3-6).

والاشارتان موجودتان فعلا في كتاب الهوّاري المصراتي، الأولى في الورقة (مخطوط م، 4 ظ: 9-11)، والثانية في الورقة (مخطوط م، 51 و: 17-18).

ولم يذكر محمد سويسي استناد ابن الغازي إلى شرح الهوّاري **لتلخيص أعمال الحساب**، ومن أهم النتائج التي حققناها في بحثنا هذا هو اكتشافنا استناد ابن الغازي في كتابه **بغية الطلاب في شرح منية الحساب** إلى كتاب الهوّاري ونقله نقلا حرفيا لفقرات عديدة، بعضها شروح وبعضها أمثلة، واثبتنا هذا الاستنتاج في جدول نقابل فيه الفقرات المنسوخة الموجودة في الكتابين (أنظر الملحق 2).
ووجدنا في **بغية الطلاب** ما لا يقل عن 111 وضعية (أمثلة ومسائل ونصوص) منقولة من كتاب الهوّاري وموزعة حسب المواضيع هكذا:

- جمع الأعداد: 15، منها اثنان من **رفع الحجاب.**
- امتحان العمليات بطرح تسعة أو سبعة: 3 ، لكنها كلها من **رفع الحجاب**.
- معنى الضرب: 2 ، وهي من **رفع الحجاب**.
- الضرب بالمحو وبالنيف وبالتسمية: 7.
- معنى القسمة: 2 ، وهي من **رفع الحجاب**.
- زوال الإشتراك: 1.
- بسط الكسور: 10.
- أعمال الكسور 15 ، منها 5 من **رفع الحجاب**.
- اعمال الجذور: 21 ، منها 2 من **رفع الحجاب**.
- مسائل الكفات: 3 ، وهي من **رفع الحجاب**.
- مسائل الجبر: 32.

وينقل ابن غازي شهادتين وردتا في كتاب الهوّاري كما هي:

الشهادة الأولى لابن الياسمين: "وقال أبو محمد ابن الياسمين: إذا سمّينا أحدهما من المجموع، أسقطنا تلك النسبة من العدد نفسه وضربنا ما بقي في الكل، فيخرج المطلوب." (ابن غازي 1983، ص. 99: 2-4). وهذه الشهادهة موجودة أيضا في كتاب الهوّاري (مخطوط م، الورقة 23 ظ: 12-15).

الشهادة الثانية لأبي محمد عبد الحق بن طاهر: "حتى قال أبو محمد عبد الحق بن طاهر: إنّ الوجه الأوّل يُسمّى عملا والأربعة الأخرى تُسمّى أقيسة". (ابن غازي 1983، ص. 201: 17-18). وهذه الشهادهة موجودة أيضا في كتاب الهوّاري (مخطوط م، الورقة 51 و: 13-14).

والغريب في استعمال ابن غازي ما يقارب من 111 مثال ومسألة منقولة من كتاب الهوّاري هو سكوته الكامل عن مصدرها رغم أنه لا يغفل عن ذكر اسم المصدر في كل الحالات الأخرى، خاصة اللغوية، التي يستشهد بها.

2.4 الهوّاري المصراتي و شكر زاده

النتيجة الهامة الثانية التي حققناها في هذا البحث هي اكتشافنا استشهاد الرياضي العثماني شكر زاده في كتابه: **أمثلة من التلخيص لابن البنا والحاوي لابن الهائم** بجل الأمثلة الموجودة في كتاب الهوّاري. فمن هو شكر زاده؟
هو السيّد فيض الله سرمد، شهر شكر زاده كان رياضيا و مؤقتا و خطاطا و شاعرا. وبعد أن درّبه في الرّياضيات والفلك الرياضي العثماني مصطفى صدقي (تـ 1756/1170)، اختصّ فيها شكر زاده ودرّس الحساب وفن الخط في مدارس أزمير حيث كان يشغل خطة مُلّا سنة 1777/1191. ونشر شكر زاده عدة رسائل وكتب، منها الكتاب المسمى: **أمثلة من التلخيص لابن البنا والحاوي لابن الهائم،** وهو عمل قام به محرره في فترة تعلّمه الحساب والجبر. وتوفى شكر زاده سنة 1202 هـ/1787 م.[1]

كتاب "أمثلة من التلخيص لابن البنا والحاوي لابن الهائم"

ألّف شكر زاده هذا الكتاب[2] على نمط تعليمي موحد:

- في رأس كل ورقة يذكر شكر زاده تعريفا لمفهوم رياضي أو قاعدة حسابية أو جبرية منقولة من **تلخيص أعمال الحساب** لابن البنا أو أحيانا من **الحاوي في الحساب** لابن الهائم.

[1] جل هذه المعلومات مستنبط من المرجع التركي "عصمانلي ماتيماتيك ليتيراتور تاريخي (OMLT)" لإكمال الدين إحسان أغلو ورمضان ششن وجواة إزغي؛ إسطنبول 1999: 248-250.و"عصمانلي ماتيماتيك ليتيراتور تاريخي (OALT)" لإكمال الدين إحسان أغلو ورمضان ششن وجواة إزغي؛ إسطنبول 1999: 524- 525.

[2] توجد نسخة واحدة من كتاب شكر زادة وهي الجزء الثاني من مخطوط أسعد أفندي رقم 3150 بالمكتبة السليمانية باسطنبول (الورقات 11و إلى 89ظ).

- ثم يوضح المفهوم والقاعدة الحسابية بأمثلة مأخوذة من كتب قديمة أو من تأليفه الخاص.

وتكمن طرافة هذا الكراس في أن كل الأمثلة وحلول كل المسائل الحسابية والجبرية ليست مكتوبة بالألفاظ والجمل بل كلها مدونة بالرموز والعلامات المستعملة في المغرب العربي منذ القرن السادس الهجري (الثاني عشر مسيحي) والتي قلّ استعمالها في الشّرق العربي وهذا لافت للنظر. ونقرأ في الورقة الأولى من المخطوط:

"قد قرأت **كتاب التلخيص** على التمام وكتبت أمثلته في هذا المقام عن أستاذي وسيدي النحرير المدقق الكامل المحقق صدقي الحاج مصطفى أفندي الأمين بدار الضرب العثمانية". (شكر زاده، كتاب "أمثلة من التلخيص"، الورقة 11و).

ثم يقترح على القارئ حل لغز حسابي للحصول على تاريخ انتهائه من كتابة هذا الكتاب. نتيجة اللغز أن شكر زاده انتهى الكتابة يوم 15 من شهر جمادى الأولى سنة 1761/1175.

محتوى كتاب "أمثلة من التلخيص لابن البنا والحاوي لابن الهائم"

جمع شكر زاده في هذا الكتاب جملة من المسائل الحسابية والجبرية في فقرات خاصة وربط كل واحدة منها بمفهوم منصوص في كتاب **التلخيص** وفي كتاب **الحاوي** حسب نمط واحد:

يبدأ المؤلف بالتذكير بالمفهوم الرياضي أو القاعدة الحسابية أو الجبرية المنقولة إما من "التلخيص" وإما من "الحاوي" و يكتبها بالحبر الأحمر.
يوضح المفهوم أو القاعدة بالاستشهاد بأحد شارحي كتاب **التلخيص** ويكتب هذا النص (القصير في أكثر الحالات) باللون الأكحل ذاكرا اسم الشارح أو الكتاب.
يختم الفقرة بنص مسألة (أو مسائل) عددية أو جبرية.
يتبع النص بحل يستعمل فيه الرموز المغربية للدلالة على الأعداد والكسور والجذور والمعادلات ذاكرا في كل حالة مصدر المسألة.

رغم الطابع الرياضي التقليدي العربي الإسلامي بالنسبة إلى المحتوى العلمي فان كتاب شكر زاده فريد من بين كتب الحساب يتميز بطرافته وجمال عرضه. فتكمن طرافته في أنه شهادة على عمل أنجزه طالب في فترة تعلمه، فيمكننا من

معرفة المراجع في عهده وغزارتها وحصر عدد هائل من الأمثلة العددية والمسائل الجبرية في بعض ورقات مع إسناد كل واحدة إلى مرجعها.
وأما جمال عرض كتاب **الأمثلة من التلخيص**، فهو نتيجة الاستعمال المكثف للرموز والعلامات المغربية وتخصيص اللون الأحمر للعمليات والأكحل للأرقام. فيكتسي الكتاب طابعا عصريا فريدا.

يحتوي كتاب شكر زاده على خمسة أبواب، ويتبع في ذلك كتاب **التلخيص** :

- العدد الصحيح ومراتبه وحسابه (ورقات 11 ظ – 34 و)
- في الكسور (ورقات 34 ظ – 39 ظ)
- حساب الجذور (ورقات 40 و – 53 و)
- حساب الكفّات (ورقات 53 ظ – 56 ظ). وأدخل المؤلف بين الورقة 56ظ والورقة 57و ورقة مضافة تحتوي على قصيدة حول الكفّات و على مسألة.
- الجبر والمقابلة (ورقات 57 و إلى 89 ظ).

مراجع شكر زاده في كتاب "أمثلة من التلخيص لابن البنا والحاوي لابن الهائم"

يستند شكر زاده إلى عدد كبير من الكتب منها 5 إحالات إلى كتاب **حاوي اللباب في شرح التلخيص** لابن مجدي (تـ. 1447/851) و 12 إحالة إلى كتاب **شرح الحاوي** لسبط المارديني (تـ. 1593/1002)، و 11 إحالة إلى كتاب **التمحيص في شرح التلخيص** لابن هيدور التادلي و 7 إحالة إلى كتاب **شرح تلخيص أعمال الحساب** للقلصادي (تـ. 1486/891)، و 4 إحالة إلى كتاب **القول المبدع في شرح المقنع** لسبط المارديني (تـ. 1506/912)، وما لا يقل عن عشر إحالة منفردة إلى مراجع أخرى.

مكانة الهوّاري المصراتي في كتاب شكر زاده

كتاب الهوّاري المصراتي هو المرجع الأساسي في كتاب **"أمثلة من التلخيص لابن البنا والحاوي لابن الهائم"** إذ شكر زاده ينقل حرفيا فقرات عديدة، بعضها شروح وبعضها أمثلة أو مسائل. وأحصينا أكثر من 111 إحالة راجعة إلى كتاب **اللباب**، وهي موزعة حسب المواضيع كما يلي:

- صورة الأعداد: 3.

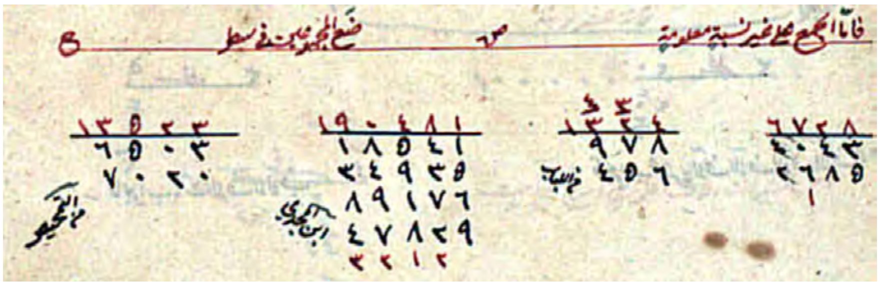
- جمع الأعداد: 15.
- طرح الأعداد: 5.
- امتحان العمليات بطرح تسعة أو سبعة: 9.
- ضرب الأعداد: 14.
- قسمة الأعداد: 6.
- زوال الإشتراك: 2.
- بسط الكسور: 9.
- أعمال الكسور (جمع – طرح – ضرب): 4.
- اعمال الجذور: 0 . كل مسائل الجذور من اختراع شكر زاده.
- مسائل الكفات: 4 .
- مسائل الجبر: 40.

نماذج من إحالات شكر زاده إلى نص الهوّاري المصراتي

نقدّم في ما يلي بعض النماذج من إحالات شكر زاده إلى **اللباب في شرح التلخيص** للهواري.

جمع عددين صحيحين

يخصص ابن البنا فقرة كاملة لتدقيق مراحل عملية جمع عددين صحيحين تحت عنوان: "الجمع على غير نسبة معلومة" (**التلخيص**، صفحة 41: 10-16). وينقل الهوّاري هذه الفقرة حرفيا ثم يشفعها بمثالين، الأول في الجمع من أوّل المراتب للعددين 4043 و 2685، والثاني في الجمع من آخر المراتب للعددين 978 و 456. وفي كلتا الحالتين يستعمل حروف الغبار ويضع المجموع فوق الأعداد ويفصله عن العددين بخط.
أما شكر زاده، فهو يكتفي بذكر عنوان طريقة العمل ويتبع العنوان بأربع صور لأمثلة عددية منقولة عن كتب مختلفة، منها كتاب **اللباب**. (المثال الثاني في الصورة التالية) وكتاب **الحاوي** لابن مجدي (المثال الثالث) وكتاب **التمحيص** لابن هيدور (المثال الرابع).

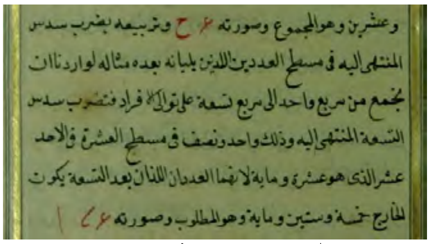


(شكر زاده، 16 ظ: 11-15)

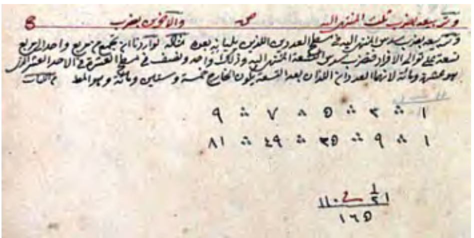
جمع الحدود في المتتاليات العددية

تفضي عملية الجمع في **التلخيص** إلى تدارس مجموع الحدود في المتتاليات العددية، فيذكر ابن البنا بإيجاز تسع قواعد خاصة بها: الجمع على توالي الأعداد، وعلى توالي الأفراد، وعلى توالي الأزواج، وكذلك جمع مربعاتها ومكعباتها (**التلخيص**، صفحة 13:42 إلى 10:43). ويخصّ الهوّاري مثالا عدديا أو مثالين لكل قاعدة، متخذا 10 أو 9 في مقام العدد العام.[1]

أما شكر زاده، فهو ينقل كل قاعدة كما جاءت في **التلخيص** ويشفعها بمثال عددي خال من أي شرح أو تعليق، كما في هذه الصورة:



(الهوّاري، نسخة م: 9 و)



(شكر زاده، 17 و: 14-21)

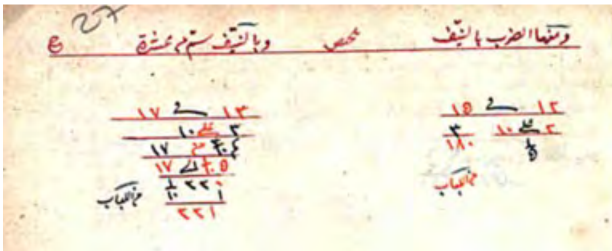
في أنواع الضرب

يتميز باب الضرب في كتب الحساب العربية القديمة بكثرة الانواع المدروسة (الضرب بالتنقيل، وبنصف تنقيل، وبالجدول، الضرب بالمحو، وبغير محو،

[1] جل أمثلة هذا الفصل من كتاب الهوّاري منقولة عن كتاب **المقالات الأربع في العدد** لابن البنا (ص. 154-156) وهي موجودة أيضا في كتاب ابن الياسمين (ص. 136) .

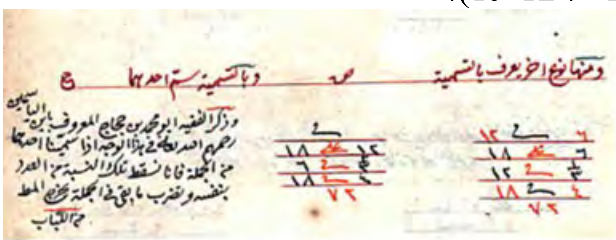
الضرب بالمربعات، إلخ).[1] وتبعا لهذا المنهج ذكر ابن البنا في كتاب **التلخيص** كل قواعد العمل الخاصة بالطرق المعروفة لضرب عددين صحيحين ولم يذكر أي مثال عددي (**التلخيص**، صفحة 13:42 إلى 10:43).
وقدم الهوّاري في **اللباب** مثالا عدديا خاصا لكل نوع من أنواع الضرب الواردة في كتاب **التلخيص** مشفوعة بصورة أو صور حسب مراحل العمل. (**اللباب**، مخطوط م، ورقات 15 ظ إلى 26 ظ).
أما شكر زاده فقد امتنع من نقل القواعد والشروح واختصر بالتذكير بعنوان كل نوع من انواع الضرب وبتقديم عدة صور من الامثلة العددية لذلك النوع، منها المستنبطة ومنها المأخوذة من المراجع المعروفة (**شكر زاده**، الورقات 22و إلى 30ظ).

مثالان من الضرب بالنيف:



(شكر زاده، الورقة 27و)

وفي أسفل نفس الورقة 27و، ينقل شكر زاده عن الهواري مثالين من الضرب بالتسمية وكذلك شهادة لابن الياسمين كما وردت في كتاب **الهوّاري** (مخطوط م، الورقة 23 ظ: 12-15):

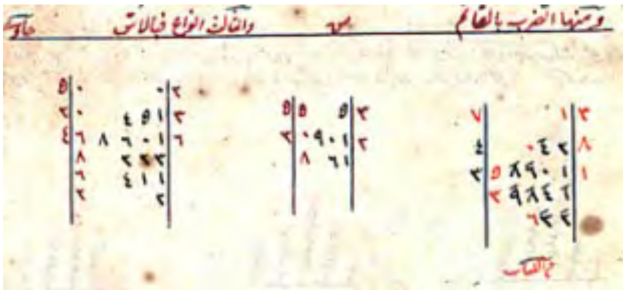


(شكر زاده، الورقة 27 و)

[1] فصّل أبلاغ في تقديمه لكتاب **رفع الحجاب** أنواع الضرب وقام بتحليل دقيق لها ولاحظ أنّ: "التنويع ناتج عن ضرورة تقديم الطريقة الأقرب في حل المسألة المطروحة ولو كانت صعبة على الفهم وتقديم الطريقة الأقرب إلى الفهم ولو كانت طويلة في العمل" (أبلاغ، صفحة 140).

الضرب بالقائم

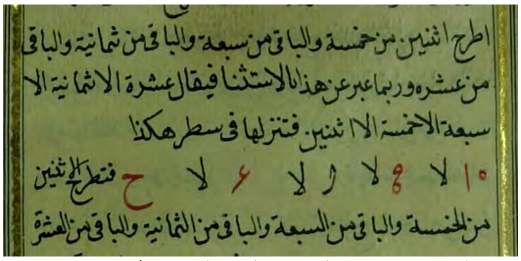
وهذه صور من الضرب بالقائم كما وردت في كتاب شكر زاده:



(شكر زاده، الورقة 25 و)

الطرح المكرر

لم ترد قواعد عمليات الطرح المكرر في **التلخيص**، فتدرك ابن البنا هذا النقص وأدرجها مُطولا في مدخل باب الطرح من **رفع الحجاب** (صفحة 245: 2 إلى 247: 2)، وأشفعها بعدة أمثلة عددية. ونقل الهوّاري عنه القواعد العامة والأمثلة المصاحبة لها.


اطرح اثنين من خمسة والباقي من سبعة والباقي من ثمانية والباقي
من عشرة وربما عبر عن هذا الاستثنا فيقال عشرة الا ثمانية الا
سبعة الا خمسة الا اثنين فتنزلها في سطر هكذا
فتطرح الاثنين
من الخمسة والباقي من السبعة والباقي من الثمانية والباقي من العشرة

(الهوّاري (نسخة م) ، الورقة 12 ظ: 7 – 11)

ولم ينقل شكر زاده القواعد العامة بل اكتفى بتحليل شامل للمثال العددي الذي جاء به الهوّاري، كما يتبيّن في هذه الصورة:

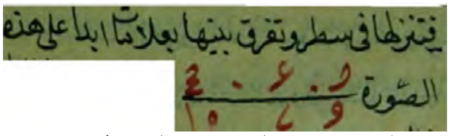
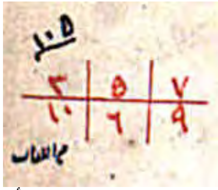
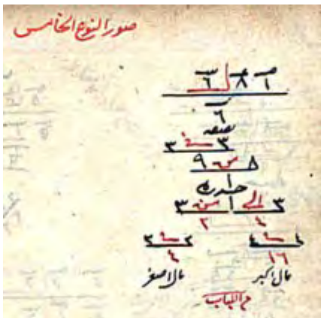
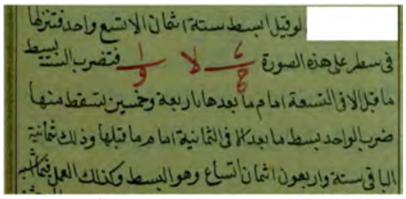
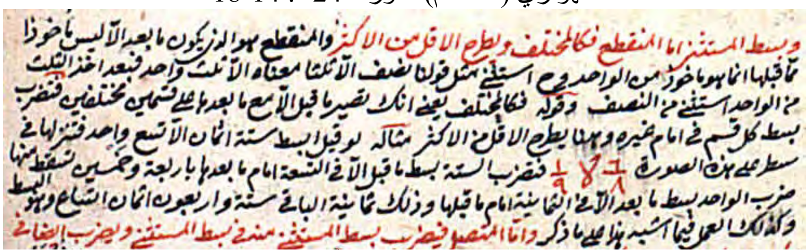
(شكر زاده، الورقة 19و : 1-20)

العلامات والرموز

قلنا أعلاه إن اختراع العلامات والرموز للدلالة على الكسور والجذور والعبارات الجبرية تزامن مع استعمال الأرقام الهندية (حساب الغبار) في الأندلس وفي المغرب. ولاحظنا أن ابن البنا في كتاب **المقالات في الحساب** وطالبه الهوّاري في كتاب **اللباب** استعملا الرموز في باب الكسور، ولكن لا توجد آثار للرموز الخاصة بالجذور أو بالعبارات الجبرية في مؤلفاتهما.

ويمتاز كتاب شكر زاده بطرافته في استعماله صور المسائل و يكمن شذوذه في أن كل الأمثلة وحلول كل المسائل الحسابية والجبرية ليست مكتوبة بالألفاظ والجمل بل كلها مدونة بالرموز والعلامات المستعملة في المغرب العربي منذ القرن السادس الهجري والتي قلّ استعمالها في الشّرق العربي وهذا لافت للنظر.

ونقدم في الجدول التالي بعض النماذج من نقل شكر زاده لمسائل الكسر والجبر المأخوذة من كتاب **اللباب** للهواري المصراتي:

<table>
<tr><td>
الهوّاري (نسخة م)، صفحة 34ظ: 3-4</td><td>
شكر زاده، ورقة 35أ</td></tr>
<tr><td>
الهوّاري (نسخة م)، الورقة و57: 10-15</td><td>
شكر زاده، 58ب</td></tr>
<tr><td colspan="2">
الهوّاري (نسخة م)، الورقة 24ظ: 14-18

شكر زاده، 35أ</td></tr>
</table>

3.4 ادماج كتاب "اللباب" في برنامج جامع الزيتونة لسنة 1875

اعتبر محمد سويسي شرح الهوّاري المصراتي شرحا نفيسا لأنه تسجيل مباشر لدرس أستاذه. فهل احتل كتاب الهوّاري المصراتي مكانة تُذكر في تعليم الحساب عند العرب؟ الجواب أنه لم يُعترف بمساهمته العلمية ولم تذكره كتب التراجم العربية القديمة فلا نعرف شيئا عن حياته وأعماله. ولعل السكوت ناتج عن أنه لم يؤلف إلا كتابا واحدا تحت رعاية شيخه الذي كان أستاذا مشهورا. ولا شك أن انتشار كتاب **تلخيص أعمال الحساب** وكتاب **رفع الحجاب** اقترن مع نشر شروح القلصادي وابن غازي لكتاب **التلخيص**. ولكن رغم غياب الهوّاري من التراجم لم تنتثر آثار كتابه **اللباب في شرح تلخيص أعمال الحساب** ولم تفن أولا لأن ابن غازي المكناسي انتحل الجزء الكبير منه، خاصة الأمثلة الطريفة وسهلة الفهم، وثانيا لأن نسخا من الكتاب انتقلت إلى تونس وإلى الشرق الاسلامي عبر القرون، فتوجد منه حاليا نسخ كتبت في المغرب وفي القسطنطنية وفي دمشق، أقدمها نسخة سنة 1345/746، أي سنة واحدة بعد وفاة محمد الهواري، أخ المؤلف، وتاريخ النسخ الأخرى يتراوح بين سنة 880هـ وسنة 1270هـ (أي 1475-1853م). وانتشار النسخ هو دليل على استعمال الكتاب كمرجع أصلي للتدريس لا يحتاج إلى شرح خاص به.

ودليل آخر على أهمية كتاب الهوّاري هو آثاره التي ضبطناها في كتاب العالم العثماني شكر زاده، والتي تشهد على مكانة شرح الهوّاري المصراتي في تعليم الحساب والجبر في القرن الثامن عشر.

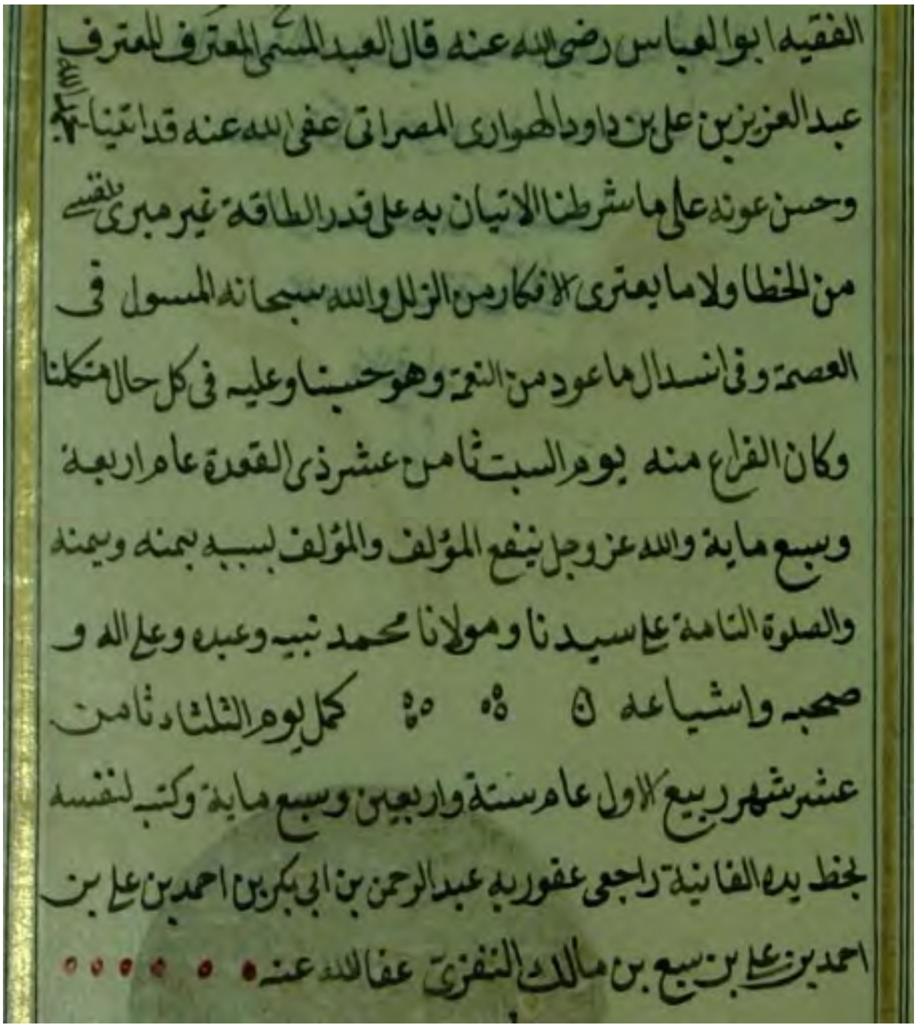
وأخيرا، أكبر شهادة على مكانة كتاب الهوّاري المصراتي عند علماء تونس برزت مع قرار الوزير الأول خير الدين باشا المتعلق بتحديث التعليم بجامع الزيتونة. وإثر هذا القرار اقترحت لجنة مكونة من شيوخ الجامع: "الفنون التي تقرأ في الجامع الأعظم والكتب التي تدرّس في تلك الفنون وفي مراتبها" ومن بين هذه الفنون: "علم الحساب وعلم الهندسة وعلم الهيئة وعلم المساحة". ومن بين الكتب التي تدرّس في المرتبة العالية: "من علم الحساب: شرح لابن غازي على المنية والتلخيص بشرح المسراتي". (المنشور المنظم للتعليم بالجامع الأعظم، الصادر في 28 ذي القاعدة 26/1292 ديسمبر 1875).

بعض الصور من مخطوطات كتاب

اللباب في شرح تلخيص أعمال الحساب

مخطوط "م"، مجموعة عارف حكمت رقم "21 حساب"، مكتبة المخطوطات بالمدينة المنورة (سنة 1345/746)

19 و



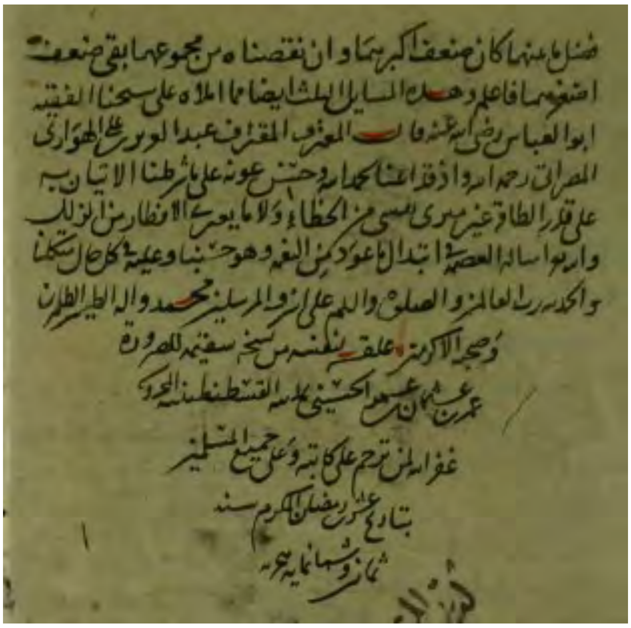
الفقيه ابو العباس رضى الله عنه قال العبد المسمى المعترف للمعترف
عبد العزيز بن على بن داود الهوارى المصراتى عفى الله عنه قد اتينا بحمد الله
وحسن عونه على ما شرطنا الاتيان به على قدر الطاقة غير مبرى لنفسى
من الخطا ولا ما يعترى الافكار من الزلل والله سبحانه المسول فى
العصمة وفى انسدال ما عود من النعمة وهو حسبنا وعليه فى كل حال متكلنا
وكان الفراغ منه يوم السبت ثامن عشر ذى القعدة عام اربعة
وسبع ماية والله عز وجل ينفع المولف والمولف لسببه بمنه ويمنه
والصلوة التامة على سيدنا ومولانا محمد نبيه وعبده وعلى اله و
صحبه واشياعه ه ه ه ه كمل يوم الثلثاء ثامن
عشر شهر ربيع الاول عام ستة واربعين وسبع ماية وكتب لنفسه
بخط يده الفانية راجى عفو ربه عبد الرحمن بن ابى بكر بن احمد بن على بن
احمد بن على بن سبع بن مالك النفزى عفا الله عنه ه ه ه ه ه ه ه

مخطوط "أ" ، مجموعة شهيد علي باشا بالمكتبة السليمانية، رقم 2/1977 (سنة 1476/880)


شرح التلخيص في الحساب للهواري

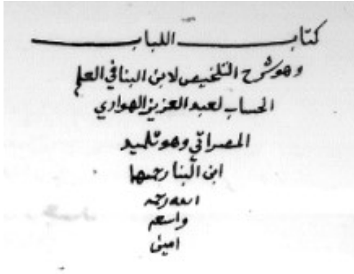
التلخيص للفاضل الكامل المحقق
المدقق أبي العباس أحمد بن محمد
الأزدي المعروف بابن البنا
رضي الله عنه وأرضاه
وجعل الجنة
مثواه

54 و



103 ظ

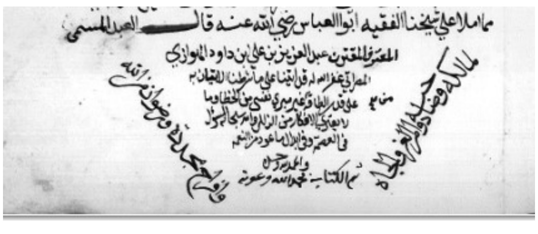
مخطوط "ب"، مكتبة مخطوطات أكسفرد، رقم Marsh 378/3
(بعد سنة 1467/872)



كتاب اللباب
وهو شرح التلخيص لابن البنا في العلم
الحساب لعبد العزيز الهواري
المصراتي وهو تلميد
ابن البنا رحمهما
الله رحمه
واسعه
امين

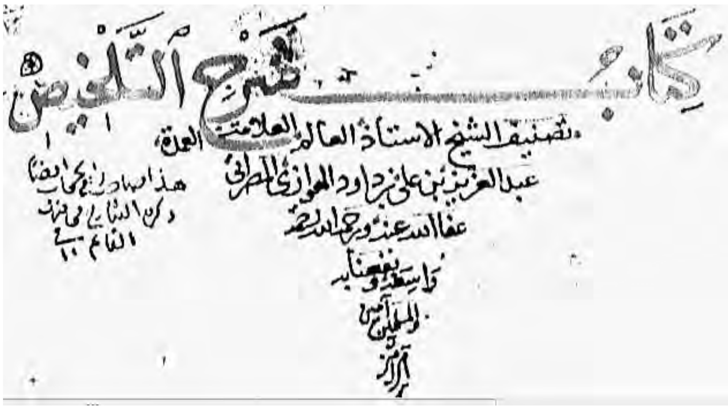
109 و

109 ظ

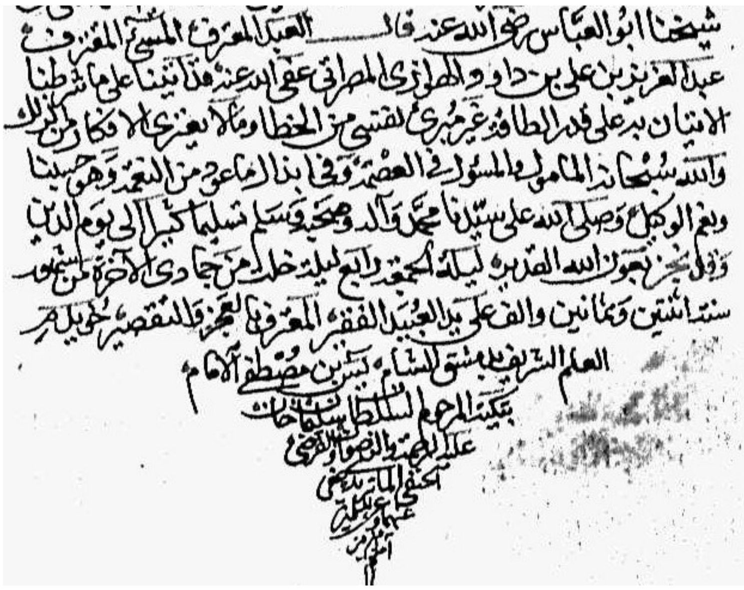


162 ظ

مخطوط المكتبة الوطنية بتونس، رقم 9940.
(قبل سنة 1082/1671)



54 و



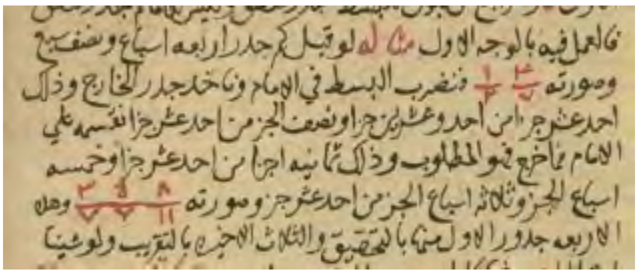
103 ظ

مخطوط مكتبة مجلس الشورى الاسلامي بتهران، رقم 2/239 (سنة 1564/972)


كتاب اللباب في شرح مختصر اعمال الحساب
الملخص
شرح القانون
تصنيف الشيخ الاستاذ عبد العزيز بن علي بن داود الهواري المصراتي
عفا الله عنه وكرمه امين امين
يارب العالمين

20 و

21 ظ



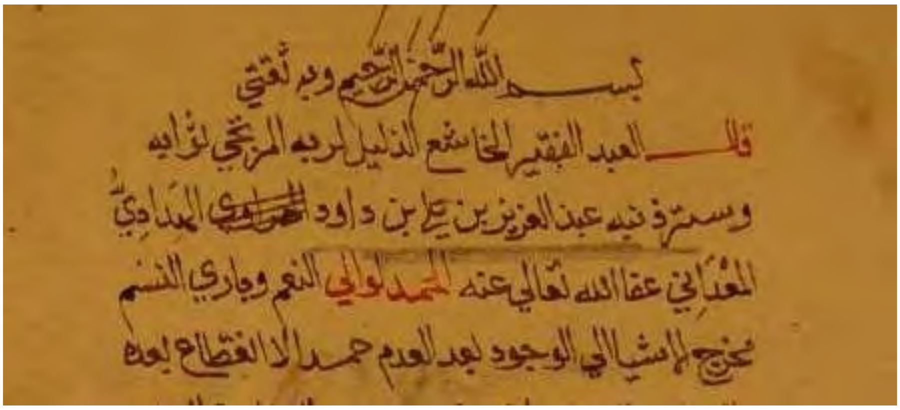
41 ظ

مخطوط رقم 1077، مكتبة المخطوطات بوزارة الأوقاف بالقاهرة (سنة 1853/1270)


هذا شرح تلخيص
ابن البناء في علم الحساب
بالتمام والكمال

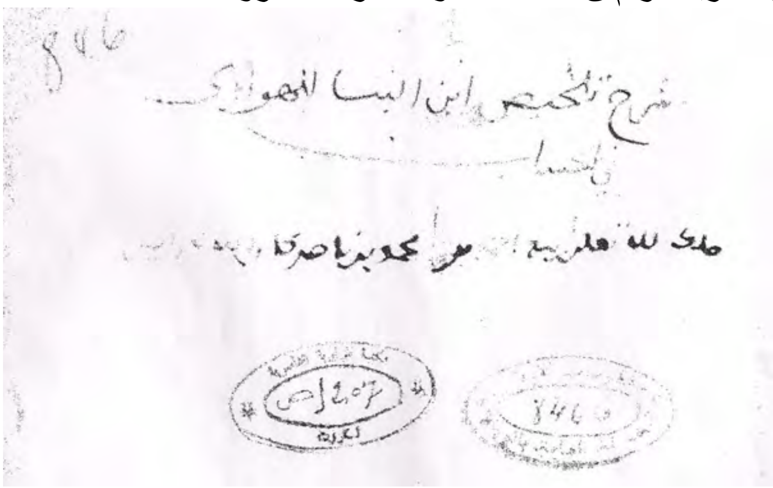
1و

1ظ

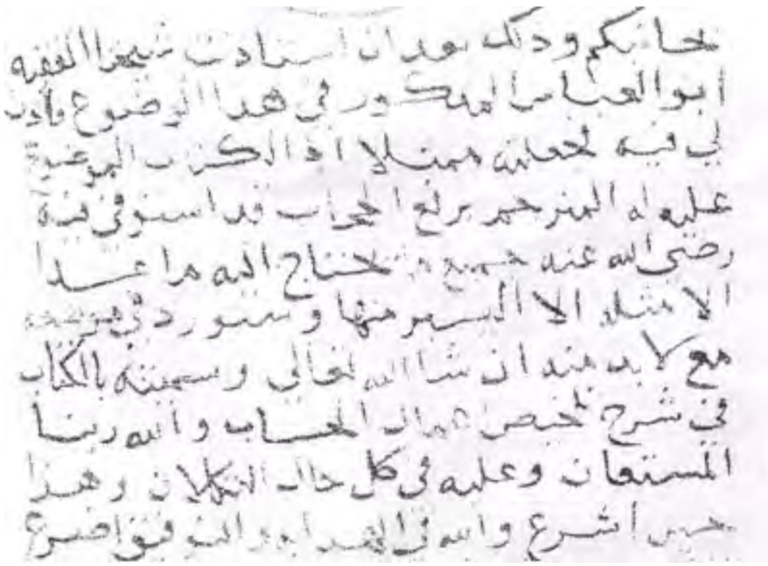

بسم الله الرحمن الرحيم وبه ثقتي
قال العبد الفقير الخاشع الذليل لربه المرتجي لثوابه
وستر ذنبه عبد العزيز بن علي بن داود الهواري
عفا الله تعالى عنه الحمد لولي النعم وباري النسم
مخرج الاشياء الى الوجود بعد العدم حمدا لا انقطاع بعده

مخطوط رقم 953 ، مكتبة المخطوطات أسكوريال باسبانية
(قبل سنة 1550/957)

مخطوط الرباط رقم ق 846، مجموعة خزانة تمكروت.



الورقة 1و



الورقة 2و

اللباب في شرح
تلخيص أعمال الحساب
لأبي العباس بن أحمد الأزدي

عبد العزيز بن علي بن داود الهواري المصراتي[1]

---

[1] **مخطوط عارف حكمت** عدد 2856، ورقة 1أ ، بمكتبة الملك عبد العزيز بالمدينة المنورة. ورمز المخطوط في هذا التحقيق: (م).
وهو المخطوط الرئيسي الذي اعتمدنا عليه في تحقيقنا والذي قابلناه مع أربع مخطوطات تمكنّا من صور شمسية منها. وترقيم الصفحات في هذا التحقيق ترجع إلى هذا المخطوط.

نستعمل كذلك في هذا التحقيق أربع نسخ أخرى من كتاب الهواري:
مخطوط شهيد علي باشا عدد 1977-1، بالمكتبة السليمانية باسطنبول. ورمز المخطوط في هذا التحقيق: (أ).
وعنوانه: "شرح التلخيص في الحساب للهوّاري. التلخيص للفاضل الكامل المُحقّق المُدقّق أبي العباس أحمد بن محمد الأزدي المعروف بابن البنا رضي الله عنه وأرضاه وجعل الجنّة مثواه". بداية النص في الورقة 54ظ.

مخطوط بمكتب أكسفورد عدد Oxford Library : MS. Marsh 378. ورمز هذا المخطوط في تحقيقنا (ب).
وعنوانه في الغلاف: "كتاب اللباب، وهو شرح التلخيص لابن البنا في علم الحساب لعبد العزيز الهواري المصراتي، وهو تلميذ ابن البنا، رحمهما الله رحمة واسعة. آمين". بداية النص في الورقة 114ظ.

مخطوط كتابخانه مجلس شوراي اسلامي، تهران عدد: 114-21. ورمز هذا المخطوط في تحقيقنا (ط).
وعنوانه في الغلاف: "شرح القانون، تصنيف الشيخ الأستاذ عبد العزيز بن علي بن داوود الهوازي المصراتي، عفا الله عنه بمنه وكرمه، آمين آمين ربي العالمين.". وشطب لفظ "القانون" وعوض بلفظ "التلخيص". وتوجد زيادة في الصفحة قبل العنوان :" كتاب اللباب في شرح التلخيص أعمال الحساب"، وبعد العنوان :" بحمد الله وحده ". و "طالعه من أوله إلى آخره ، داعيا الله بالبقاء والعلو في العلم والارتقاء، الفقير إلى الله الغني به، علي بن ناصر الدين بن الطرابلسي الحنفي، عفا عنهم الله، في سنة 972". بداية النص في الورقة 10ظ.

مخطوط بالمكتبة الوطنية بتونس عدد 9940. ورمز هذا المخطوط في تحقيقنا (ت).
وعنوانه في الغلاف: "كتاب شرح التلخيص، تصنيف الأستاذ العالم العلّامة العمدة، عبد العزيز بن داود الهوازي المصراتي، عفا الله عنه ورحمه الله رحمة واسعة، ونفعنا به والمسلمين، آمين". بداية النص في الورقة 1ظ .

< التـــــــــوطئـة > [1]

مخطوط (م): ورقة 1 ظ
مخطوط (أ): ورقة 54 ظ
مخطوط (ب): ورقة 114 ظ

[1] في (ط) وفي (ت) : الطوطئة غائبة.

بسم الله الرحمن الرحيم،

قال العبد الخاضع[1] الذليل لربّه، المرتجي ثوابه، وسَتر ذنبه، عبد العزيز بن علي بن داود[2] الهَواري[3] المصراتي عفا الله عنه.

الحمد لله، وليّ النعم وبارئ النسم، مخرج الأشياء إلى الوجود بعد العدم، حمدا لا انقطاع لعده. والصلوات التّامة على محمد نبيه وعبده. والرضى عن السّلف المُقتفين سُنن الشرع[4]، الواقفين عند حدّه.

وصلاة الدعاء لمولانا أمير المسلمين بن أمير المسلمين أبي يعقوب بالنصر العزيز من عنده.

وبعد، أطال الله بقاء سيدنا الوزير المُعَظّم السني السّري، المُبارك المُحترم، الفقيه العالم العلامة، المُكرّم ذي المناقب العَلِيّة والهمم الفاضلة، (الأبية السنية بسراج)[5] الدّولة الميمونة المرينية[6]، أبي محمد عبد الله بن عمادنا الشيخ الفقيه الصالح الناسك الزاهد الحسن السيرة والمسالك[7] المرحوم المقدس الأطهر أبي مَدْيَنْ، جعله الله من المُشفعين في المَحْشر، (خلد الله في المتقين)[8] ذكرهم وحفظ من عور الأيام وردهم وصدرهم.

فإني استخرت الله عز وجل في أن أشرح كتاب شيخنا الفقيه الأوحد (السّريّ الأمجد المشارك الأوجد)[9] أبي العباس أحمد بن محمد ابن عثمان الأزدي، أبقى الله بَرَكَتَهُمْ وأدامَ بِمَنِّه حِفْظَهُمْ[10] حُظْوَتَهُمْ ورفعَتهم، المُسمّى بتلخيص أعمال الحساب، أطرِزُه بِاسْمِكُم[11] وأحَلّيه بِحَليتكم، ليكون |2| و| وسيلة للتبرّك بكم واللياذ[12] بجانبكم.

[1] في (ب) : "الخاشع".
[2] في (أ) : "داوود".
[3] في (ب) : "الهدادي المعداني". ويكتب الناسخ في الحاشية: " نقل السيوتي أن الهدادي، بفتح الهاء وبالدالين المهملين والتخفيف للدال الاول، نسبة إلى هداد، بطن من الأزدي. والمعداني، بفتح الميم والعين المهملة، نسبة إلى معدان جدة."
[4] في (أ) : "الشريعة".
[5] في (أ) : "سراج".
[6] في (أ) : سقط لفظ "المرينية".
[7] في (أ) وفي (ب) : "المآثر".
[8] في (أ) : "خلد في المتقين ". وفي (ب) : "وأحلّه في المتقين الدين ".
[9] في (أ) وفي (ب) : "الاجل الامجد".
[10] في (أ) وفي (ب) : سقط لفظ "حفظهم".
[11] في (أ) وفي (ب) : "بالحِكَم".
[12] في (ب): "العبادة بجانبكم". في (أ) : "واسناد بجنابكم"، ولا يستقيم المعنى.

وذلك بعد أن استأذنت شيخنا الفقيه أبا العباس المذكور في هذا الموضوع، فأذن لي فيه.
فجعلته مُمَثلا، إذ الكتاب الموضوع عليه ذيّله[1] المُترجم **برفع الحجاب،** قد استوفى فيه، رضي الله عنه، جميع ما يحتاج إليه ما عدا الأمثلة إلا اليسير منها. وسنُورده في موضعه مع ما لا بد منه، إن شاء الله تعالى. وسميته **باللباب**[2] **في شرح تلخيص أعمال الحساب**.

والله ربّنا المُستعان وعليه في كلّ حال التكلان. وهذا حين أشرع، وإليه في الهداية والتوفيق أضرع.
(آخر الطوطئة)

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[1] في (م) وفي (ب) : "له".
[2] في (أ) و في (ب): "بالكتاب".

([1]) قال شيخنا الفقيه الأستاذ الإمام الحبر السّني العلّامة الحافظ أبو العباس أحمد بن محمد بن عثمان الأزدي، (أبقى الله جلاله وذكره وحفظ كماله)[2] :

([3]) الغرض في هذا الكتاب تلخيص أعمال الحساب وتقريب أبوابه ومعانيه وضبط قواعده ومبانيه.

وهو يشتمل على جزأين : الأوّل في أعمال العدد المعلوم، والثاني في القوانين التي يمكن بها الوصول إلى معرفة المجهول المطلوب من المعلوم المفروض، إذا كانت بينهما وصلة تقتضى ذلك.

ومن الله سبحانه[4] أسأل العون والتوفيق، والإرشاد إلى سواء الطريق.

الجزء الأول : في العدد المعلوم

وهو ينقسم إلى ثلاثة أقسام : الأوّل في أعمال الصحيح، والثاني في أعمال الكسور[5]، والثالث في أعمال الجذور.

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[1] هذه بداية الشرح، نجدها بعد الطوطئة في النسخ (م) و (أ) و (ب)، وفي أعلى الورقة 54ظ من المخطوط (ت)، وفي أعلى الورقة 10ظ من المخطوط (ط).

[2] في النسخ الأخرى : "عفا الله عنه". ويلاحظ محمد سويسي في تحقيقيه للتلخيص (1969: 35) أنّه يظهر من هذه الزيادة أنّ "التلخيص جملة أعمال ألقاها ابن البنا على تلاميذه، وعلّق كل عليها بما حفظه من شروح شيخه".

[3] هذه بداية كتاب **تلخيص أعمال الحساب** لابن البنا. نرمز لهذا الكتاب بحرف (س) وكل أرقام الصفحات ترجع إلى تحقيق محمد سويسي (تونس 1969).

[4] في (ط) : سقط لفظ "سبحانه".

[5] في (أ) : "المكسور".

القسم الأول :

في الصحـــــيح

وما يتعلّق به من الأعمال بحسب مقصدنا ستة أبواب :

الباب الأوّل في أقسام العدد ومراتبه

العدد ما تألّف من الآحاد. وهو ينقسم بحسب مأخذه قسمين : صحيح و كسر.

مثال الصحيح : خمسة عشر وكثمانية عشر وشبهه[1]، ومثال الكسر: نصف وكثلاثة أثمان ونصف ثمن، وكتسع وربع، وكستة أسباع سبعة أثمان، [2 ظ] وكخمسة أسداس إلا تسعا.

والصحيح على ضربين : زوج و فرد.

فالزوج ما في أوّله اثنان، أو أربعة، أو ستّة، أو ثمانية، وكذلك كلّ ما ليس في أوّله آحاد، مثل العشرة، والخمسين، وشبه ذلك. والفرد ما في أوّله واحد، أو ثلاثة، أو خمسة، أو سبعة، أو تسعة.

(ثم إن الزوج)[2] على ثلاثة أنواع : زوج الزوج، وزوج الفرد، وزوج الزوج والفرد.

أمّا زوج الزوج، فهو كلّ عدد يتنصف وكلّ واحد من نصفيه يتنصف إلى أن ينتهي به التّنصيف إلى الواحد.

مثاله : إثنان وثلاثون، فإنّها تتنصف ونصفها ستّة عشر، وكلّ واحد من الستتي عشر يتنصف ونصفها ثمانية، ونصف الثمانية أربعة، ونصف الأربعة اثنان، ونصف الاثنان واحد، ونحو ذلك.

وأمّا زوج الفرد، فهو كلّ عدد يتنصّف أوّل وهلة[3] إلى عدد فرد غير الواحد.

مثاله أربعة عشر، فإنّها تتنصّف ونصفها سبعة، عدد فرد غير الواحد. وشبه ذلك من الأعداد. فعلى هذا الإثنان هي من نوع[4] الأوّل. فاعلمه.

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[1] في (أ) وفي (ط) : "شبه ذلك".
[2] في (س) وفي (ب) وفي (ت): "والزوج".
[3] في (ط): "وينتهي".
[4] في (أ) وفي (ط) : سقط لفظ "نوع".

وأمّا زوج الزوج والفرد، فهو كلّ عدد يتنصّف وكلّ واحد من نصفيه يتنصّف إلى أن ينتهي به[1] التنصيف إلى عدد فرد غير الواحد.

مثاله : ثمانية وعشرون، فإنّها تتنصّف، ونصفها أربعة عشر[2]، وكلّ واحد من الأربعتي[3] عشر تتنصّف، ونصفها سبعة، عدد فرد، كما تقدّم. فمن حيث انقسمت أوّلا بالزّوجية أشبهت زوج الزوج، ومن حيث انتهت إلى عدد فرد غير الواحد أشبهت زوج الفرد. فتأمّل ذلك.

(ثم الفرد)[4] على نوعين: أوّل وفرد الفرد.

أمّا الفرد الأوّل، فهو كلّ عدد لا يعُدّه إلّا الواحد، مثل: الأحد عشر، والتسعة والعشرين، وشبه ذلك. وتسمّى الأجزاء الصمّ والبسيطة أيضًا، على ما نبيّن في عمل الغربال.

وأمّا فرد الفرد، فهو كل عدد تعدّه أعداد أفراد مثل خمسة عشر، [3 و] فإنها مُركّبة من ضرب[5] ثلاثة في خمسة، وشبه ذلك.

قلتُ[6] نحتاج أن تقدّم ههنا[7] مُقدّمة نذكر فيها أسماء المُركّبات على اختلافها ونُمثّلها.
فنقول العدد بالنسبة إلى التركيب إمّا زوج وإمّا فرد. والزوج على ضربين : إمّا أوّل بسيط غير مُركّب (وهي الاثنان وحدها، وإمّا مُركّب)[8]، وهو ثلاثة أنواع :

مُركّب من عددين مُتساويين، ويُسمّى مُربّعًا أو مجذورًا، وكلّ واحد من العددين يُسمّى[9] ضلعًا أو جذرًا.

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[1] في (ط) : سقط لفظ "به".
[2] في (ط) : زيادة لفظ "يتنصفا".
[3] في (ط) : "الأربعة".
[4] في (س) وفي (ب) وفي (ت) : "والفرد".
[5] في (م) : سقط لفظ "ضرب".
[6] في (م) : سقط لفظ "قلت".
[7] في (ب) : سقط لفظ "ههنا". وفي النسخ الاخرى : "هنا".
[8] في (م) : سقط الجملة "وهي ... مركب".
[9] في (م) : سقط لفظ "يسمى".

مثاله : ستة وثلاثون، فإنّها مُركبة من ستة في ستة. فجملة الستة والثلاثين تُسمّى مُربّعًا ومجزورًا. وكلّ واحد من الستتين تُسمّى[1] ضلعًا وجذرًا.
ومُركب من عددين مُختلفين، فأكثر. ويُسمّى مُسطّحًا. وكلّ واحد من تلك الأعداد يُسمّى[2] ضلعًا.

مثال المُركّب من عددين مختلفين[3]: ثمانية عشر، فإنّها مُركّبة من ثلاثة في ستة، أو اثنين في تسعة. فجملة الثمانية عشر تُسمّى مُسطحًا. وكلّ واحدة من الاثنين والتسعة أو الثلاثة والستة تُسمّى ضلعًا.

ومثال المُركّب من ثلاثة أعداد: أربعة وعشرون، فإنّها مُركّبة من ثلاثة في أربعة في اثنين. فجملة الأربعة والعشرين تُسمّى مُسطحًا. وكل واحدة من الثلاثة والأربعة والاثنين يُسمّى ضلعًا.
وكذلك ما فوق ذلك.

ومُركّب من ثلاثة أعداد مُتساوية، ويُسمّى مُكعبّا. وكلّ واحد من تلك الأعداد ضلعًا وكعبًا.

مثاله : أربعة وستون، فإنّها مُركّبة من ضرب[4] أربعة في أربعة في أربعة. فجملة الأربعة والستين تُسمّى مُكعبًا. وكلّ واحدة من تلك الأربعات تُسمّى ضلعًا وكعبًا. وكذلك ما أشبه ذلك. وقد يُسمّى المُكعّب كعبًا باسم ضلعه.

والفرد هو أيضا على ضربين، إمّا بسيط، وقد تقدم، وإمّا مُركّب، وهو ثلاثة أنواع، كالزوج:

سواء مركب من عددين متساويين. ويُسمّى مربعًا ومجذورًا، وكلّ واحد من ذينك العددين ضلعًا وجذرًا.

مثاله : خمسة وعشرون، [3 ظ] فإنّها مُركّبة من خمسة في خمسة،

[1] في (م) وفي (ط) : سقط لفظ "يسمى".
[2] في (م) وفي (ط) : سقط لفظ "يسمى".
[3] في (أ) وفي (ط) : سقط لفظ "مختلفة".
[4] في (م) وفي (ط) : سقط لفظ "ضرب".

ومُركّب من عددين مُختلفين، فأكثر. ويُسمّى مُسطّحًا، وكلّ واحد من تلك الأعداد ضلعًا.

مثال المُركّب من عددين مُختلفين[1]: خمسة وثلاثون، فإنّها مُركّبة من خمسة في سبعة.

ومثال المُركّب من ثلاثة أعداد مُختلفة[2]: خمسة ومائة، فإنّها مُركّبة من ثلاثة في خمسة في سبعة.
وكذلك أيضًا ما فوقه.

ومركبة من ثلاثة أعداد مُتساوية وتُسمّى مُكعبًا. وكلّ واحد من تلك الأعداد يُسمّى[3] ضلعًا ومكعبًا.

مثاله : سبعة وعشرون، فإنّها مُركّبة من ثلاثة في ثلاثة في ثلاثة.

فصل[4]

فصار الضلع والكعب بمعنى واحد كما أنّ الجذر والضلع أيضًا بمعنى واحد لم يختلفا إلا بالعموم والخصوص لدى الإطلاق ويستويان لدى التقييد. واخذ ضلع المُكعّب طويل العمل قليل الجدوى ولأجل هذا لم يذكره، رضي الله عنه. إلّا أنّه يوجد[5] بطريق الحلّ، وهو قريب. تحلّ المُكعّب إلى أعداده التي تركّب منها، وتلفق منها بالتركيب ثلاثة أعداد متساوية يكون أحدها هو الضلع المطلوب. فتأمله تصب إن شاء الله.

ولما كان العدد يتزايد إلى غير[6] نهاية جعل له ثلاث مراتب.

[وسميت مراتب لأنّ بعضها يلي بعضا وآحاد كلّ مرتبة أعظم من آحاد التي قبلها وأصغر من آحاد التي بعدها.][7]

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[1] في (م) : سقط لفظ "مختلفين".
[2] في (م) : سقط لفظ "مختلفة".
[3] في (م) وفي (ط) : سقط لفظ "يسمى".
[4] في (أ) : سقط لفظ "فصل".
[5] في النسخ الأخرى : "يؤخذ".
[6] في (م) : سقط لفظ "غير".
[7] في (م) : الجملة "وسميت مراتب ... بعدها" منقولة من **رفع الحجاب**، صفحة 212: 9-10.

وتُسمّى **أيضا منازل، [باعتبار حلول العدد فيها]** [1] **تدور عليها منازل كل العدد.**

في كلّ مرتبة منها تسعة أعداد.

فالمرتبة الأولى من واحد إلى تسعة، وتُسمّى مرتبة الآحاد.

وصُورتها هكذا: 1 ، 2 ، 3 ، 4 ، 5 ، 6 ، 7 ، 8 ، 9.

وقد نظّمها بعضُهم، فقال:

ألف وحاء ثمّ حجّ بعده عــو * وبعد العـو عينٌ تُـــــرسـم
هــاء وبعد الهاء شكل ظاهر [4] و * يبدو كمخطاف كذلك تنظمّ
صفــران ثامنها وألــــــف * والواو تاسعها كذلك تـــفهم

و لقد أحسن فيها وأبدع في قوله صفران ثامنها، لأنّه أفادنا معرفة الصفر.

والثانية من عشرة إلى تسعين، وتُسمّى مرتبة العشرات.

وصُورها هكذا : 10 ، 20 ، 30 ، 40 ، 50 ، 60 ، 70 ، 80 ، 90.

والثالثة من مائة إلى تسعمائة، وتُسمّى مرتبة المئين.

وصُورها هكذا : 100 ، 200 ، 300 ، 400 ، 500 ، 600 ، 700 ، 800 ، 900.

وللعدد اثنا عشر اسمًا بسيطا يتركب منها جميع أسمائه. فالتسعة الأولى منها هي التي للآحاد، والعاشر للعشرات، والحادي عشر للمئين، والثاني عشر للآلاف، وهي بمنزلة الآحاد.

في كونها في أوّل المراتب الثلاث، كما كانت الآحاد أوّل[2] قبل عشراتها ومئينها.

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[1] في (م) : الجملة "باعتبار حلول العدد فيها" منقولة من **رفع الحجاب**، صفحة 212: 10.
[2] في (م) : سقط لفظ "أول".

ومن هنا يعود الدور.

أي من المرتبة الرابعة التي هي الآلاف، فنقول آحاد، عشرات، مئون، وهي كلّها آلاف، لم تختلف مع المراتب الثلاثة الأوّل إلّا في لفظ الألف. وكذلك المراتب الثلاث التي هي آلاف الآلاف، هي أيضا آحاد، عشرات، مئون ، لم تختلف مع التي قبلها إلّا في النطق بالألف مرتين. وكذلك المراتب الثلاث ثلث الرّابعة هي مع التي قبلها على الصفة المذكورة. وكذلك على توالي الأعداد. [4 ظ] فأعلمه.

مثال من ذلك : خمسة وعشرون ومائتان وأربعة وثمانون ألفا ومائة ألف وسبعة وستون ألف ألف وثلاثمائة ألف ألف وتسعة آلاف ألف ألف، وصُورتها هكذا: 9367184225. فالأربعة والثمانون ألفا والمائة ألف هي أحاد عشرات مئون، كالمراتب الثلاث الأوّل، لا فرق إلّا في لفظ الألف خاصة. وكذلك السبعة والستون ألف ألف والثلاثمائة ألف ألف هي آحاد عشرات مئون، غير أنّها مباينة لما تقدم في كونها ألف ألف. وكذلك التسعة آلاف ألف ألف هي آحاد عشرات مئون، لم تختلف فيما تقدم إلّا في تكرار آلاف ثلاث مرات. فاعلم.

ويُعرف كل عدد من جهة أسّه واسمه. والأسّ عبارة عن مرتبة العدد.
فأسّ الآحاد واحد،
أي هي في المرتبة الأولى،
وأسّ العشرات اثنان،
أي هي في المرتبة الثانية،
وأسّ المئين ثلاثة،
أي هي في المرتبة الثالثة،
وعلى هذا ما بعد ذلك.

مثال من ذلك : خمسة وعشرون وسبعمائة وأربعة وثمانون ألفا، وصورتها هكذا : 84725. فتجد الخمسة وهى من الآحاد كما تقدم في المرتبة[1] الأولى، وذلك هو أُسّها وأسّ أمثالها من الآحاد. وكذلك العشرون من العشرات وهي في المرتبة الثانية، وذلك هو أُسّها وأسّ أمثالها من العشرات. فلو كان مثلا ثلاثون

[1] في (م) : في هذه الفقرة يستعمل لفظ "مرتبة" وأما النسخ الأخرى فهي تستعمل لفظ "منزلة".

أو ستون أو ثمانون لجعلتها مكان العشرين إذ العشرات أسّها اثنان أي هي في المرتبة الثانية كما تقدم. وكذلك السبعمائة في المرتبة الثالثة وذلك هو أسّها [5 و] وأسّ أمثالها. وكذلك الأربعة الآلاف هي المنزلة الرابعة وذلك هو أسّها وأسّ أمثالها. وكذلك الثمانون ألفا هي في المرتبة الخامسة وذلك هو أسّها وأسّ أمثالها. (فافهم ذلك. وكذلك)[1] مثله فيما فوق ذلك وفي نظائره من الأعداد.

والاسم عبارة عن العدد الذي يحل في مرتبة ما.
فاسم الواحد آحاد، والاثنين عشرات والثلاثة مئون.

مثال منه: ثلاثة وأربعون ومائة، وصُورتها : 143. فتجد ثلاثة منازل. أسّ الأولى واحد واسم هذا الواحد آحاد لأنّ المرتبة مرتبة الآحاد، وأسّ الثانية اثنان واسم هذه الاثنين عشرات لأنّ المرتبة مرتبة العشرات، وأسّ الثالثة ثلاثة واسم هذه الثلاثة مئون لأنّ المرتبة مرتبة المئين. فافهم.

فصل في معرفة أسّ العدد المكرّر
تضرب عدد التكرار في ثلاثة، و تزيد على الخارج أسّ نوع ذلك العدد، يكون المطلوب.
(والتكرار هو عدد نطقك بالألف)[2].

مثاله : لو قيل لنا ما أسّ عشرة آلاف ألف؟ فتجد عدد التكرار اثنين، تضربها في ثلاثة (بستّة. وكذلك إن كان التكرار أكثر من اثنين أو أقلّ، فلا بدّ من ضربه في ثلاثة)[3] أبدًا، ثمّ تحمل على الستة المذكورة، أس نوع العشرة آلاف الألف[4]، وذلك اثنان. لأنّه قد تقدم أن آحاد آلاف الألف بمنزلة الآحاد، والعشرات بمنزلة العشرات، والمئين بمنزلة المئين، يكون المجموع ثمانية، هو أسّ العدد المفروض، فاعلمه. فيكون وضعها سبعة أصفار وواحد، على هذه الصورة: 10000000.

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[1] في (م) : "وكذلك فافهم". وفي (ط) : "فافهم".
[2] في (س) : سقطت الجملة "والتكرار ... بالألف"، ويعتبرها ناسخ (ت) من إضافات الهواري.
[3] في (ط) : سقطت الجملة " بستّة. وكذلك إن كان التكرار أكثر من اثنين أو أقلّ، فلا بدّ من ضربه في ثلاثة".
[4] في (م) : سقط لفظ "الالاف".

وعكسه، إذا كانت معك منازل كثيرة[1] وأردت اسمها فاقسمها على ثلاثة قسمة يبقى لك منها ثلاثة أو أقلّ، فما خرج فهو عدد[2] التكرار للعدد المستدل عليه بالباقي.

مثاله : لو قيل لنا ما اسم العدد الواقع في المنزلة[3] العاشرة؟ فنقسم العشرة التي هي عدد المنازل على ثلاثة يخرج لنا ثلاثة ويبقى واحد. وسوا أيضا كانت المنازل |5| ظ| أكثر من عشرة أو أقل فلا بدّ من قسمتها على ثلاثة أبدًا. فالثلاثة الخارجة عدد التكرار لاسم الواحد الباقي واسم الواحد الباقي آحاد، فالعشرة مراتب هي أسّ لآحاد آلاف آلاف الآلاف. وصورتها : 1000000000.
ولو يبقي من القسمة اثنان لكانت عشرات آلاف آلاف ألف، أو ثلاثة لكانت مئين[4] آلاف آلاف ألف. فاعلمه.

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[1] في (م) وفي (ط) : سقط لفظ "كثيرة".
[2] في (س) : سقط لفظ "عدد".
[3] في النسخ الاخرى : "مرتبة".
[4] في (ط) : سقط لفظ "مئين".

الباب الثاني في الجمع

الجمع ضم الأعداد بعضها إلى بعض ليلفظ بها بلفظ واحد.

وهو ينقسم على[1] خمسة أضرب:
أحدها الجمع على غير نسبة معلومة
والثاني الجمع على تفاضل معلوم.

وهو ينقسم على قسمين:
[تفاضل في الكيف، وهو الذي تكون أعداده على نسبة هندسية، فتكون الأعداد متفاضلة بأعداد مختلفة وهي متشابهة[2] في الكيف عند نسبة بعضها إلى بعض مثل نسبة النصف أو الثلث أو غير ذلك.][3]
وهذا القسم على ضربين لأنّ النسبة إمّا أن تكون نسبة النصف، وهي المعنيّ بها في قوله "وإمّا الجمع على التفاضل في مثل بيوت الشطرنج وأشباهها"، وذلك أنّ الواحد الذي في البيت الأوّل هو نصف الاثنين التي في البيت الثاني، والاثنان هي نصف الأربعة التي في البيت الثالث، وكذلك إلى آخرها."
وإمّا أن تكون نسبة الثلث، أو الربع، أو السبع، أو غير ذلك من النسب، وهي المعنيّ بها في قوله : "وإن كانت أعداد على تفاضل آخر". ويأتي مثاله بعد في موضعه، إن شاء الله تعالى.
[والقسم الثاني: تفاضل في الكم، وهو الذي تكون أعداده على نسبة عددية مثل الأعداد على تواليها التي تتفاضل بالواحد، والأفراد[4] على تواليها التي [6 و] تتفاضل باثنين اثنين، ونحو ذلك][5].
فتكون[6] الأعداد متفاضلة بأعداد متساوية وهي مختلفة في الكيف عند نسبة بعضها إلى بعض. وهو المعنيّ بها في قوله: "وإن تفاضلت الأعداد بعدّة معلومة دون التضعيف".
وإنّما قال، رضي الله عنه : "تفاضل معلوم"، ولم يقل: "نسبة معلومة" لأنّه [متى أطلق لفظ النسبة فالمشهور فيه الهندسية][1].

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[1] في (م) سقط لفظ : "على".
[2] في (أ) وفي (ب) : "متساوية"، وفي (ت) وفي **رفع الحجاب** : "الأفراد".
[3] الفقرة : "تفاضل في الكيف ... غير ذلك" منقولة من **رفع الحجاب**، صفحة 214: 3-5.
[4] في (أ) وفي (ب) : "الازواج" .
[5] الفقرة : "فالقسم الثاني: تفاضل في الكم ... ونحو ذلك" منقولة من **رفع الحجاب**، صفحة 214: 13-15.
[6] في (أ) : "وإلا فتكون".

والثالث الجمع على توالي الأعداد و مُربّعاتها و مُكعّباتها،
والرابع الجمع على توالي الأفراد و مُربّعاتها و مُكعّباتها،
والخامس الجمع على توالي الأزواج و مُربّعاتها و مُكعّباتها.

تنبيه : [وجمع الأعداد على تواليها والأفراد على تواليها والأزواج على تواليها إنّما هو من النسبة العددية. وهذه الأضرب الثلاثة الأخيرة إنّما المقصود فيها جمع المُربّعات والمُكعّبات خاصة، وقدّم فيها جمع الأعداد، وإن كان من الضرب الذي قبله، كالتوطئة لجمع المُربّعات والمُكعّبات، ولأنّه عمل ملخص خاص من ذلك العام على اقرب مأخذ][2].

فأمّا الجمع على غير نسبة معلومة، فالمقصود به أن تجمع عددا من منازل كثيرة إلى عدد كذلك، وينبغي أن يوضع أحد المجموعين في سطر ويوضع تحته المجموع الآخر، كلّ منزلة تحت نظيرتها، ثمّ تجمع كلّ منزلة من أحد المجموعين إلى نظيرتها من الآخر. وإن لم يوجد لها نظيرة فتكون كأنّها الجواب المجتمع منها ومن نظيرتها لو كانت لها نظيرة، فما اجتمع فهو الجواب.
و يبتدأ بالجمع من أوّل المراتب أو من آخرها. والاختيار الابتداء من أوّلها، فهو أرتب.

مثال من ذلك بالجمع من أوّلها : أردنا جمع ثلاثة وأربعين وأربعة آلاف إلى خمسة وثمانين وستمائة وألفين. فنضع المجموع في سطر وتحته المجموع إليه في سطر آخر يوازيه، [6 ظ] كما ذكر، على هذه الصورة :

4043
2685 .

فنجمع الخمسة التي (في أوّل السطر الأسفل إلى نظيرتها من السطر الأعلى)[3] وهي الثلاثة بثمانية. ننزلها فوقهما لأنها آحاد من نوعهما. ثمّ نجمع الثمانية التي في مرتبة العشرات من الأسفل إلى نظيرتها من الأعلى، وهي الأربعة، باثني

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[1] الجملة : "متى أطلق لفظ النسبة فالمشهور فيه الهندسية" منقولة من **رفع الحجاب**، صفحة 214: 18 إلى 215: 1.
[2] الفقرة : "جمع الاعداد ... على أقرب مأخذ" منقولة من **رفع الحجاب**، صفحة 226: 4-7.
[3] في (ط) وفي (ت) : سقط لفظ " في السطر الأسفل إلى نظيرتها من الأعلى ".

عشر. فنثبت الاثنين فوقهما، لأنّها من نوعهما أيضا. ونضيف العشرة بواحد إلى الستة التي في مرتبة المئين من الأسفل بسبعة، لأنّ كلّ مرتبة فهي آحاد للتي بعدها وعشرات للتي قبلها وليس للسبعة نظيرة من الأعلى.
فكأنّها المجتمع من تلك المرتبة ومن (التي هي)[1] نظيرتها لو كان فيها شيء. فنثبت السبعة فوق الصفر. ثمّ نجمع الاثنين التي في مرتبة الآلاف من الأسفل إلى نظيرتها من الأعلى، وهي الأربعة، بستة. نثبتها فوقها. وقد تمّ[2] العمل.
فيكون المجتمع: ثمانية وعشرين وسبعمائة وستة آلاف، وهذه[3] صورة ذلك: 6728.

مثال منه آخر بالجمع من آخرها : أردنا جمع ثمانية وسبعين وتسعمائة إلى ستة وخمسين وأربعمائة. فننزلها في سطرين، كما ذكرنا، على هذه الصورة :

978
456 .

فنجمع الأربعة التي في مرتبة المئين من السطر الأسفل إلى نظيرتها من السطر[4] الأعلى، وهي التسعة بثلاثة عشر. نثبت الثلاثة فوقهما والعشرة بواحد بعد الثلاثة. ثمّ نجمع الخمسة التي في مرتبة العشرات من الأسفل إلى نظيرتها من الأعلى وذلك سبعة باثني عشر. فنثبت الاثنين فوقهما، ونضيف العشرة بواحد إلى الثلاثة التي فوق المجموعين بأربعة. نثبتها مكانها. ثم نجمع الستة التي في مرتبة الآحاد [7 و] من الأسفل إلى نظيرتها من الأعلى، وهي الثمانية، بأربعة عشر. فنثبت الأربعة فوقهما، ونضيف العشرة بواحد إلى[5] الاثنين، التي فوق المجموعين أيضا، بثلاثة. نثبتها مكانها. وقد تمّ العمل.
فيكون المجتمع[6] أربعة وثلاثين وأربعمائة وألفا، وصورة ذلك : 1434.

وغاية ما يفيد[7] الجمع (مرتبة)[8] واحدة.

[1] في (م) وفي (أ) : سقطت "التي هي".
[2] في (م) : "تقدم".
[3] في (م) : سقط لفظ "هذه". وفي (ط) : "وهو كذلك."
[4] في (م) : سقط لفظ "سطر".
[5] في (م) : "فوق".
[6] في (م) : "مجموع".
[7] في (س) وفي (أ) : "يفيده"
[8] في المخطوطات الأخرى : " منزلة "

مثاله : أردنا جمع تسعة إلى تسعة، وهما أقصى عدد يقع في تينك المنزلتين من السّطرين. فنقول تسعة إلى تسعة بثمانية عشر. وصورتها هكذا : 18. أفاد الجمع منزلة واحدة.

واختبار الجمع أن تطرح أحد سطريه من الجواب يبقى الآخر.

مثاله : لو أردنا اختبار هذه المسألة لطرحنا التسعة، التي هي أحد المجموعتين، من ثمانية عشر، التي هي الجواب، تبقى التسعة الأخرى. فافهم.

وأمّا الجمع على التفاضل في مثل بيوت الشطرنج وأشباهها على أن يكون في البيت الأوّل واحد، ثمّ يتدرج التضعيف من أوّله إلى آخر المفروض، فهو أن تزيد على الواحد الذي في البيت الأول واحدا ليكون ذلك ما في البيت الثاني، ثمّ تضرب ذلك[1] في نفسه، فما بلغ فهو ما في (البيت)[2] الثاني وما قبله بزيادة واحد، [ثمّ تضرب ذلك في نفسه أيضًا، فما بلغ فهو ما في (البيت)[3] الرابع وما قبله بزيادة واحد][4]، ثمّ لا تزال تضرب الخارج في نفسه وتضاعف البيوت للخارج حتى تنتهي إلى المفروض، وتسقط الواحد من المجموع، فما بقي فهو المطلوب.

مثاله : لو أردنا جمع ما في ستة عشر بيتا منها [7 ظ] مع كونها على الصفة المذكورة لحملنا على الواحد الذي في البيت الأوّل واحدا باثنين، نضربها في نفسها بأربعة، وهي ما في البيت الثاني وما قبله بزيادة واحد، وهي أيضًا ما في البيت الثالث وحده، لأنّ البيوت هنا بمنزلة المنازل للعدد ولاشكّ أنّا إذا ضربنا منزلة ما في منزلة ما كان[5] أسّ الخارج عدد أسّ المضروبين إلّا واحدا أبدًا، على ما نبين في الضرب. فلما ضربنا ما في الثاني في مثله كان الخارج ما في الثالث، لا محالة له[6]. فاعلمه.
ثمّ نضرب أيضًا الأربعة في نفسها بستة عشر، وهي ما في الرابع وما قبله بزيادة واحد، وهي أيضًا ما في الخامس وحده. ثمّ نضرب الستة عشر في نفسها

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[1] في (م) : سقط لفظ "ذلك".
[2] في (س) وفي (م) وفي (ط) : سقط لفظ "البيت".
[3] في (م) وفي (س) : سقط لفظ "البيت".
[4] في (ت) وفي (ط) : سقطت الجملة: "ثم نضرب ... واحد".
[5] في (م) وفي (ط) : سقط لفظ "كان".
[6] في (م) وفي (ط) : سقط لفظ "له".

بستة وخمسين ومائتين (وصُورتها : 256)[1]، وهي ما في الثامن وما قبله بزيادة واحد، وهي أيضًا ما في التاسع وحده. ثمّ نضرب أيضًا الستة والخمسين والمائتين في نفسها بستة وثلاثين وخمسمائة وخمسة وستين ألفا، (وصُورتها : 65536)[2]، وهي ما في السادس عشر وجميع ما قبله بزيادة واحد، وهي أيضًا جميع ما في السابع عشر وحده.

فتبين أنّ كلّ عدد في بيت يزيد على مجموع ما قبله بواحد، فتسقط الواحد الزائد، يبقى خمسة وثلاثون وخمسمائة وخمسة وستون ألفا، وهو المطلوب. وكذلك العمل لو كان المطلوب أكثر أو أقلّ. فاعلمه. وصورة[3] ذلك مُجدولا[4] :

1	2	4	8	16	32	64	128
256	512	1024	2048	4096	8192	16384	32768

8 و وإن اختلف الوضع فاضرب الباقي في الأول يكن المطلوب. (واختلاف الوضع هو أن يكون في البيت الأول غير الواحد)[5].

مثاله : لو أردنا جمع ما في ثمان[6] بيوت وفي الأوّل أربعة. لفرضنا نحن أنّ في البيت الأوّل واحدًا واستخرجنا الجملة كما تقدم وأسقطنا الواحد، وذلك معنى قوله "الباقي"، فيكون ذلك خمسة وخمسين ومائتين، وصُورتها : 255، فتضربها في الأربعة التي في البيت الأول يكون ذلك عشرين وألفا، (وصُورته)[7] : 1020، وهو المطلوب.

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[1] في (ط) وفي (ت) : الصورة غائبة.
[2] في (ط) وفي (ت) : الصورة غائبة.
[3] في (م) : "وصفة".
[4] في (أ) وفي (ب) : صورة الجدول هذه:

1	2	4	8	6
16	32	64	128	5
256	512	1024	2048	5
4096	8192	16384	32768	3
				6

[5] في (س) : سقطت الجملة : "واختلف ... غير الواحد"، ويعتبرها ناسخ (ت) من إضافات الهواري.
[6] في (ط) وفي (ت) : "تسعة". وهذا خطأ.
[7] في (أ) : سقطت كلمة: "وصورته".

وإن كانت أعداد على تفاضل آخر، فاضرب أصغرها في فضل الأكبر عليه واقسم على الفضل بين الأصغر والعدد الذي يليه وزد الخارج على الأكبر. يكن الجواب.

مثاله : لو أردنا جمع خمسة أعداد، مثلا، على نسبة الثلثين مثل ستة عشر، وأربعة وعشرين، وستة وثلاثين، وأربعة وخمسين، وواحد وثمانين. فننزلها في سطر ونفرق بينها على هذه الصورة[1] :

.81 ∴ 54 ∴ 36 ∴ 24 ∴ 16

فنضرب الأصغر، وهو الستة عشر في فضل الواحد والثمانين عليه، لأنّها أكبر الأعداد، وذلك خمسة وستون، يكون الخارج أربعين وألفا، وصورتها : 1040. فنقسمها على الفضل بين الستة عشر والعدد الذي يليه[2]، وهي الأربعة والعشرون، وذلك ثمانية، فيخرج من القسمة ثلاثون ومائة، (وصورتها 130)[3]. فنزيدها على الأكبر، يكون ذلك أحد عشر ومائتين، وصورتها : 211. وهو المطلوب.

[وهذا العمل عام لجميع الأعداد التي على نسبة هندسية بإطلاق[4]، كانت على نسبة النّصف أو غيره. والذي قبله خاص، والخاص أصل في بابه كما هو العام أصل لأنّ استعمال العام |8 ظ| في موضع الخاص يصير غير مفيد لأجل[5] التطويل. ومن هذا العمل العام يتبيّن لك أنّ كلّ عدد من بيوت الشطرنج يزيد على مجموع ما قبله بواحد][6]. (فاعلمه)[7].

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[1] في (م) وفي (أ) : غابت النقط في الصورة. أما في (ت) فقد عوضت بأسطر مستقيمة.
[2] في (أ) : "بعده".
[3] في (ت) وفي (ط) : سقطت العبارة "وصورته 130".
[4] في (م) : "باي اطلاق". وفي (ط) وفي (ت) : سقطت "كانت".
[5] في **رفع الحجاب** : إلا أن التطويل على ما نذكر في باب الضرب". وفي (م) : "لا التطويل". وهذا فيه نقص.
[6] الفقرة: "وهذا العمل ... بواحد"، منقولة من **رفع الحجاب**، صفحة 220: 8-12.
[7] في (أ) وفي (ب) : سقطت العبارة: "فاعلمه".

(وإن تفاضلت الأعداد بعدّة معلومة دون التّضعيف فاضرب التفاضل في عدّة الأعداد إلّا واحدا، فما خرج فاحمل عليه العدد الأوّل، فما بلغ فهو آخر الأعداد، فاجمعه مع الأوّل واضربه في نصف عدّة الأعداد، يكون الجواب المطلوب[1].)[2]

مثاله : أردنا جمع ستة أعداد، (أوّلها عشرة، وهو الطرف الأصغر)[3]، وتتفاضل بثلاثة إلى آخرها.
فالمجهول في هذه المسألة شيئان: الطرف الأكبر وهو آخرها، والجملة. فنضرب الثلاثة، التي هي التفاضل، في (الخمسة، عدد الأعداد إلّا واحدا)[4] بخمسة عشر، فنحمل عليها العشرة، العدد الأوّل، يكون ذلك خمسة وعشرين، وهو آخر الأعداد، فنجمعه أيضًا مع الأول، يكون خمسة وثلاثين. فنضربها في ثلاثة، نصف عدّة الأعداد، بخمسة ومائة، وصورتها 105، وهو المجتمع من تلك الأعداد، أعني جملتها.
ومن هذا الوجه يتبين لك جميع[5] سائر الوجوه (اللازمة عنه. فاعلمه.)[6]

وأمّا الجمع على توالي الأعداد، فهو أن تضرب نصف المنتهى إليه في المنتهى إليه وواحد.

مثاله : أردنا أن نجمع من واحد إلى عشرة على التوالي. فنحمل على العشرة، المنتهى إليه، وواحدا. فيكون ذلك أحد عشر، فنضربها في نصف العشرة بخمسة وخمسين، وهو المجموع، وصورته : 55.

وتربيعه بضرب ثلثي المنتهى إليه وزيادة ثلث واحد في المجموع.

مثاله : لو أردنا أن نجمع من مُربّع واحد إلى مُربّع عشرة على التوالي، فنأخذ ثلثي العشرة المنتهى إليه |9 أ| بستة وثلثين ونزيد عليها ثلث واحد، أبدًا، بسبعة. فنضربها في المجموع، وذلك خمسة وخمسون، يكون الخارج خمسة وثمانين وثلاثمائة، وهو المطلوب، وصورتها : 385.

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[1] في (م) وفي (ط) : سقط لفظ "المطلوب".
[2] في (س) : سقطت الجملة "وإن تفاضلت ... المطلوب".
[3] في (أ) : "أولها، وهو الطرف الأصغر، عشرة". وفي (ت) وفي (ط) : "مثلا، أولها عشرة، وهو المطلوب الأصغر".
[4] في (أ) وفي (ط) : "الستة، عدّة الأعداد، إلا واحدا".
[5] في (م) : سقط لفظ "جميع".
[6] في (أ) : "اللازمة ".

وتكعيبه بتربيع المجموع.

مثاله : لو أردنا أن نجمع من مُكعّب واحد إلى مُكعّب عشرة على التوالي، فنضرب المجموع، الذي هو خمسة وخمسون، في نفسه يكن الخارج خمسة وعشرين وثلاثة آلاف، وهو المطلوب. وصورته : 3025.

وأمّا الجمع على توالي الأفراد، فهو أن تربّع نصف المنتهى إليه المؤلف مع الواحد.

مثاله : لو أردنا أن نجمع من واحد إلى تسعة على توالى الأفراد، فنحمل واحدا على التسعة المنتهى إليه بعشرة، فنضرب نصف العشرة في نفسه بخمسة وعشرين، وهو المجموعٍ، وصورتها : 25.

وتربيعه بضرب سدس المنتهى إليه في مُسطّح العددين اللذين يليانه بعده.

مثاله : لو أردنا أن نجمع من مُربّع واحد إلى مُربّع تسعة على توالي الأفراد، فنضرب سدس التسعة المنتهى إليه، وذلك واحد ونصف، في مسطح العشرة في الأحد عشر، الذي هو عشرة ومائة، لأنّهما العددان اللذان بعد التسعة. يكون الخارج خمسة وستين ومائة، وهو المطلوب، وصورته : 165.

وتكعيبه بضرب المجموع في ضعفه إلا واحدا.

مثاله : لو أردنا أن نجمع من مُكعّب واحد إلى مُكعّب تسعة على توالى الأفراد. فنضرب المجموع، وذلك خمسة وعشرون، في ضعفها إلّا واحدا، الذي هو تسعة وأربعون، يكون الخارج خمسة وعشرين ومائتين وألفا، وهو المطلوب، وصورته[1] [9 ظ] : 1225.

وأمّا الجمع على توالي الأزواج، فهو أن تحمل على المنتهى إليه اثنين أبدا وتضرب نصف المجتمع في نصف المنتهى إليه.

[1] في (أ) : سقطت الصورة.

مثاله : لو أردنا أن نجمع من اثنين إلى عشرة على توالى الأزواج، فنحمل اثنين على العشرة المنتهى إليه باثني عشر فنضرب نصفها في نصف العشرة بثلاثين، وهو المجموع، وصُورته : 30.

وتربيعه بضرب ثلثي المنتهى إليه وثلثي واحد في المجموع.

مثاله : لو أردنا أن نجمع من مُربّع اثنين إلى مُربّع عشر على توالى الأزواج، فنأخذ ثلثي العشرة المنتهى إليه بستة وثلثين ونزيد عليها ثلثي واحد أبدا بسبعة وثلث، فنضربها في المجموع، الذي هو ثلاثون، يكون الخارج عشرين ومائتين، وهو المطلوب. وصورته : 220.

(وإن شئت)[1]، فتضرب سدس المنتهى إليه في مُسطّح العددين اللذين يليانه بعده.

مثاله : لو أردنا أن نجمع من مُربّع[2] اثنين إلى مُربّع[3] اثنا عشر، فنأخذ سُدس الاثني عشر المنتهى إليه باثنين، فنضربها في مُسطّح الثلاثة عشر[4] في الأربعة عشر، الذي هو اثنان وثمانون ومائة، لأنّهما العددان اللذان بعد الاثني عشر. يكون الخارج أربعة وستين وثلاثمائة، وهو المطلوب. وصورته : 364.

وتكعيبه بضرب المجموع في ضعفه.

مثاله : لو أردنا أن نجمع من مُكعّب اثنين إلى مُكعّب عشرة على توالى الأزواج، فنضرب المجموع، الذي هو ثلاثون، في ضعفها، الذي هو ستون، يكون الخارج ثمانمائة وألفا، [10 و] وهو المطلوب. وصُورته : 1800.

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[1] في (س) : " أو ".
[2] في (أ) : "مُكعّب" ، وهو خطأ.
[3] في (أ) : "مُكعّب" ، وهو خطأ.
[4] في (أ) : سقط لفظ: "عشر".

[ومتى كان الابتداء من غير الواحد (في هذه الأقسام الثلاثة)[1] فتجمع من واحد إلى المنتهى إليه، ثمّ من واحد إلى العدد الذي قبل المبتدأ وتسقط الأقلّ من الأكثر والاثنان في الأزواج بمنزلة الواحد.

فتفهم كلّ هذا وتدبّره تصب، إن شاء الله تعالى.][2]

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[1] في **رفع الحجاب** : سقطت " في هذه الأقسام الثلاثة ".

[2] في (س) : سقطت الفقرة "ومتى كان ... إن شاء الله تعالى "، ويعتبرها ناسخ (ت) من تأليف الهواري. وهي منقولة من **رفع الحجاب**، صفحة 228: 13-16.

وأما ناسخ (ب)، فهو أضاف في الحاشية الفقرات التالية:

" في جمع الأعداد العادة على تواليها أن تجمع من أربعة إلى عشرة، فاجمع من واحد إلى عشرة، يكون خمسة وخمسون، ثم تجمع من واحد إلى العدد الذي ابتدأت به، وهو أربعة، يكون ستّة. تسقطها من الأكثر، وهو خمسة وخمسون، يبقى تسعة وأربعون. وهو المطلوب. ومثاله في جمع الأفراد على تواليها : أردت أن تجمع من خمسة إلى تسعة، يكون خمسة وعشرون، ثم تجمع من واحد إلى العدد الذي ابتدأت به أفرادا، ليكون أربعة. تسقطها من خمسة وعشرين، يبقى واحد وعشرون. ومثاله في جمع الأزواج على تواليها الواردة: أن تجمع من ستّة إلى عشرة، يكون ثلاثون، ثم تجمع من اثنين إلى العدد الذي ابتدأت به أزواجا، ليكون ستّة. تسقطها من ثلاثين، يبقى أربعة وعشرون. وهو المطلوب". (مخطوط أكسفورد، ورقة 9 ب)

الباب الثالث في الطرح

الطرح هو طلب الباقي بعد إسقاط أحد العددين من الآخر. وهو على ضربين : (ضرب يطرح الأقل من الأكثر مرّة واحدةّ)[1]، وضرب يطرح الأقلّ من الأكثر أكثر من مرّة واحدة حتى يفنى الأكثر أو يبقى منه فضلة أقلّ من الأقلّ، وهذا الضرب يُسمّى الامتحان بالطرح.

فالضرب الأول ينبغي أن تضع المطروح منه في سطر وتحته المطروح، على صفة الجمع، وتطرح كلّ منزلة من نظيرتها، إن وجدت لها نظيرة. وإن لم تجد لها نظيرة، أو يكون فيها أقلّ من المطروح فاطرح المطروح منه من المطروح، فما بقي[2] إطرحه من المرتبة التي بعدها، وتضع الباقي في الموضع الذي تعطيه مرتبة المنازل.

وإن شئت فاحمل على النظيرة عشرة أبدًا، وتطرح من المجتمع، وتزيد واحدا في المرتبة الثانية من المطروح، ثمّ اصنع كذلك حتى تأتي على جميع المطروح والمطروح منه.

وتبدأ بالطرح من أوّل المراتب[3] أو من آخرها، والاختيار الابتداء من آخرها (على خلاف)[4] الاختيار في الجمع.

مثال منه بالطرح من آخرها : لو أردنا طرح ثمانية وستين وتسعمائة وأربعة آلاف من خمسة وثلاثين وخمسة آلاف، فننزلهما في سطرين متوازيين[5]، كما تقدّم في الجمع، على هذه الصورة[6] :

5035
. 4968

فنطرح الأربعة التي في (مرتبة الآلاف)[7] [10 ظ] من نظيرتها من المطروح منه، وهي خمسة. فيبقى منه واحد، نثبته فوقها، ثمّ نطرح التسعة التي في مرتبة

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[1] في (م) : سقطت الجملة "ضرب يطرح ... مرّة واحدة".
ويعتبرها (ت) من زيادات الهواري.
[2] في (ب) وفي (ط) وفي (ت) : "بقي منه".
[3] في (ط) في (ت) : "المنازل".
[4] في (أ) و (ب) : "بخلاف".
[5] في (ط) وفي (ت) : "متساويين". وهذا خطأ.
[6] في (ط) : 4968
.5035
[7] في (ب) : سقط لفظ "مرتبة". وفي (ت) : "مرتبة المئين"، وهذا خطأ.

المئين من نظيرتها أيضا، وليس فيها شيء سوى الصفر، والصفر ليس بشيء. [فنطرح ما[1] ليس بشيء (من هذا)[2] المطروح منه، من التسعة المطروحة، تبقى تسعة نطرحها][3] من الواحد الذي على الخمسة، لأنّه عشرة بالنّسبة إلى الصفر، كما تقدّم. فيبقى منها واحد، نثبته فوق الصفر. وليس على الخمسة شيء. ثمّ نطرح الستة التي في مرتبة العشرات من نظيرتها أيضا، وهى ثلاثة، أقلّ من الستة المطروحة، فنطرح الثلاثة المطروح منها من الستة المطروحة، يبقى منها ثلاثة. نطرحها من العشرة التي على الصفر، تبقى منها سبعة، نثبتها فوق الثلاثة. وليس على الصفر شيء. ثمّ نطرح الثمانية، التي في مرتبة الآحاد، من نظيرتها أيضًا، وهي خمسة، أقلّ من الثمانية. فنطرح الخمسة المطروح منها، من الثمانية المطروحة. يبقى منها ثلاثة. نطرحها من السبعين التي فوق الثلاثة. الباقي منها سبعة وستون. فنثبت السبعة فوق الخمسة والستين بستة فوق الثلاثة، وهو الباقي، وصورة[4] ذلك:

0067
. $\overline{5035}$
4968

مثال منه آخر بالطرح من أوّلها : لو أردنا طرح تسعة وستين وأربعمائة وثلاثة آلاف من ثلاثة وأربعين وخمسمائة وستة آلاف، فننزلها في سطرين متوازيين كما ذُكر، على هذه الصورة[5] :

6543
. 3469

فنطرح التسعة التي في مرتبة الآحاد من نظيرتها من المطروح منه وهى ثلاثة، | 11 و | أقلّ من التسعة. فنحمل على الثلاثة المطروح منها عشرة بثلاثة عشر، فنطرح منها التسعة، الباقي أربعة. نثبتها فوق الثلاثة، ثمّ نحمل على الستة التي في مرتبة العشرات من المطروح واحدا بسبعة. نطرحها من نظيرتها أيضًا

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[1] في (أ) وفي (ب) : سقط لفظ "ما".
[2] في (م) : "وهو". وفي (ب) : "هذا".
[3] في (ط) وفي (ت) : سقطت الجملة وعوضت بالجملة "فنطرج التسعة".
[4] في (ط) : 67.
[5] في (ط) : 3469 / 6543.

وهي أربعة، أقلّ من السبعة، فنحمل على الأربعة عشر بأربعة عشر[1]، فنطرح منها السبعة الباقي بسبعة[2]. نثبتها فوق الأربعة. ثمّ نحمل واحدا على الأربعة التي في مرتبة المئين بخمسة، نطرحها من نظيرتها أيضًا وهي خمسة، تفنى بها. فنثبت فوق الخمسة صفرا. ثمّ نطرح الثلاثة التي في مرتبة الآلاف بغير زيادة عليها. لم يبق شيء من التي قبلها من نظيرتها أيضًا، وهي الستة. تبقى منها ثلاثة، نثبتها فوق الستة. وقد تمّ العمل. فيكون الباقي أربعة وسبعين[3] وثلاثة آلاف، وصورة ذلك[4] :

$$\begin{array}{r} 3074 \\ \overline{6543} \\ 3469 \end{array}.$$

وغاية ما يحط الطرح[5] منزلة واحدة.

مثاله : إذا قيل لك اطرح واحدا من عشرة. الباقي تسعة. انحط من المطروح منه منزلة. فافهمه.

واختبار الطرح بأن تجمع الباقي إلى المطروح فيخرج المطروح منه أو تطرح الباقي من المطروح منه يبقى المطروح.

مثاله : في المسألة المذكورة آنفا[6]، فإنّ الباقي منها تسعة تجمعه إلى الواحد، الذي هو المطروح، بعشرة، وهي المطروح منه. وإن طرحنا الباقي، وهو التسعة، من المطروح منه، وهو العشرة، فيبقى واحد، مثل المطروح.

[ويلزم من (الوجه الأوّل)[7] عمل الطرح بالجمع بأن تطلب عددا إذا زدته على المطروح كان مثل المطروح منه. والابتداء فيه يكون من أوّل المراتب. (وبه يعمل أصحاب الرشم بالرّومي)[8].

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[1] في (أ) : سقط "بأربعة عشر".
[2] في (ت) : سقطت عبارة: "بسبعة".
[3] في (م) : "تسعين". وهذا خطأ.
[4] في (أ) وفي (ط) وفي (ت) : وصورة ذلك: 3074.
[5] في (أ) : "يحطه". وسقط لفظ "الطرح" في كل النسخ.
[6] في (س) : "أيضًا".
[7] في **رفع الحجاب**، صفحة 245: "وجه اختبار الطرح بالجمع".
[8] في **رفع الحجاب** : غابت الجملة "وبه يعمل بالرشم الرومي". وفي (أ) : "أصحاب الرسم بالرومي". وفي (ب) وفي (ت) : " أصحاب الرومي".

11 ظ ومتى كان معك طرح عدد من عدد، والباقي من عدد آخر كذلك، فتعمل بهذا الباب فيها على التوالي.

وإن شئنا ركّبنا المسألة من جمع وطرح، نأخذ الأزواج من المطروحات، أعني الثاني والرابع والسادس كذلك، ونجمعها مع المطروح منه، ونجمع الأفراد من المطروحات، أعني الأول والثالث والخامس كذلك، ونسقط ما يجتمع منها من المجموع الأوّل.
وابتداء العدد في المطروحات إنّما يكون فيها مما يلي المطروح منه إلى آخرها.][1]

[مثاله : اطرح اثنين من خمسة، والباقي من سبعة، والباقي من ثمانية، والباقي من عشرة.
وربّما عبّر عن هذا بالاستثناء. فيقال: عشرة إلّا ثمانية إلّا سبعة إلّا خمسة إلّا اثنين. فننزلها في سطر هكذا[2]:

10 إلا 8 إلا 7 إلا 5 إلا 2.

فنطرح الاثنين من الخمسة، والباقي من السبعة، والباقي من الثمانية، والباقي من العشرة. يبقى ستة، وهو المطلوب.

(وإن شئنا فنجمع أزواج[3] المطروحات، وهي : السبعة والاثنان مع العشرة، المطروحَ منها، يكون ذلك تسعة عشرة. ثمّ نجمع أفراد[4] المطروحات، وهي : الثمانية والخمسة، يكون ثلاثة عشر. نطرحها من التسعة عشر[5]، يبقى ستة)[6].
وإن شئنا فننظر بين ثلاثة منها متوالية، فنطرح الأوسط من[7] المجموع طرفيها، ويبقى الباقي عددا واحدا. (فنضعه في موضع المطروح)[8] وننظر بينه وبين اثنين من الباقية كذلك، مثل العشرة والثمانية والسبعة، فنسقط الثمانية من سبعة

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[1] في (أ): يعتبر الناسخ الفقرتين "ويلزم ... إلى آخرها" من تأليف ابن البنا في **التلخيص**. ولكن لا شيء في النسخ الاخرى يدل على ذلك. وفي الحقيقة الفقرتان منقولتان من **رفع الحجاب**، صفحة 245: 8-15**.** ونلاحظ أن الهواري أبدل ضمير الخطابة من "شئنا" إلى "شئت".
[2] في (ط) وفي (ت): "10 ∴ 8 ∴ 7 ∴ 5 ∴ 2".
[3] في (ب) : "وان شئنا فنجمع أزواج". وفي (ط) وفي (ت): "وان شئنا فنجمع أفراد".
[4] في (ب) : "نجمع أفراد". وفي (ط) وفي (ت): "أزواج".
[5] في (ت) : سقط لفظ: "عشر".
[6] في (أ) : الجملة: "وإن شئنا، نجمع ... تبقى ستة" غائبة.
[7] في (م) : سقط لفظ "من".
[8] في **رفع الحجاب** وفي (م) وفي (ط) : سقطت الجملة.

عشر، مجموع الطرفين، يبقى تسعة، وهي مع الاثنين والخمسة ثلاثة أعداد متوالية[1]. نسقط الخمسة من مجموع الطرفين، يبقى ستة. [12 و]

أو ننظر أوّلا بين السبعة والخمسة والاثنين، فنسقط الخمسة من التسعة، مجموع الطرفين، يبقى أربعة، وهي مع الثمانية والعشرة ثلاثة أعداد متوالية، فنسقط الثمانية المتوسّطة من (مجموع الطرفين)[2]، يبقى ستة.
أو ننظر بين الثمانية والسبعة والخمسة أولا، فنسقط السبعة المتوسّطة من مجموع طرفيها، يبقى ستة، هي متوسطة بين العشرة والاثنين. فنسقطها من مجموعهما، يبقى ستة.

وإن شئنا فنطرح الثمانية من العشرة ونزيد الباقي على السبعة ونطرح منها الخمسة، ونزيد الباقي على الاثنين، يكون ستة.

وعلة ذلك أن طرح النّاقص من الزائد يكون ناقصا، وطرح الناقص من ناقص مثله يكون زائدا.

فلهذا كان الناقص الثاني والرابع والسادس من الأزواج أبدًا زائدة، لأنّ كلّ واحد منها هو ناقص من ناقص، والأفراد أبدًا ناقصة لأنّها ناقصة من زائد.][3] فاعلمه.

والضرب الثاني فيه ثلاثة طروح، وهي التي كثر استعمالها في اختبار الأعمال. أحدها طرح تسعة والثاني طرح ثمانية والثالث طرح سبعة.

فطرح تسعة يبقى من كلّ عقد واحد، فتأخذ العدد من مراتبه، كأنّه آحاد، فتطرحه[4] تسعة تسعة.

مثاله : لو أردنا طرح خمسة وثلاثين وأربعمائة وستة آلاف، فننزل العدد في سطر هكذا : 6435، فنجمع الخمسة إلى الثلاثة بثمانية، نجمعها إلى الأربعة باثني عشر، نطرح منها تسعة، الباقي ثلاثة. نجمعها مع الستة بتسعة وهي منطرحة. فالجواب [12 ظ] بطرح.

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[1] في (م) : سقط لفظ "متوالية".
[2] في **رفع الحجاب** وفي (ب): "مجموع طرفيها".
[3] الفقرات: "مثاله : اطرح ... ناقصة من زائد."، منقولة من **رفع الحجاب**، صفحة 245: 10 إلى 247: 2.
[4] في (س) : "فتطرح".

(وطرح ثمانية)[1] يبقى من كلّ عشرة اثنان ومن كلّ مائة أربعة، وأزواج المئين وما فوقها منطرحة. فيبقى من أفراد المئين أربعة، وتضرب العشرات في اثنين، (لأنّ الباقي من كلّ عشرة اثنان)[2]، وتجمع ذلك مع الأربعة ومع الآحاد، وتطرحه ثمانية ثمانية.

مثاله : لو أردنا طرح ثلاثة وتسعين وثلاثمائة وخمسة آلاف، فننزل العدد في سطر هكذا : 5393، فخمسة آلاف منطرحة، ويبقى من ثلاثمائة أربعة، نحفظها ونضرب التسعين في اثنين بثمانية عشر نجمعها إلى الأربعة المحفوظة مع الثلاثة الآحاد، يكون خمسة وعشرين، وبطرحها ثمانية ثمانية. يبقى واحد، وهو الجواب.

وأمّا طرح سبعة، فإنّه[3] يبقى من كلّ عشرة ثلاثة، ومن كلّ مائة اثنان، ومن كلّ ألف ستة، ومن كلّ عشرة آلاف أربعة، ومن كلّ مائة ألف خمسة، ومن كلّ ألف ألف واحد، ومن ثمّ يعود الدور. فتختبره بهذه الحروف[4] أ جـ ب و د ه مُكرّرة تحت المنازل.

الألف واحد والجيم ثلاثة والباء اثنان والواو ستة والدّال أربعة والهاء خمسة. (وقد نظّم ذلك بعضهم في بيت، فقال :

ثلاث واثنتان وست وأربع * وخمس وفرد ذاك طرح لسبعة)[5]

(فإن شئت وضعت الحروف كما ذكر وإن شئت وضعتها بحروف الغُبار)[6].

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[1] في (ب) و في (ت) : "وأما الطرح بالثمانية". وفي (ط) : " وطرح الثمانية".

[2] في (م) الجملة موجودة، وتليها فقرة ليست في محلها إذ تخص الطرح بالسبعة وبيت شعر مشهور، وهذه القرة غائبة فيما بعد.

وفي (س) وفي (ط) وفي (ت) : الجملة "لأنّ الباقي من كلّ عشرة اثنان" غائبة.

[3] في (ت) : سقطت "إنّه ".

[4] في (س) : "الأحرف".

[5] في (م) : غابت الفقرة " وقد نظّم ذلك بعضهم في بيت، فقال : ثلاث واثنتان وست وأربع * وخمس وفرد ذاك طرح لسّبعة ".

[6] في (س) : سقطت الجملة "وإن شئت ... غبار"، ويعتبرها ناسخ (ت) من إضافات الهواري.

وتضرب كلّ منزلة فيما تحتها من عدد لفظ الحروف[1] وتطرح بسبعة[2]. وتلقي بقيته فوقها. ثمّ تجمع ما في كلّ منزلة من الباقيات كالآحاد، وتطرحه سبعة سبعة.

مثاله : لو أردنا طرح خمسة وثلاثين وأربعمائة وستة وثمانين ألفا وسبعمائة ألف وثلاثة [13 و] وعشرين ألف ألف، فتضعها في سطر وتمدّ عليها خطا وتضع الحروف تحتها، كلّ حرف تحت عدد منها على التوالي. فإن فنيت الحروف ولم يفنى العدد فتكرّر الحروف بقدر الباقي. وذلك معنى قوله: ("ومن ثم يعود الدور" أي بعد الهاء مع بقاء العدد)[3] يعود تكرار الحروف بقدر الباقي منه. وكذلك أبدًا على هذه الصورة:

2	3	7	8	6	4	3	5
ج	أ	ه	د	و	ب	ج	أ

فنضرب الخمسة في عدد الألف الذي تحتها بخمسة، نثبتها فوق الخطّ على الخمسة. ثمّ نضرب أيضًا الثلاثة في عدد الجيم الذي تحتها بتسعة. نطرحها بسبعة، الباقي اثنان. نثبتها فوق الخط على الثلاثة. ثمّ نضرب الأربعة في عدد الباء بثمانية، يبقى منها واحد. نثبته على الأربعة. ثمّ نضرب الستة في عدد الواو بستة وثلاثين، يبقى منها واحد. نثبته على الستة. ثمّ نضرب الثمانية في عدد[4] الدالّ باثنين وثلاثين، يبقى منها أربعة. نثبتها على الثمانية. ثمّ نضرب السبعة في الهاء بخمسة وثلاثين، تفنى. فنثبت صفرا على السبعة. ثمّ نضرب الثلاثة في الألف المُكرّرة أيضًا بثلاثة. لا تنطرح. فنثبتها على الثلاثة. ثمّ نضرب الاثنين في الجيم بستة، لا تنطرح أيضًا. نثبتها على الاثنين. وصورته :

6	3	0	4	1	1	2	5
2	3	7	8	6	4	3	5
ج	أ	ه	د	و	ب	ج	أ

[1] في (س) : "عدد الحرف".
[2] في (أ) وفي (ط) : "سبعة سبعة".
[3] في (أ) : سقطت الجملة "ومن ثمّ يعود الدور ... بقاء العدد".
[4] في (م) وفي (ط) : سقط لفظ "عدد".

فنجد الباقيات : خمسة واثنين وواحد وواحد وأربعة وثلاثة وستة. فنجمعها كذلك[1] كالآحاد، |13 ظ| ونطرح بسبعة. فما بقي، فهو الجواب، وذلك واحد.

وإن شئت، فاضرب ما في المنزلة الأخيرة في ثلاثة[2]، وتطرحه سبعة سبعة، وتحمل البقاي على ما قبله.[3] وإن لم يكن في المنزلة التي قبله عدد فتضرب البقية المحمولة في ثلاثة، وتطرحه بسبعة. وافعل كذلك حتى تنتهي إلى الآحاد.

مثاله : أردنا طرح أربعة وستين وثمانية وخمسين ألفا، فننزلها في سطر هكذا : 58064، ثمّ نضرب ما في المنزلة[4] الأخيرة وذلك خمسة في ثلاثة بخمسة عشر. نطرحها بسبعة. الباقي واحد. نحمله على الثمانية التي قبل بتسعة. نضربها في ثلاثة بسبعة وعشرين. الباقي منها ستة. وليس في المنزلة التي قبل شيء، فكأنّ هذه الستة المجتمع من البقية ومن المنزلة لو كان فيها شيء. فنضربها في ثلاثة بثمانية عشر. الباقي منها أربعة. نحملها على الستة التي قبلها أيضًا بعشرة. نضربها في ثلاثة بثلاثين. الباقي منها اثنان. نحملها على الأربعة التي في منزلة الآحاد بستة. وقد تمّ العمل. فتكون الستة هي الجواب. فاعلمه.

وإن شئت، فاجعل المنزلة[5] الأخيرة عشرات وتضيف إليها ما قبلها بآحاد، وتطرح بسبعة. ثمّ تجعل الباقي عشرات وتضيف إليها ما قبلها بآحاد، وتطرح كذلك.

مثاله : في مثالنا المُتقدّم، نجعل الخمسة الأخيرة عشرات ونضيف إليها الثمانية التي قبلها بآحاد بثمانية وخمسين، ونطرح بسبعة. الباقي إثنان. نجعلها عشرات ونضيف إليها الصفر الذي قبلها، إذ ليس ثمّ عدد، بعشرين. |14 و| الباقي منها ستة. نجعلها عشرات ونضيف إليها الستة التي قبلها بستة وستين. الباقي منها ثلاثة. نجعلها عشرات ونضيف إليها الأربعة التي قبلها بأربعة وثلاثين. الباقي منها ستة، وهو الجواب.

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[1] في (م) وفي (ط) : سقط لفظ "كذلك".
[2] في (أ) : زيادة "في ثلاثة، لأن الباقي من كل عشرة ثلاثة".
[3] في (ط) وفي (ت) : زيادة "وتضرب أيضًا في ثلاثة. وتلقي بقيته فوقها. ثمّ تجمع ما في كلّ منزلة من الباقيات كالآحاد وتطرحه سبعة سبعة ، وتحمل الباقي على ما قبله".
[4] في (م) : سقط لفظ "المنزلة".
[5] في (أ) وفي (ب) وفي (ط) وفي (ت) : "المرتبة".

فصل في وجه الامتحان بهذه الطروح

أمّا الجمع، فتطرح كلّ سطر منه، وتجمع الباقي منها، وتطرحه. فما بقي فهو الجواب. فتطرح المجتمع (في المسألة)[1]، يوافق الجواب.

مثاله : جمعنا ثلاثة وأربعين، (وصورتها: 43)[2] إلى أربعة وستين، (وصورتها: 64)[3]. فاجتمع[4] سبعة ومائة، وصورتها : 107.

قلنا أن نختبر هذه المسألة وغيرها من المسائل بطرح تسعة، أو ثمانية، أو سبعة. لكن الجمهور كثيرا ما يستعملون طرح سبعة.
فكأنا أرادنا أن نختبر به المسألة المذكورة وكذلك سائر الأمثلة التي تأتي، فنجمع الواحد الباقي من المجموع (إلى الواحد الباقي من المجموع إليه)[5] بإثنين، وهو الجواب. ولو كان هذا المُجتمع من الباقيات مما يمكن أن يطرح لطرحناه أيضًا وباقيه هو الجواب. ثمّ نطرح المجتمع. يبقى منه إثنان، مُوافقة للجواب. فاعلمه.

وأمّا الطرح، فتطرح المطروح منه وتحفظ الباقي، ثمّ تطرح المطروح وتسقط بقيته من المحفوظ. وإن كان أقلّ فزد عليه الطرح وأسقط [6] من المجتمع، يبقى الجواب. فاطرح الباقي من المسألة، يوافق الجواب.

أو تجمع بقية المطروح إلى بقية الباقي، يوافق بقية المطروح منه.

مثاله : طرحنا أربعة وسبعين، (وصورتها: 74)[7]، من ستة وتسعين، (وصورتها: 96)[8]، | 14 ظ | فيبقى اثنان وعشرون، (وصورتها: 22)[9]. فإذا

[1] في (أ) وفي (ب) وفي (ط): سقطت العبارة : "في المسألة".
[2] في (أ) وفي (ط) وفي (ت) : لا إشارة إلى الصورة.
[3] في (أ) وفي (ط) وفي (ت) : لا إشارة إلى الصورة.
[4] في (أ) وفي (ط) وفي (ت) : "يكون الجواب".
[5] في (أ) : "والمجموع اليه".
[6] في (أ) وفي (ب) : "وأسقطه".
[7] في (أ) وفي (ط) وفي (ت) : لا إشارة إلى الصورة.
[8] في (أ) وفي (ط) وفي (ت) : لا إشارة إلى الصورة.
[9] في (أ) وفي (ط) وفي (ت) : لا إشارة إلى الصورة.

أردنا اختبارها، نحفظ الخمسة الباقية من المطروح منه، ثمّ نطرح المطروح، تبقى منه أربعة. نسقطها من الخمسة المحفوظة، يبقى واحد، وهو الجواب.

ولو كانت الأربعة الباقية من المطروح أكثر من الخمسة الباقية من المطروح منه لزدنا على الخمسة، العدد المطروح به، إن كانت[1] تسعة فتسعة، أو ثمانية فثمانية[2]، أو سبعة[3] فسبعة. ثمّ نطرح الباقي. يبقى منه واحد، موافق للجواب.
ولو جمعنا الأربعة، بقية المطروح، إلى الواحد، بقية الباقي، لكانت مثل الخمسة، بقية المطروح منه. فاعلمه.

وأمّا الضرب، فتطرح المضروبين وتضرب باقي أحدهما في باقي الآخر، وتطرح[4]. فما بقي فهو الجواب. فتطرح خارج الضرب، يوافق الجواب.

مثاله : ضربنا اثني عشر، (وصورتها: 12)[5] ، في ستة عشر، (وصورتها: 16)[6]. الخارج من[7] الضرب اثنان وتسعون ومائة، وصورتها[8] : 192.
فنضرب الخمسة الباقية من المضروب في الاثنين الباقية من المضروب فيه بعشرة. يبقى منه ثلاثة، وهو الجواب. ثمّ نطرح الخارج من[9] الضرب، يبقى منه ثلاثة، مثل الجواب.

وهذا عام في الصحيح والكسور بعد بسطها [أي تصير المسألة كلها من نوع كسر واحد][10].

[مثاله : لو قيل اضرب ثلثا في أربعة عشر وربع، يخرج من[11] الضرب أربعة وثلاثة أرباع، (على ما يأتي في عمل الكُسور، إن شاء الله تعالى)[12].

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[1] في (م) وفي (ط) : سقط لفظ "كانت".
[2] في (م) وفي (ط) : سقط لفظ "فثمانية".
[3] في (م) وفي (ط) : سقط لفظ "سبعة".
[4] في (ط) : سقط لفظ "تطرح".
[5] في (أ) وفي (ط) وفي (ت) : لا إشارة إلى الصورة.
[6] في (أ) وفي (ط) وفي (ت) : لا إشارة إلى الصورة.
[7] في (م) : "في".
[8] في (م) : "193". وهذا خطأ.
[9] في (م) : "في".
[10] في (س) : سقطت الجملة : "أي تصير المسألة كلها من نوع كسر واحد"، ويعتبرها ناسخ (ت) من إضافات الهواري. وهي في الحقيقة منقولة من **رفع الحجاب**، صفحة 248: 11-12.
[11] في (م) : "في".
[12] في **رفع الحجاب** : سقطت الجملة : "على ما يأتي في عمل الكُسور، إن شاء الله تعالى".

فإذا أردنا اختباره، ضربنا الثلث، وهو الباقي من أحد المضروبين، في الربع، الباقي من المضروب الثاني، يخرج ثلث ربع. ويبقى منه بعد بسطه واحد، وهو ثلث ربع، وهو[1] |15 و| الجواب. ثمّ نبسط خارج المضروب، يكون تسعة عشر ربعا. يبقى منه خمسة أرباع. نبسطها أثلاثا بضربها في ثلاثة بخمسة عشر. يبقى منها واحد، وهو ثلث ربع، مساو للجواب في كميته وكيفيته][2].

وأمّا القسمة والتسمية، فتطرح الخارج والمقسوم عليه أو المُسمّى منه وتضرب باقي أحدهما في باقي الآخر وتطرحه[3]. فما بقي فهو الجواب. فتطرح المقسوم أو المُسمّى، يوافق الجواب.
وهذا العمل أيضًا[4] عام في الصحيح والكسور بعد بسطها.

مثاله في القسمة : قسمنا ثمانية وثمانين وأربعمائة وألفا (وصورتها : 1488)[5]، على اثني عشر. يخرج من القسمة أربعة وعشرون ومائة، (وصورتها : 124)[6]. فإذا أردنا اختباره ضربنا الخمسة الباقية من الخارج في الخمسة الباقية من المقسوم عليه بخمسة وعشرين. يبقى منه أربعة، وهو الجواب. ثمّ نطرح المقسوم. يبقى منه أربعة، موافقة للجواب.

ومثاله بالكسر : [ولو قيل لك[7] اقسم خمسة أسداس وثلاثة أرباع على نصف، لخرج من القسمة ثلاثة وسدس. ويبقى منه بعد بسطه خمسة أسداس، نضربه في واحد، الذي هو بسط النصف، المقسوم عليه، يكون خمسة أنصاف سدس. فنضربها في اثنين لتصير أرباع سدس، موافقة في البسط لبسط المقسوم. يكون عشرة. ويبقى منها ثلاثة أرباع سدس، وهو الجواب. ثمّ نطرح بسط المقسوم، وهو ثمانية وثلاثون ربع سدس. يبقى منها ثلاثة أرباع سدس، وهو مثل الجواب][8].

|15 ظ|

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[1] في (أ) وفي (ب) : "وذلك".
[2] الفقرة : "مثاله: لو قيل اضرب ثلثاً ... في كيفيته وكميته" منقولة من **رفع الحجاب**، صفحة 248: 13-17.
[3] في (م) : سقط لفظ "وتطرحه".
[4] في (س) وفي (أ) وفي (ط) : سقط لفظ "أيضًا".
[5] في (ط) وفي (ت) : لا إشارة إلى الصورة.
[6] في (ط) وفي (ت) : لا إشارة إلى الصورة.
[7] في (أ) : سقط لفظ "لك".
[8] الفقرة : "لو قيل اقسم خمسة ... وهو مثل الجواب" منقولة من **رفع الحجاب**، صفحة 248: 18 إلى 249: 4.

ومثال التسمية : لو قيل سمّ أحد عشر من خمسة عشر. يخرج من التسمية ثلاثة أخماس وثلثا خمس. ويبقى منه بعد بسطه أربعة، نضربها في الواحد الباقي من المُسمّى منه، بأربعة، وهي الجواب. ثمّ نطرح المُسمّى، يبقى منه أربعة، مثل الجواب. [[1]]

ومثاله بالكسر : [لو قيل لنا[2] سمّ سدسين وثلثي سدس من خمسة أثمان وثلث ثمن. تخرج لنا ثلثان. والباقي من بسطها اثنان. نضربها في الاثنين الباقيين من بسط المُسمّى منه، يكون أربعة أثلاث ثلث ثمن. ثمّ في الستة[3]، تكون أربعة وعشرين ثلث ثلث سدس ثمن، ويبقى منه ثلاثة، وهو الجواب. ثمّ نطرح بسط المسمى يبقى منه واحد، نضربه في الثلاثة ثمّ في الثمانية، يكون أربعة وعشرين ثلث ثلث سدس ثمن، ويبقى منه ثلاثة، مثل الجواب.
فلا بد أن ترجع المسألة كلّها إلى أدق[4] كسر فيها، وهو الجزء المُسمّى من جميع أئمّتها. وهو معنى البسط على ما يأتي بيانه، بحول الله تعالى][5].

[1] في (ط) : زيادة في حاشية الورقة (20ظ : 3) : "قلت، وهذا لا يظهر به وجه الاختبار الا إذا كان باقي المسمى منه واحدا، كفي هذا المثال والا فلا. ألا ترى أنك لو اختبرت هذا المثال بطرح تسعة لكان باقي الخارج بعد بَسْطِه اثنين، وبان المسمى منه ستة، والباقي بعد ضرب أحدها في الآخر وطرح الحاصل بالتسعة ثلاثة، وليست موافقة لبقية المسمى لانه اثنان فقط. فتأمله. بن ط". (يحتمل أن "بن ط" يرجع إلى علي بن الطرابسي الذي راجع النص باكمله.)

[2] في (م) وفي (أ) : سقط لفظ "لك".

[3] في (م) وفي (ت) : "الثمانية"، وهو خطأ. أما في **رفع الحجاب** وفي (أ) : "السدس" وهو مقبول.

[4] في (أ) وفي (ط) وفي (ت) : "أدنى".

[5] الفقرة : "لو قيل سمّ سُدسين ... بحول الله تعالى" منقولة من **رفع الحجاب**، صفحة 249: 5-11.

الباب الرابع في الضرب وتقريب مُلَحِه

الضرب عبارة عن تضعيف أحد العددين بقدر ما في الثاني من الآحاد.

[وهذا الباب قسمان :
قسم ينزل[1] المضروب فيه، كلّ واحد منه مثل الواحد من المضروب. فيكون التضعيف فيه ظاهرا في اللفظ والمعنى.
[والقسم الثاني، يكون جميع ما في المضروب فيه من الآحاد مساويا للواحد من المضروب. فتكون آحاد المضروب فيه، هي عدد ما في واحد المضروب من الأجزاء. وهذا القسم يُسمّى بالصرف. والتضعيف فيه إنّما هو باللفظ دون المعنى][2].
[فإذا قيل ثلاثة [16 و] رجال، لكلّ واحد منهم خمسة دراهم. فتضرب خمسة في ثلاثة بخمسة عشر درهمًا. وهو تضعيف في اللفظ وفي المعنى.
وإذا قيل خمسة دراهم، كم ثلثا فيها؟ فتضرب خمسة في ثلاثة بخمسة عشر ثلثا. فهذا تضعيف في اللفظ خاصة وأمّا المعنى فإنّ خمسة عشر ثلثا هي الخمسة بعينها][3].

وهو ينقسم على[4] ثلاثة أضرب[5]. (وذلك بالنسبة إلى العمل)[6]. الأوّل الضرب بالتنقيل، والثاني بنصف تنقيل والثالث بغير تنقيل.

فالضرب الأوّل، وهو الضرب بالتنقيل، هو الممحو المُسمّى بالنائم. وهو أن تضع المضروب والمضروب فيه في سطرين وتكون أوّل مرتبة من المضروب فيه تحت آخر مرتبة من المضروب. ثمّ تضربها في جميع مراتب المضروب فيه وتبدأ بكتابة الخارج من هناك[7]، مارًا على السَّطر، مُتّصلا بسطر المضروب. ثمّ تنقل العدد المضروب فيه على وضعه تحت المنزلة التي تلي تلك قبلها. ثمّ تضربها في جميع منازل الأسفل، على المثال الأوّل، (وكلّما

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[1] في (أ) وفي (ب) : "يكون".
[2] في (أ) : سقطت كل الفقرة: "والقسم الثاني ... دون المعنى". وفي (ط) : "ينزل."
[3] الفقرة : "وهذا الباب قسمان ... بعينها" منقولة من **رفع الحجاب**، صفحة 255: 3-12.
[4] في (أ) سقط لفظ : "على".
[5] موجود في (م) و(س) وغائب في باقي النسخ.
[6] في (س) سقطت الجملة : "وذلك بالنسبة إلى العمل"، ويعتبرها ناسخ (ت) من إضافات الهواري.
[7] في (ب) : "هنالك".

ضربت في عدد جمعت)[1] الخارج مع ما على رأس ذلك العدد من الخارج قبل وتضعه كما يجب. وهذا العمل عام في جميع مسائل الضرب.

مثاله: لو[2] أردنا ضرب ثلاثة وأربعين في أربعة وخمسين. فنضع الثلاثة وأربعين، المضروبة، في سطر، والأربعة والخمسين، المضروبة فيها، في سطر آخر. وتكون أوّل منزلة[3] من المضروب فيه تحت آخر منزلة[4] من المضروب، كما ذكر، على هذه الصورة:

```
4  3
5  4
```

.

فنضرب [16 ظ] ما في آخر (منزلة من)[5] المضروب، وهي الأربعة، في الخمسة، التي هي آخر منزلة من المضروب فيه، بعشرين. فنثبت صفرا فوق الخمسة، والعشرين باثنين بعد الصفر. ثمّ نضربها أيضًا في الأربعة التي تحتها بستة عشر. نثبت الستة مكانها، والعشرة بواحد مكان الصفر، وكلّ ذلك مُتّصل بسطر المضروب. ثمّ نقهقر المضروب فيه منزلة، وذلك تحت الثلاثة الأربعة[6]، (والخمسة تحت الستّة)[7] (على هذه الصورة)[8]:

```
_________
2  1  6  3
      5  4
```

ثمّ نضرب الثلاثة المقهقر تحتها في جميع[9] المضروب فيه أيضًا ونجمع ذلك مع ما فوقه، نضربها أولا في الخمسة بخمسة عشر. فنضيفها إلى الستة عشر التي فوقها بأحد وثلاثين. فنثبت الواحد مكان الستة والثلاثين بثلاثة مكان العشرة. ثمّ نضربها أيضًا في الأربعة باثني عشر. نثبت الاثنين مكانها ونضيف العشرة بواحد إلى الواحد الذي على المرتبة الثانية باثنين. نثبتها مكانها. وقد تمّ العمل. فيكون الخارج اثنين وعشرين وثلاثمائة وألفين، وصورته : 2322.

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[1] في (أ) : " كما ضربت في عدد جميع ".
[2] في (م) وفي (ط) : سقط لفظ "لو". ونلاحظ أن ليس لناسخي المخطوط منهجا واضحا، تارة يستعملون "لو" وتارة لا. إنا سنضع "لو" في كل الحالات القادمة.
[3] في (أ) وفي (ط) وفي (ت) : "مرتبة".
[4] في (أ) وفي (ط) وفي (ت) : "مرتبة".
[5] في (أ) وفي (ط) : "مرتبة". وفي (ت) : سقط اللفظ "مرتبة".
[6] في (م): إعادة " الثلاثة تحت الاربعة" للتأكيد.
[7] في (أ) : "والاربعة تحت الخمسة". وفي (ت) : "وتحت الاربعة الخمسة". والنسختان خاطئتان.
[8] في (م) وفي (ب) وفي (ط) : الصورة غير موجود.
[9] في (ب) : سقط لفظ "جميع".

ومنه نوع آخر يعرف بالقائم، وهو أن تجعل سطري المضروبين[1] قائمين وتكون أوّل مرتبة من المضروب فيه بإزاء آخر مرتبة من المضروب. وتصنع في ضربهما[2] كما صنعت بالنائم من نقل ومحو.

مثاله : أردنا ضرب اثنين وأربعين في سبعة وثلاثين. فننزل المضروب في سطر قائم، كما ذكر، والمضروب فيه[3] كذلك. وتكون أوّل (منزلة منه موازية لآخر منزلة من المضروب)[4]، على هذه الصورة: [17 و]

	2
7	4
3	

ثمّ نضرب آخر منزلة من المضروب، وهي أربعة، في جميع منازل المضروب فيه، كما تقدّم. نضربها أوّلا في الثلاثة باثني عشر، ننزل الاثنين بإزاء الثلاثة متصلة بسطر المضروب، كما تقدّم، والعشرة بواحد تحت الاثنين. ثمّ نضربها أيضًا في السبعة، الموازية لها، بثمانية وعشرين. نثبت الثمانية مكانها ونضيف العشرين باثنين إلى الاثنين، التي تحت تلك المرتبة بأربعة. ننزلها مكانها. ثمّ نقهقر المضروب فيه منزلة، السبعة موازية للاثنين، والثلاثة موازية للثمانية، على هذه الصورة :

7	2
3	8
	4
	1

فنضرب أيضًا الاثنين في جميع المضروب فيه، كما تقدّم. نضربها أوّلا[5] في الثلاثة بستة. نضيفها للثمانية الموازية لها بأربعة عشر. فنثبت الأربعة مكان الثمانية، والعشرة بواحد، نضيفها إلى الأربعة التي تحت[6] تلك المرتبة بخمسة. نثبتها مكانها. (ثمّ نضرب الاثنين أيضًا في السبعة بأربعة عشر. فنثبت الأربعة

[1] في (س) : "الضرب" وفي (أ) وفي (ط) وفي (ت) : "الضريبين".
[2] في (أ) و في (ب) : "ضربه"، وفي (ط) وفي (ت): "ضربها".
[3] في (ب) : سقط لفظ "فيه"، وهو خطأ.
[4] في (م) : "من المضروب فيه" لفض "فيه" زائد، وفي (أ) : "مرتبة من المضروب فيه بئزاء آخر مرتبة من المضروب"، وفي (ب) وفي (ط) وفي (ت) : مرتبة من المضروب فيه موازية لآخر مرتبة من المضروب".
[5] في (أ) : ناقصة.
[6] في (أ) وفي (ط) : "في".

مكانها والعشرة بواحد. نضيفها إلى الأربعة التي تحت[1] تلك المرتبة بخمسة. نثبتها مكانها)[2]. وقد تمّ العمل. فيكون الخارج أربعة وخمسين وخمسمائة وألفا، وصورتها : 1554.

(والضرب الثاني هو)[3] الضرب بنصف تنقيل. ولا يتصور إلّا في العددين المتماثلين. (وصورته [17 ظ] أن تضع أحد العددين المتماثلين)[4] في سطر، وتجعل بين مراتبه علامات بنقط. ثمّ تضرب آخر منزلة في نفسها وتثبت الخارج فوقها. ثمّ تضعّفها وتنقلها في موضع العلامة التي[5] قبلها. ثمّ تضرب ما في المنزلة التي قبلها في المنقول وفي نفسه، وترسم ما خرج من كلّ مضروب على رأسه. ثمّ تضعّف تلك المرتبة التي ضربت كما فعلت أوّلا. ثمّ تنقله في موضع العلامة التي قبله. ثمّ تنقل المضاعف أوّلا على حسبه.
ثمّ تضرب ما في المنزلة التي قبل العلامة المنقول في موضعها في جميع المضاعف، ثمّ في نفسه، كما فعلت أوّلا.
ثمّ لا تزال تفعل[6] كذلك من التضعيف والنقل والضرب حتى تأتي على جميع السّطر.

مثال ذلك : لو أردنا ضرب ثلاثة وستين وأربعمائة في مثلها. فننزلها في سطر ونفرّق بين الأعداد بنقط، كما ذكر، على هذه الصورة : 3 ∴ 6 ∴ 4.
فنضرب الأربعة الأخيرة في نفسها بستة عشر. نثبت الستة فوق الأربعة، والعشرة بواحد بعد الستة. ثمّ نضعّفها بثمانية وننقلها في موضع العلامة التي قبلها. فتكون على هذه الصورة :

```
      6
1  4  ∴  6  ∴  3
         8
```

ثمّ نضرب الستة المقهقرة بإزائها في الثمانية المنقولة بثمانية وأربعين. نثبت الثمانية فوق الثمانية المضروب فيها ونضيف الأربعين بأربعة إلى الستة التي بعد تلك المرتبة بعشرة. فنثبت صفرا مكانها ونضيف العشرة بواحد [18 و]

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[1] في (أ) : "في".
[2] في (م) : إعادة الجملة "ثمّ نضرب ... مكانها" خطأ.
[3] في (أ) وفي (ط) : "والثاني وهو".
[4] في (أ) : "تضعه". وفي (ط) : "وصورته أن تضعه"
[5] في (م) : "التي ما".
[6] في (ت) : سقط لفظ "تفعل".

إلى الواحد الذي بعد تلك المرتبة باثنين. نثبتها مكانها، ثمّ نضرب أيضًا الستة في نفسها بستة وثلاثين. نثبت الستة الخارجة فوقها ونضيف الثلاثين بثلاثة إلى الثمانية التي بعد تلك المرتبة بأحد عشر. نثبت الواحد، مكان الثمانية، والعشرة بواحد، مكان الصفر، ونقهقر الستة أيضًا مضاعفة باثني عشر. نثبت الاثنين موضع العلامة التي قبل، والثمانية نضيفها إلى العشرة[1] الحادثة من التضعيف بواحد. تكون تسعة. ننزلها موضع الستة، على هذه الصورة :

```
2 1   1 6
  4  ∴ 6 ∴  3
        8 9  2
```

ثمّ نضرب أيضًا الثلاثة المقهقرة بإزائها في جميع المنقول وفي نفسها، كما تقدّم. نضربها أوّلا في التسعة بسبعة وعشرين. نحمل عليها الستة عشر التي فوقها بثلاثة وأربعين. نثبت الثلاثة، مكان الستة (التي فوقها بثلاثة)[2] والأربعين بأربعة، مكان العشرة. ثمّ نضربها أيضًا في الاثنين بستة. نثبتها فوقها. ثمّ نضربها أيضًا في نفسها بتسعة. نثبتها فوقها. وقد تمّ العمل.
فيكون الخارج تسعة وستين وثلاثمائة وأربعة عشر ألفا ومائتي ألف، وصورته: 214369. فافهم وقس (عليه ما أشبهها من المسائل تصب بحول الله تعالى)[3].

والضرب الثالث، هو الضرب بغير تنقيل ويتنوع أنواعًا كثيرةً.

فمنها الضرب بالجدول، وصورته أن تعمل سطحًا مُربعًا وتجدوله طولًا وعرضًا بقدر ما | 18 ظ | **في العددين المضروبين من المنازل، وتقطر مُربّعاته بأقطار واحدة[4] من الميمنة السّفلى إلى الميسرة العليا، وتضع المضروب على رأس المُربّع، وتُقابل بكل منزلة منه جدولا. ثمّ تضع المضروب فيه (عن يسار المُربّع أو عن يمينه)[5]، هابطا معه، وتقابل بكلّ منزلة منه جدولا أيضًا. ثمّ تضرب منزلة بعد منزلة من المضروب (في جميع منازل المضروب فيه)[6] وتجعل الخارج لكلّ منزلة في المُربّع الذي يتقاطعان عليه، (ومعنى التقاطع**

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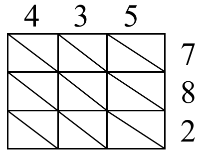
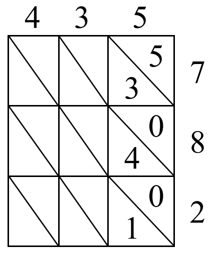
[1] في (م) : "الثمانية"، وهو خطأ.
[2] في (أ) وفي (ب) وفي (ط) وفي (ت) : سقطت الجملة "التي ... بثلاثة".
[3] في (أ) وفي (ط) وفي (ت) : "عليه ما أشبهه". وفي (ب) سقطت الجملة.
[4] في (س) وفي (ب) : "آخذة".
[5] في (م) : إعادة الجملة "على يسار ... عن يمينه" زائدة.
[6] في (م) : سقطت "في جميع منازل المضروب فيه".

حيث يلتقيان. تجعل الآحاد فوق القطر)[1] والعشرات تحته. ثمّ تبدأ بالجمع من الركن الأيمن الأعلى. فتجمع ما بين الأقطار بلا محو، وتضع كلّ عدد في مرتبته، وتحمل عشرات كلّ مجموع إلى القطر الذي بعده، تؤلفها بالجمع مع ما فيه، فما اجتمع لك فهو الخارج.

مثاله : لو أردنا ضرب خمسة وثلاثين وأربعمائة في سبعة وثمانين ومائتين. فنضع المُربّع كما ذكره (والمضروب فيه على يسار المُربّع أو عن يمينه)[2] والمضروب فوق المربّع، (كل عدد منه فوق جدول من المربّع، وكأنه وضعنا المضروب فيه عن يمين المربّع)[3] كذلك أيضًا، على هذه الصورة:

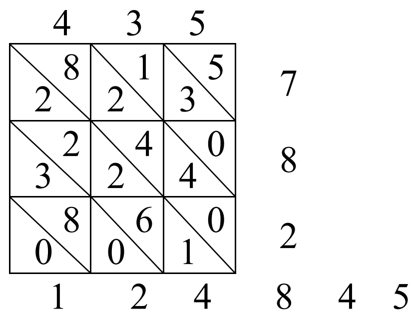


فنضرب الخمسة في السبعة بخمسة وثلاثين. ننزل الخمسة فوق قطر المُربّع الذي تلاقيا فيه الضربان والثلاثين بثلاثة تحته.
ثمّ نضرب[4] أيضًا الخمسة [19 و] في الثمانية بأربعين. ننزل الصفر فوق قطر المُربّع الذي تلاقيا فيه والأربعين بأربعة تحته.
ثمّ نضرب الخمسة أيضًا في الاثنين بعشرة. ننزل الصفر فوق قطر المُربّع الذي تلاقيا فيه والعشرة بواحد تحته، على هذه الصورة[5] :



[1] في (س) : "تجعل الآحاد فوق القطر". وفي (أ) : "بحيث تجعل الآحاد فوق القطر". وفي (ط) وفي (ت) : "ومعنى التقاطع أن يلتقيان تحت الاحاد فوق القطع".
[2] في (م) : سقطت الجملة "والمضروب فيه ... عن يمينه".
[3] في (أ) وفي (ط) : سقطت الجملة "وكل عدد ... عن يمين المربّع".
[4] في (س) وفي (أ) : سقط لفظ "نضرب".
[5] في (أ) وفي (ط) وفي (ت) : غابت الإشارة إلى الصورة وكذلك الصورة.

ثمّ نفعل كذلك بالثلاثة، نضربها في جميع منازل المضروب فيه وننزل الخارج من كلّ واحد منهما في المُربّع الذي تلاقيا فيه، كما تقدّم. ونفعل كذلك بالأربعة الباقية من المضروب. ونتمّ الضرب. فتكون الأعداد الواقعة في المُربّع بجملتها على هذه الصورة :



ثم نبدأ بالجمع. فنرفع الخمسة التي فوق القطر الأوّل، وهو الرّكن الأيمن الأعلى كما ذُكر. ثمّ نجمع أيضًا ما بين القطر الأوّل والثاني، وذلك ثلاثة وواحد، يكون المجموع أربعة. نرفعها بعد الخمسة المرفوعة أوّلا، ثمّ نجمع أيضًا ما بين القطر الثاني والثالث، وذلك أربعة، وأربعة واثنان وثمانية. فيكون الجميع ثمانية عشر. فنرفع الثمانية بعد الأربعة المرفوعة ونجمع العشرة بواحد مع[1] ما بين القطر الثالث والرابع، وذلك واحد وستة واثنان واثنان واثنان، فيكون المجموع أربعة عشر. فنرفع [19 ظ] الأربعة بعد الثمانية المرفوعة. ونجمع العشرة بواحد مع[2] ما بين القطر الرابع والخامس، وذلك ثمانية وثلاثة باثني عشر. نرفع الاثنين بعد الأربعة المرفوعة والعشرة (بواحد بعدها)[3]، وقد تمّ العمل.
فيكون مجموع ذلك كلّه هو المطلوب. وذلك خمسة وأربعون و ثمانمائة وأربعة وعشرون ألفا ومائة ألف. وصورتها : 124845.

ومنها الضرب بالقائم، وهو أن تخطّ خطّين قائمين بينهما فسحة وترسم المضروبين على جنبيهما، ثم تضرب مرتبة بعد مرتبة من أحدهما في جميع مراتب الآخر، وتجعل الخارج في الفسحة بين الخطين حيث توجبه مرتبة الأسوس.

---

[1] في (أ) : سقط لفظ: "مع".
[2] في (أ) : سقط لفظ: "مع".
[3] في (أ) وفي (ط) وفي (ت) : " بعدها بواحد ".

مثاله : أردنا ضرب ثلاثة وثمانين ومائة في سبعة وأربعين وثلاثمائة.
فننزل المضروب في سطر قائم والمضروب فيه كذلك موازيا للمضروب. وبعد المضروب خط وقبل المضروب فيه خط آخر، كما ذكر، على هذه الصورة :

7	3
4	8
3	1

فنضرب الثلاثة التي هي أوّل المضروب في جميع مراتب[1] المضروب فيه : نضربها أوّلا في السبعة بأحد وعشرين. (ثم ننزل الواحد)[2] أمام الثلاثة بعد الخط، والعشرين باثنين تحته. ثمّ نضربها أيضًا في الأربعة باثني عشر. نضيف إليها الاثنين، الذي في المرتبة الثانية من الخارج، بأربعة عشر. ننزل الأربعة أمام الاثنين، والعشرة بواحد تحتها. ثمّ نضربها أيضًا في الثلاثة بتسعة، ونضيفها إلى الواحد، [20 و] الذي في المرتبة الثالثة من الخارج، بعشرة. ننزل الصّفر أمام الواحد، والعشرة بواحد تحته. وصورة ذلك :

7			1	3
4		4	2	8
3	0	1		1
	1			

ثمّ نضرب الثمانية أيضًا، التي هي ثاني مرتبة من المضروب، في جميع[3] منازل[4] المضروب فيه أيضًا[5]. نضربها أوّلا في السبعة بستة وخمسين. نضيفها إلى الأربعة، التي في المرتبة الثانية من الخارج، بستين. ننزل الصفر أمام الأربعة والستين بستة أمام الصّفر الذي بعد الثانية. ثمّ نضربها أيضًا في الأربعة باثنين وثلاثين، نضيفها إلى الستة[6]، التي هي ثالث مرتبة من الخارج، بثمانية وثلاثين. ننزل الثمانية أمامها. ونضيف الثلاثين بثلاثة إلى الواحد، الذي بعد تلك المرتبة، بأربعة[7]. ننزله أمامها. ثمّ نضربها أيضًا في الثلاثة بأربعة وعشرين.

---

[1] في (أ) وفي (ط) وفي (ت) : "منازل".
[2] في (أ) وفي (ط) وفي (ت) : "نثبت الواحد فوق"
[3] في (أ) : سقط لفظ "جميع".
[4] في (م) : سقط لفظ "منازل".
[5] في (م) : سقط لفظ "أيضًا".
[6] في (م) : "المرتبة".
[7] في (م) : "وصور ة ذلك"، وتليها صورة.

نضيفها إلى الأربعة، التي هي رابع مرتبة من الخارج، بثمانية وعشرين. ننزل الثمانية أمامها والعشرين باثنين بعد تلك المرتبة[1].
ثمّ نضربه (أيضًا في)[2] الواحد، الذي هو ثالث مرتبة من المضروب، في جميع المضروب فيه أيضًا: نضربه أوّلا في السبعة بسبعة. نضيفها إلى الثمانية، التي هي ثالث مرتبة من الخارج، بخمسة عشر. نثبت الخمسة أمامها ونضيف العشرة بواحد إلى الثمانية، التي هي بعد تلك المرتبة، بتسعة. ننزلها أمامها. ثمّ نضربها (أيضًا في)[3] الأربعة بأربعة. نضيفها إلى التسعة، التي هي رابع مرتبة من الخارج، بثلاثة عشر. ننزل الثلاثة أمامها، ونضيف العشرة بواحد إلى الاثنين، التي بعد تلك المرتبة، بثلاثة. ننزلها أمامها. ثمّ نضربه أيضًا في الثلاثة بثلاثة، نضيفها إلى الثلاثة، التي هي خامس مرتبة [20 ظ] من الخارج، بستة. ننزلها أمامها. وقد تمّ العمل. وصورة ذلك[4] :

7							1	3
4					0	4	2	8
3		5	8	6	0	1		1
	3	9	8	4	1			
	6	3	2					

فيكون الخارج (واحدا وخمسمائة وثلاثة وستين ألفا)[5]. وهو السّطر[6] القائم الذي يلي المضروب فيه. فاعلمه.

ومنها الضرب بالنائم، وهو أن تجعل المضروبين في سطرين متوازيين، ثمّ تضرب كلّ منزلة[7] من أحدهما في (كل منزلة)[8] الآخر (وتجعل الخارج حيث توجبه مرتبة الأسوس.
وتبدأ بالضرب من أوّل المنازل أو من آخرها.)[9] ويُسمّى هذا النّوع الضرب بالأس.

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[1] في (م) : سقط لفظ "منازل".
[2] في (م) : سقط "أيضًا في".
[3] في (م) : سقط "أيضًا في".
[4] في (ط) وفي (ت) : صورة العملية غائبة.
[5] في (أ) وفي (ط) وفي (ت) : "63501".
[6] في (ط) وفي (ت) : "أصل السطر".
[7] في (ت) : "مرتبة".
[8] في (ت) : "جميع مراتب".
[9] في (ط) : سقطت الجملة "وتجعل الخارج حيث توجبه مرتبة الأسوس، وتبدأ بالضرب من أوّل المنازل أو من آخرها.".

قلت نحتاج أن نقدّم هنا مقدمة لهذا النوع، وبها[1] يعمل أيضًا في الضرب بالرُّومي. وهي أنّ الخارج من ضرب الآحاد آحاد لأنّه متى ضرب عدد في عدد. فإنّ أسّ الخارج مجموع أسّي العددين[2] المضروبين إلاّ واحدا، على ما بيّنه المُؤلف، رضي الله عنهُ، في **"رفع الحجاب"**، في باب الجمع منه. ولا شكّ أنّ أسّ المضروب واحد وأسّ المضروب فيه واحد، ومجموعهما اثنان، نسقط منهما واحدا، يبقى واحد. واسم الواحد على ما تقدّم آحاد. فالخارج إذًا من ضرب الآحاد في الآحاد آحاد، وكذلك أيضًا ضربها في العشرات عشرات وفي المئين مئون، وضرب العشرات في العشرات مئون وفي المئين آلاف، والمئين في المئين عشرات آلاف. فإذا، تمّت | 21 و | المقدّمة.

(فكأنا أردنا ضرب)[3] ثلاثة وخمسين ومائتين في سبعة وثمانين وتسعمائة. فننزلهما في سطرين متوازيين، كما ذُكر ، على هذه الصورة :

2	5	3
·9	8	7

فنضرب الثلاثة التي هي في منزلة[4] الآحاد من المضروب في جميع مراتب المضروب فيه: نضربها أوّلا في السبعة بأحد وعشرين. ننزل الواحد في مرتبة الآحاد، على ما تقدّم، والعشرين باثنين بعدها. ثمّ نضربها أيضًا في الثمانية بأربعة وعشرين. نضيف إليها الاثنين، التي في مرتبة[5] العشرات من الخارج، إذ هما من جنس واحد على ما تقدّم، بستة وعشرين. نثبت الستة فوق الاثنين والعشرين باثنين بعدها. ثمّ نضربها أيضًا في التسعة[6] بسبعة وعشرين، نضيف إليها الاثنين، التي في مرتبة المئين[7] من الخارج، إذ هما أيضًا[8] من جنس واحد، بتسعة وعشرين. نثبت التسعة فوق الاثنين، والعشرين باثنين بعدها. وهذه صورة ذلك :

[1] في (أ) وفي (ب) : "ورُبّما يفعل". وفي (ت) : "ورُبّما العمل"، وفي (ط) : "مقدمة: وبها العمل."
[2] في (م) : سقط "العددين". وفي (ط) : "أس الضربين".
[3] في (ب) وفي (ط) وفي (ت) : "فإنا إذا ضربنا".
[4] في (ت) : "مرتبة".
[5] في (أ) : "منزلة".
[6] في (م) : "سبعة"، وهو خطأ.
[7] في (أ) وفي (ت) : "العشرات"، وهو خطأ.
[8] في (أ) سقط لفظ "أيضا".

```
    9 6
2 2 2 1
-------
  2 5 3
  9 8 7
```

ثمّ نضرب أيضًا الخمسة، التي في مرتبة العشرات من (المضروب، في جميع مراتب)[1] المضروب فيه:
نضربها أوّلا في السبعة بخمسة وثلاثين، نضيف إليها الستة، التي في مرتبة العشرات من الخارج، إذ هما أيضًا من جنس واحد، بواحد وأربعين. نثبت الواحد فوق الستة ونضيف الأربعين بأربعة إلى التسعة، التي بعد تلك المرتبة، بثلاثة عشر. نثبت الثلاثة فوق التسعة ونضيف العشرة بواحد إلى الاثنين، التي بعد تلك المرتبة، بثلاثة. ننزلها فوق الاثنين. ثمّ نضربها [21 ظ] أيضًا في الثمانية بأربعين، نضيف إليها الثلاثة، التي في مرتبة المئين من الخارج، بثلاثة وأربعين. نثبت الثلاثة مكانها ونضيف الأربعين بأربعة إلى الثلاثة، التي بعد تلك المرتبة، بسبعة. فنثبتها فوق الثلاثة. ثمّ نضربها أيضًا في التسعة بخمسة وأربعين، نضيف[2] إليها السبعة، التي في مرتبة الآلاف من الخارج، باثنين وخمسين. نثبت الاثنين فوق السبعة والخمسين بخمسة بعدها.
وصورة ذلك[3]:

```
  2
  7 3 1
  3 9 6
5 2 2 2 1
---------
    2 5 3
    9 8 7
```

ثمّ نضرب أيضًا الاثنين، التي في مرتبة المئين من المضروب، في جميع مراتب[4] المضروب فيه: نضربها أوّلا في السبعة بأربعة عشر، نضيفها إلى الثلاثة، التي في مرتبة المئين من الخارج، إذ هما من جنس واحد أيضًا[5]، بسبعة عشر. نثبت السبعة فوق الثلاثة ونضيف العشرة بواحد إلى الاثنين، التي بعد

[1] في (م) : سقطت الجملة "المضروب في جميع مراتب"، وهو خطأ.
[2] في (م) : سقط لفظ "نضيف".
[3] في (أ) وفي (ت): الصورة غائبة.
[4] في (أ) : "منازل".
[5] في (ب) : سقط لفظ "أيضا".

تلك المرتبة، بثلاثة. نثبتها فوق الاثنين. ثمّ (نضربها أيضًا)[1] في الثمانية بستة عشر. نضيفها إلى الثلاثة، التي في مرتبة الآلاف من الخارج، بتسعة عشر. نثبت التسعة فوق الثلاثة ونضيف العشرة بواحد إلى الخمسة، التي بعد تلك المرتبة بستة. نثبتها فوق الخمسة. ثمّ نضربها أيضًا في التسعة بثمانية عشر، نضيفها إلى الستة، التي في مرتبة عشرات الآلاف من الخارج، بأربعة وعشرين. فنثبت الأربعة فوق الستة والعشرين باثنين بعدها. وقد تمّ العمل.
وصورة ذلك[2] : [22 و]

		9			
		3			
		2	7		
	4	7	3	1	
	6	3	9	6	
2	5	2	2	2	1
			2	5	3
			9	8	7

فيكون الخارج من الضرب أحد عشر وسبعمائة وتسعة وأربعين ألفا ومائتي ألف. وهو المطلوب.
ولو شئنا الابتداء من آخر المنازل لاتّبعنا الطريقة. فاعلمه.

ومنها نوع آخر، يشترط فيه أن تكون مراتب المضروبين متساوية وتكون[3] أعداد (كلّ مرتبة من مراتب)[4] كلّ سطر متساوية أيضًا. وكيفيته في الوضع مثل كيفية الممحو[5]، ثمّ تضع تحت أوّل مرتبة من مراتب السّطر الأعلى واحدا، وتحت الثانية إثنين، وكذلك (تتزايد بواحد)[6]، حتى تنتهي إلى آخر منازل المضروب. فيكون ما تحتها مشتركا بينها وبين أوّل منزلة[7] من المضروب فيه. (فمن المنزلة الثانية من المضروب فيه تبدأ بنقصان واحد واحد حتى تنتهي إلى آخر منزلة من المضروب فيه)[8]. فتكون الأعداد المكتوبة بجملتها

[1] في (م) : "نضرب أيضًا الاثنين".
[2] في (أ) وفي (ط) وفي (ت) : "249711".
[3] في (م) : سقط لفظ "تكون".
[4] في (م) وفي (أ) وفي (ب) وفي (ط) : سقطت العبارة " مرتبة من مراتب".
[5] وفي (م) : "المجموع". وفي (أ) وفي (ت) : "المحو". وفي (ط) : "وكيفيته مثل كيفية المحو".
[6] في (ت) : "بتزايد واحد"
[7] في (ت) : "مرتبة"
[8] في (س) : سقطت الجملة "فمن المنزلة ... فيه".

سطرا ثالثا. (وهي أسوس منازل)[1] المضروب مستقيمة، وأسوس منازل المضروب فيه معكوسة. ثمّ تضرب عدد منزلة من المضروب في عدد منزلة من المضروب فيه، فما خرج يضرب في السّطر الحادث عن الكتابة. فما خرج فهو المطلوب.
ويُسمّى هذا النوع الضرب بالتضعيف.

مثاله : لو قيل اضرب أربعة وأربعين وأربعمائة في ثلاثة وثلاثيين وثلاثمائة.
فننزلهما (في سطرين، كما ذُكر. تكون)[2] على هذه الصورة :

		4	4	4
3	3	3		

ثمّ نضع تحت الأربعة الأولى واحدا وتحت الثانية [22 ظ] اثنين وتحت الثالثة ثلاثة، وهي أوّل منزلة من المضروب فيه.
ثم نضع تحت الثلاثة الثانية الثانية اثنين، لأنّه من ثمّ تبدأ نقصان واحد واحد، كما ذُكر، وتحت الثلاثة الثالثة واحدا على هذه الصورة :

		4	4	4
3	3	3		
1	2	3	2	1

وهي، كما ذُكر، أسوس منازل المضروب مستقيمة، وأسوس منازل المضروب فيه معكوسة، لأنّ أسّ الثلاثة، التي في أوّل المضروب فيه، واحد، وقد وضعناه تحت الثلاثة، الثالثة التي في آخر منزلة منه.
ثمّ نضرب عدد منزلة من المضروب، وهي في المثال أربعة، في عدد منزلة من المضروب فيه، وهي في المثال ثلاثة، باثني عشر. فنضربها في السّطر الحادث: نضربها أوّلا في الواحد، الذي هو آخر السّطر، باثني عشر. نثبت الاثنين فوقه والعشرة بواحد بعده. ثمّ نضربها أيضًا في الاثنين، الرابعة في السّطر، بأربعة وعشرين. نثبت الأربعة فوق الاثنين ونضيف العشرين باثنين إلى الاثنين، التي بعد تلك المرتبة، بأربعة. نثبتها فوقها. ثمّ نضربها أيضًا في الثلاثة، الثالثة في السَطر، بستة وثلاثين. نثبت الستة فوق الثلاثة ونضيف

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[1] في (أ) : "وهي أسوس، فأسوس منازل".
[2] في (م) : "كما ذكرنا، فيكونان".

الثلاثين بثلاثة إلى الأربعة، التي بعد تلك المرتبة، بسبعة. نثبتها فوقها. ثمّ نضربها أيضًا في الاثنين، الثانية في السَطر، بأربعة وعشرين. نثبت الأربعة فوق الاثنين ونضيف العشرين باثنين إلى الستة، التي بعد تلك المرتبة، بثمانية. نثبتها فوقها. ثمّ نضربها أيضًا في الواحد، الذي في أوّل السّطر، باثني عشر. نثبت الاثنين فوقه ونضيف العشرة [23 و] بواحد إلى الأربعة، التي بعد تلك المرتبة، بخمسة. نثبتها فوقها. وقد تمّ العمل، وصورة ذلك[1]:

	4	7	8	5	
1	2	4	6	4	2
	1	2	3	2	1

فيكون الخارج من الضرب اثنين وخمسين وثمانمائة وسبعة وأربعين ألفا ومائة ألف. وهو المطلوب.

ومنها الضرب بالنيف، وهو أن تُسمّـي ما زاد على العشرة في أحد المضروبين، من العشرة، ثمّ تأخذ تلك النسبة من صاحبها، فتحملها عليه وتجعلها عشرات. وإن كان في النسبة كسور أخذتها من العشرة وجعلتها في موضع الآحاد.

مثاله : لو أردنا ضرب اثني عشر في خمسة عشر، لسميّنا[2] الاثنين الزائدة على العشرة في المضروب من العشرة، تكون خُمسًا. فنأخذ خُمس الخمسة عشر، المضروب فيها، بثلاثة. نحملها عليها بثمانية عشر ونجعلها عشرات. تكون مائة وثمانين. (وهو الخارج، وهو الضرب المطلوب، وصورته: 180)[3].
(ولو سميّنا الخمسة، الزائدة على العشرة من المضروب فيه، من العشرة تكون نصفًا، نأخذ نصف الاثني عشر بستة، نحملها عليها بثمانية عشر أيضا. نجعلها عشرات. تكون مائة[4] وثمانين)[5]، وهو الخارج، كما تقدم.

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[1] في (أ) وفي (ط) وفي (ت) : "147852".
[2] في (أ) وفي (ط) وفي (ت) : "نسبنا".
[3] في (أ) : "وهو الخارج"، وفي (ب) : "والخارج هو المطلوب، وصورته: 180". وفي (ط) : "وهو الخارج من الضرب"، وفي (ت) : "وهو الخارج كما تقدم".
[4] في (م) : سقط لفظ "مائة"، وهو خطأ.
[5] في (ت) : سقطت الجملة "ولو سمّينا ... وثمانين".

مثال منه آخر[1] : لو أردنا ضرب ثلاثة عشر في سبعة عشر، لسميّنا[2] الثلاثة الزائدة على العشرة في المضروب من العشرة، تكون ثلاثة أعشار. نأخذ ثلاثة أعشار سبعة عشر، المضروب فيها،| 23 ظ | بخمسة وعُشر. نحملها عليها باثنين وعشرين وعُشر، نجعلها عشرات ونأخذ الكسر من العشرة، وذلك عشر بواحد. نجعله في موضع الآحاد، على ما ذُكر، يكون ذلك واحدا وعشرين ومائتين. وهو المطلوب. (وصورته : 221)[3].

ومنها نوع آخر، يُعرف بالتّسمية، وهو أن تجمع المضروبين، ثمّ تُسمّـي أحدهما من الجملة. ثمّ تأخذ تلك النسبة من صاحبها، (وتضربها)[4] في الجملة، فيخرج المطلوب.

مثاله : لو أردنا ضرب ستة في اثني عشر، لجمعناهما بثمانية عشر، ثمّ نسمّي أحدهما منه[5]. فكأنا سمّينا منه الستة، تكون ثلثا. نأخذ ثلث المضروب فيه، وهو الاثنا عشر، بأربعة. نضربها في الجملة.
[فما خرج، فهو المطلوب. وذلك اثنان وسبعون، (وصورته : 72)[6].
ولو سميّنا[7] الاثني عشر من الجملة لكان ثلثين، فنأخذ ثلثي الستة بأربعة، نضربها في الجملة][8]، يخرج المطلوب، كما تقدّم.

وذكر الفقيه أبو محمد بن حجاج، المعروف **بابن الياسمين،** في هذا الوجه أنّا إذا سمّينا أحدهما من الجملة، فإنّا نسقط تلك النسبة من العدد بنفسه ونضرب ما بقي في الجملة، يخرج المطلوب.

ومنها نوع آخر يُعرف بالتّسمية أيضًا. تسمّـي أسهل المضروبين، من أيّ عقد مفرد شئت، أو تقسمه عليه، فما خرج من (التسميّة أو القسمة)[9] ضربته في الآخر، فما خرج أخذت لكلّ واحد منه العقد المقسوم عليه. فما اجتمع[10]، فهو

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[1] في (م) : سقط لفظ "آخر".
[2] في (أ) وفي (ت) : "نسبنا".
[3] في (أ) وفي (ط) وفي (ت) : لا إشارة إلى الصورة.
[4] في (أ) : "ثمّ تضربها".
[5] في (أ): سقط لفظ : "منه".
[6] في (أ) وفي (ط) وفي (ت) : لا إشارة إلى الصورة.
[7] في (أ) وفي (ت) : "نسبنا".
[8] في (ت) : سقطت الفقرة "فما خرج ... نضربها في الجملة".
[9] في (س) : سقطت العبارة "القسمة أو التسمية". وفي (ط) : " القسمة أو التسمية".
[10] في (س) : "فما ارتفع من ذلك".

المطلوب. فإذا لم تصح قسمة أحدهما أو تسميته إلا بزيادة شيء عليه أو نقصانه (منه)[1] فعلت ذلك. ثمّ تضرب الزيادة في ما لم تزد عليه |24 و| وتنقص المجتمع من الخارج. وإن كنت عملت بالنقصان فزد المجتمع على الخارج.
(والعقد المفرد هو ما في أوّل كلّ مرتبة، مثل العشرة والمائة وشبه ذلك)[2].

مثال المسألة : أردنا ضرب أربعة وعشرين في ثمانية. فنُسمّي الثمانية[3]، المضروب فيها، من أيّ عقد مفرد شئنا. فكأنّه عشرة، يكون أربعة أخماس. نضربها في الأربعة وعشرين. يخرج لنا تسعة عشر وخُمسًا. نأخذ لكلّ واحد منها عشرة، العقد المُسمّى منه، بمائة وتسعين وخُمس واحد من العشرة، باثنين. نضيفه إليه، فما كان فهو المطلوب. وذلك اثنان وتسعون ومائة، (وصورته : 192)[4].

ومثال منه آخر : إذا أردنا ضرب (اثنى عشر)[5] في خمسة عشر. فنقسم الخمسة عشر على العشرة، [يخرج واحد ونصف. نضربه في (الاثني عشر)[6]][7]، يخرج من الضرب (ثمانية عشر)[8]. نأخذ لكلّ واحد منه العشرة، العقد المقسوم عليه، يكون مائة وثمانين[9]، وهو المطلوب.
ولو شئنا لأسقطنا من (الخمسة عشر)[10] الخمسة وقسمنا العشرة الباقية على العشرة. يخرج واحد، نضربه في (الاثني عشر باثني عشر)[11]. نجعلها عشرات ونحمل عليها ضرب الخمسة المنقوصة في (الاثني عشر)[12]، لكونها لم تنقص منها. يكون المجتمع مائة وثمانين[13]، وهو المطلوب، كما تقدّم.

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1 في (س) : سقط لفظ "منه".
2 في (س) : سقطت الجملة "والعقد ... وشبه ذلك" ، ويعتبرها ناسخ (ت) من إضافات الهواري.
3 في (أ) : سقط لفظ "الثمانية".
4 في (أ) وفي (ط) وفي (ت) : لا إشارة إلى الصورة.
5 في (أ) : " عشرة"
6 في (أ) : " العشرة"
7 في (م) : سقطت الجملة " يخرج واحد ونصف. نضربه في الاثني عشر. "
8 في (أ) : "عشرة"
9 في (أ) : "خمسون"
10 في (م) وفي (ط) : سقطت عبارة "من الخمسة عشر".
11 في (أ) : "عشرة بعشرة".
12 في (أ) : "عشرة".
13 في (أ) : "خمسون".

مثال منه آخر: أردنا ضرب ثلاثة في خمسة عشر. فنزيد على الثلاثة اثنين بخمسة. نسمّيها من العشرة، يكون نصفًا. نضربه في الخمسة عشر، يخرج من الضرب سبعة ونصف. نأخذ لكلّ واحد منها عشرة، العقد، ونصف العشرة بخمسة وسبعين، ونسقط [24 ظ] منها ضرب الاثنين الزائدة في الخمسة عشر، لأنّها لم يزد عليها. فما بقي فهو المطلوب، وذلك خمسة وأربعون، (وصورته : 45)[1].

ومنها ضرب التسعات، وهو بشرط أن تكون مراتب السّطرين متساوية وأحدهما فيه التسعات والثاني تستوي أعداده. وصفة العمل أن تضع السّطرين متوازيين، أحدهما تحت الآخر، وتُعلّم فوقهما بنقط بعدد ما فيهما من المنازل، وتضرب عدد منزلة من أحدهما في عدد منزلة من الثاني، وتجعل آحاد الخارج في أوّل العلامات وعشراته في وسط باقي العلامات وتنظر ما بين التسعة والعدد المضروب فيه. فتُعمّر به ما بين العددين الخارجين، أعني الآحاد والعشرات، وتُعمّر باقي العلامات بالعدد الذي هو خلاف التسعة. فما كان فهو الجواب.

مثاله : أردنا ضرب[2] أربعة وأربعين وأربعمائة في تسعة وتسعين وتسعمائة. فننزلهما في سطرين متوازيين، كما ذُكر[3]، ونعلم فوقهما علامات بقدر ما فيهما من المنازل، وذلك ستة، على هذه الصورة :

.	.	.	.	.	.
			4	4	4
			9	9	9

فنضرب عدد منزلة من المضروب في عدد منزلة من المضروب فيه أيضًا[4]، وهي تسعة (في أربعة)[5] بستة وثلاثين.
ننزل الستة في أوّل العلامات، ويبقى منها خمسة. فننزل الثلاثين بثلاثة في العلامة الثالثة من الخمسة، لأنّها الوسطى منها، ثمّ نأخذ فضل ما بين التسعة والأربعة، وذلك خمسة، فنعمّر بها العلامتين اللتين بين الستة الآحاد، والثلاثة

[1] في (أ) وفي (ط) وفي (ت) : لا إشارة إلى الصورة.
[2] في (ت) : "ضربنا"
[3] في (ت) : ناقصة
[4] في (م) وفي (ط) وفي (ت) : سقط لفظ "أيضًا".
[5] في (م) : سقطت عبارة "في أربعة"، وهو خطأ.

العشرات، ونعمّر (العلامتين الباقيتين)[1] بالأربعة، وذلك معنى قوله : "بالعدد الذي هو خلاف التسعة"، [25 و] على هذه الصورة[2] :

4	4	3	5	5	6
			4	4	4
			9	9	9

فالخارج هو العدد الذي فوق المضروبين، وذلك ستة وخمسون وخمسمائة وثلاثة وأربعون ألفا وأربعمائة ألف. فاعلمه وقس عليها أمثالها.

ومن ضرب[3] التسعات نوع آخر، وليس[4] فيه شرط، بل تكون أعداد أحد السّطرين تسعات وأعداد السّطر الآخر كيف ما كانت، ومنازله كذلك كيف[5] ما كانت.
والعمل فيه أن تزيد من الأصفار على مراتب السّطر الآخر مثل عدد مراتب التسعات، ثمّ تنقص من المجتمع، العدد الذي هو غير التسعات يبقى الجواب.

مثاله : أردنا ضرب تسعة وتسعين وتسعمائة في أربعة وخمسين وثلاثمائة وتسعة ألف. فنزيد على المضروب فيه من الأصفار بعدد مراتب التسعات، وذلك ثلاثة. فيصير إذا أربعة وخمسين ألفا وثلاثمائة ألف وتسعة آلاف ألف، وصورته : 9354000. فنسقط منه العدد الذي هو غير التسعات، وهو المضروب فيه. فما بقي فهو المطلوب. وذلك ستة وأربعون (وستمائة وأربعة)[6] وأربعون ألفا وثلاثمائة ألف وتسعة آلاف ألف، وصورته[7] : 9344646.

ومنها نوع آخر يُعرف بالتّربيع، وهو أن تأخذ نصف مجموع المضروبين، وتُربّعه، وتنقص من الخارج مُربّع نصف الفضل بينهما، فما بقي فهو الخارج من الضرب.

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[1] في (ط) وفي (ت) : "باقي العلامات".
[2] في (ت) وفي (أ) : "443556".
[3] في (ت) : "ضروب".
[4] في (س) وفي (ب) : زيادة لفظ "يشترط".
[5] في (س) وفي (ط) : "كم كانت".
[6] في (م) : سقط "وستمائة وأربعة"، وهو خطأ.
[7] في (ط) : غابت الصورة.

مثاله : أردنا ضرب سبعة عشر (في تسعة عشر)[1].
فنأخذ نصف مجموعهما، وذلك ثمانية عشر، نضربها في نفسها، وهو معنى التّربيع، كما تقدّم، بأربعة وعشرين وثلاثمائة. نسقط منها واحدا، الذي هو مُربّع نصف الفضل [25 ظ] بين المضروبين، يبقى ثلاثة وعشرون وثلاثمائة. وهو المطلوب. (وصورته: 323)[2].

ومنها نوع آخر يُعرف بالتّربيع أيضًا، وهو أن تضرب مُربّع أحد المضروبين فيما يخرج من (نسبة الآخر إلى)[3] الذي ربّعته، أو تقسم مُربّع أحدهما على الخارج من قسمة الذي ربعّته على الآخر.
مثاله : أردنا ضرب (خمسة وعشرين في خمسة عشر. فنضرب مُربّع الخمسة والعشرين، وذلك)[4] خمسة وعشرون وستمائة، في ثلاثة أخماس، التي هي نسبة الخمسة عشر إلى الخمسة والعشرين، فما خرج فهو المطلوب. وذلك خمسة وسبعون وثلاثمائة. وصورته: 375)[5].
ولو قسمنا مُربّع الخمسة عشر، وذلك خمسة وعشرون ومائتان، على الثلاثة الأخماس، الخارج من قسمة الخمسة عشر، التي ربّعنا، على الخمسة والعشرين، لخرج[6] المطلوب، مثل الأوّل. فاعلمه[7].

ومنها نوع آخر، وهو أن تضرب الفضل بين المضروبين في أكثرهما وتسقط الخارج من مُربّع أكبرهما، أو تضرب الفضل في أصغرهما و تزيد الخارج على مُربّع أصغرهما. فما كان فهو الخارج المطلوب.

مثاله : لو قيل اضرب ستة وثلاثين في أربعة عشر. فنضرب الاثنين والعشرين التي هي فضل ما بين المضروبين، في ستة وثلاثين، أكبر المضروبين، يخرج لنا اثنان وتسعون وسبعمائة. نسقطها[8] من مُربّع الأكبر[9]، وذلك ستة وتسعون ومائتان وألف[10]. فما بقي فهو المطلوب، وذلك أربعة وخمسمائة،

---

[1] في (م) : سقط "في تسعة عشر"، وهو خطأ.
[2] في (أ) وفي (ط) وفي (ت) : لا إشارة إلى الصورة.
[3] في (م) : سقط لفظ "إلى"، وهو خطأ. وفي (ط) وفي (ت) : "نسبته إلى"
[4] في (م) : سقطت الجملة "خمسة ... وذلك"، وهو نقص.
[5] في (أ) وفي (ط) وفي (ت) : لا إشارة إلى الصورة.
[6] في (أ) وفي (ط) وفي (ت) : "فيخرج".
[7] في (أ) : "فاعلم ذلك".
[8] في (أ) وفي (ط) وفي (ت) : "فننقص ذلك".
[9] في (أ) وفي (ط) وفي (ت) : "أكبر الاسمين".
[10] في (م) : زيادة في غير محلها "وذلك ستة وتسعون ومائة".

(وصورته: 504)[1]. أو نضرب الاثنين والعشرين، الفضل، في الأربعة عشر، أصغر المضروبين، يخرج ثمانية وثلاثمائة. نحملها على مُربّع الأصغر، وذلك ستة وتسعون [26 و] ومائة. فما اجتمع فهو المطلوب، وهو مثل الأوّل. فاعلمه.

وإن ضربت عددا ذا أصفار في عدد ذي أصفار، فاضرب بعضها في بعض مجرّدتين من الأصفار، ثمّ تكسو الخارج جملة الأصفار. فما كان فهو المطلوب، (لأنّ ضرب العدد في الصفر أو الصفر في العدد هو هو، وهو تصفير العدد، أو تضعيف الصفر، وكلّ ذلك ليس بعدد، فعلامته صفر أبدًا)[2].

مثاله : لو قيل اضرب ثلاثين في أربعين ومائة. فنزيل[3] الصفر عن الثلاثين ونرفعه. تبقى ثلاثة. ونزيله[4] أيضًا عن الأربعين ومائة ونرفعه. يبقى أربعة عشر. نضربها في الثلاثة ونحمل على الخارج الصفرين المرفوعين. فما كان فهو المطلوب، وذلك مائتان وأربعة آلاف، وصورته : 4200.

وغاية مراتب الخارج مجمُوع مراتب المضروبين، (لأنّ أقصى ما يكون في المنزلة الواحدة تسعة. وضرب تسعة في تسعة إنّما هو واحد وثمانون، أفاد[5] الضرب منزلة واحدة هي عشرات)[6].

واختباره[7] أن تقسم الخارج على أحد المضروبين، يخرج الثاني.

مثاله : في المثال المتقدّم[8]، نقسم الواحد والثمانين على التسعة، التي هي أحد المضروبين، يخرج (تسعة وهو المضروب الثاني)[9].

---

[1] في (أ) وفي (ط) وفي (ت) : لا إشارة إلى الصورة.
[2] في (س) : سقطت الجملة "لأن ضرب العدد ... صفرا أبدا"، ويعتبرها ناسخ (ت) من إضافات الهواري.
[3] في باقي النسخ : "فننزع".
[4] في باقي النسخ : "و ننزع".
[5] في (م) : "فان".
[6] في (س) : سقطت الجملة "لأن أقصى ... هي عشرات"، ويعتبرها ناسخ (ت) من إضافات الهواري.
[7] في باقي النسخ : "فاختبار الضرب".
[8] في (م) : "قبل".
[9] في باقي النسخ : "المضروب الثاني، وهو تسعة".

ولابد للطالب من حفظ التجزئة[1] وإتقانها. وهي :

(إذا ضُرب عدد في واحد أو ضُرب واحد)[2] فيه، فذلك العدد على حاله لا يتضاعف.

مثل اثنين في واحد باثنين أو واحد في خمسة بخمسة.

واثنان في اثنين بأربعة، وفي ما بعدها بزيادة اثنين اثنين.

وذلك على توالي الأعداد، فيكون ضرب |26 ظ| اثنين في ثلاثة بستة، وفي أربعة بثمانية، وفي خمسة بعشرة. وكذلك فيما بعد ذلك إلى العشرة، لأنّه مُنتهى ما تصل إليه التجزئة في العُرف.

وثلاثة في ثلاثة بتسعة، وفي ما بعدها بزيادة ثلاثة ثلاثة،

يعني على توالي الأعداد أيضًا. فيكون ضربها في أربعة باثني عشر، وفي خمسة بخمسة عشر. وكذلك إلى العشرة.

وأربعة في أربعة بستة عشر، وفي ما بعدها بزيادة أربعة.

وخمسة في خمسة بخمسة وعشرين، وفي ما بعدها بزيادة خمسة.

وستة في ستة بستة وثلاثين، وفي ما بعدها بزيادة ستة.

وسبعة في سبعة بتسعة وأربعين، وفي ما بعدها بزيادة سبعة.

وثمانية في ثمانية بأربعة وستين، وفيما بعدها بزيادة ثمانية.
(وكلها على توالي كما تقدم).[3]

---

[1] في (م) : "التجربة".
[2] في (ت) : "إذا ضربت عددا في واحد أو ضربت واحدا فيه".
[3] في (س) وفي باقي النسخ : سقطت الجملة "وكلها على توالي كما تقدم ".

وتسعة في تسعة بواحد وثمانين، وفي عشرة بتسعين.
وعشرة في عشرة بمائة.

فاعلمه (تصب بحول الله تعالى)[1].

[1] في (م) : "تصب بحول الله تعالى" زيادة.

الباب الخامس : في القسمة

القسمة هي حلّ المقسوم إلى أجزاء متساوية يكون عددها مثل ما في المقسوم عليه من الآحاد.
(وهذا خاص بالكم المنفصل)[1]

ويُراد بالقسمة أيضًا[2] (نسبة أحد العددين من الآخر)[3].
(وهو أيضًا خاص بالكم المتصل)[4].

والجمهور يريدون (بالقسمة أيضًا)[5] على الإطلاق معرفة ما يجب للواحد الصحيح من آحاد المقسوم عليه من جملة المقسوم.

[(فللقسمة معنيان: أحدهما المرسوم أوّلا، وهو خاص بقسمة الجنس على غير جنسه، كالدراهم على الرجال. والثاني المرسوم ثانيا، وهو خاص بقسمة الجنس على جنسه. فصار لفظ القسمة مشتركا بين المعنيين. ولا ينبغي أن ترسم على الإطلاق كما فعله الجمهور. فمعنى ما ذكره الجمهور هو المعنى الأوّل خاصة، وتركوا ذكر المعنى الثاني. فصار [27 و] رسمهم على الإطلاق يُوهم العُموم للمعنين[6] جميعا، أو أنّ القسمة إنّما هي بمعنى واحد، وليس الأمر كذلك.)[7]

مثال القسمة بالمعنى الأوّل : اقسم خمسة عشر درهما على ثلاثة رجال. فنحلّ[8] الخمسة عشر إلى ثلاثة أجزاء مثل عدة ما في المقسوم عليه من الآحاد. فيكون كلّ جزء من خمسة دراهم، وهو ما يجب للواحد الصحيح من تلك الثلاثة المقسوم عليها.

---

1 في (أ) : سقطت الجملة "وهذا خاص بالكم المنفصل". وهي منقولة حرفيا من **رفع الحجاب**، صفحة 263: 3.
2 في (س) وفي (ب) : سقط لفظ "أيضا".
3 في (أ) : "نسبة العددين"، وهو خطأ.
4 في (أ) : سقطت الجملة. وفي (ب) : "وهذا خاص أيضًا بالكم". وهي منقولة حرفيا من **رفع الحجاب**، صفحة 263: 4. وفي (ت) : "وهذا أيضًا خاص بالكم المنفصل"، وهو خطأ.
5 في (ب) : سقطت عبارة "بالقسمة أيضا". في (س) وفي (ط) وفي (ت) : سقطت عبارة "أيضا".
6 في (ط) وفي (ت) : "على المعنين"
7 في (س) : سقطت الفقرة "فللقسمة معنيان ... وليس الأمر كذلك"، ويعتبرها ناسخ (ت) من شرح الهواري. وهي منقولة حرفيا من **رفع الحجاب**، صفحة 263: 4-9.
8 في (ط) وفي (ت) : "نجعل".

ومثالها بالمعنى الثاني : اقسم خشبة من خمسة عشر شبرا على خشبة من ثلاثة أشبار.
فالمُراد هنا : كم في المقسوم من أمثال المقسوم عليه؟ فنفصّل (المقسوم بأمثال)[1] المقسوم عليه. فيكون في المقسوم (من أمثال المقسوم عليه)[2] خمسة أجزاء، كلّ جزء منها مثل المقسوم عليه.
فقد صار الخارج من عمل[3] القسمة في المعنيين جميعا خمسة. لكن آحاد هذه الخمسة الخارجة في[4] المعنى الأوّل غير آحاد الخمسة الخارجة في[5] المعنى الثاني، لأنّها في المعنى الأوّل عدّة ما في قسم من أقسام المقسوم من الآحاد، وفي المعنى الثاني عدّة أقسام المقسوم. فصار المقسوم في المعنى الأوّل معلوم عدّة الأقسام التي يتفصّل إليها. وما في كلّ قسم منها هو الذي يُعلم بالقسمة.
وصار المقسوم في المعنى الثاني معلوم ما في كل قسم من أقسامه من الآحاد، وعدّة الأقسام التي ينفصل إليها هي التي تُعلم بالقسمة.
فالمعنى الثاني على عكس المعنى الأوّل][6]. فاعلمه.

والقسمة على نوعين : قسمة كثير على قليل وقسمة قليل على كثير[7]. وقسمة القليل على الكثير تختصّ باسم التّسمية. (وأبقوا لفظ القسمة على قسمة الكثير على القليل)[8].

والعمل العام [27 ظ] **في قسمة الكثير على القليل هو أن تضع المقسوم في سطر وتضع تحته المقسوم عليه. واحذر أن يكون الكثير تحت القليل. واطلب عددا تضعه[9] تحت أوّل مرتبة من مراتب[10] المقسوم عليه وتضربه في جملة مراتبه يفنى به المقسوم كله، أو تبقى منه بقية أقلّ من المقسوم عليه، فتسمّيها منه.**

---

[1] في (ط) في (ت) : "من المقسوم أمثال".
[2] في (م) وفي (ط) وفي (ت) : سقطت عبارة "من أمثال المقسوم عليه".
[3] في (م) وفي (ط) وفي (ت) : سقط لفظ : "عمل".
[4] في (أ) وفي (ط) وفي (ت) : "من".
[5] في (أ) وفي (ط) وفي (ت) : "من".
[6] الفقرات "فللقسمة معنيان ... على عكس المعنى الأول."، وهي منقولة حرفيا من **رفع الحجاب**، صفحة 263: 4 إلى 264: 8.
[7] في (س) : "قسمة قليل على كثير وقسمة كثير على قليل".
[8] في (س) : سقطت الجملة : "وأبقوا لفظ القسمة على قسمة الكثير على القليل" ويعتبرها ناسخ (ت) من شرح الهواري.
[9] في (أ) وفي (ط) وفي (ت) : "تنزله"
[10] في (ب) وفي (س) : "منزلة من منازل"

مثال ذلك : لو أردنا قسمة خمسة وأربعين ومائتين على اثني عشر. فننزل المقسوم في سطر والاثني عشر، المقسوم عليها، (في سطر)[1] تحته: الاثنان تحت الأربعة والعشرة بواحد تحت الاثنين، على هذه الصورة :

2	4	5
.1	2	

فصار المقسوم عليه تحت الأربعة والعشرين.

فلو كانت أقلّ، مثل أحد عشرة، أوعشرة، أو شبه ذلك، (لأنزلنا آخر منزلة من المقسوم عليه تحت آخر منزلة من)[2] المقسوم.
في مثالنا ننظر عددا ننزله تحت الاثنين من المقسوم عليه، لأنّها أوّل منزلة منه، نضربها في جملته، نفني بها الأربعة والعشرين التي فوقه، أو يبقى منه أقلّ من الاثني عشر. نجده[3] اثنين. فنضرب بعدد[4] الاثنين أوّلا في الواحد، الذي هو آخر مرتبة من المقسوم عليه، باثنين. نفني بها الاثنين التي فوقه. ثمّ نضربها أيضًا في الاثنين التي هي (تحتها)[5] بأربعة. تفنى بها الأربعة التي فوقها. ثمّ نقهقر الاثني عشر منزلة تحت الخمسة، وننظر عددا ننزله تحت الاثنين من المقسوم عليه نضربه فيه، فيفنى ما فوقه. فلا نجد شيئا، (نجعل عوضه)[6] صفرًا، و تبقى خمسة على الاثنى عشر. نسمّيها منها، تكون سُدسين ونصف سُدس، (نحملها على الصحيح، فما كان فهو المطلوب. وذلك: عشرون وسدسان ونصف سدس.
وصورته: 28 و

$20\frac{2}{6}\frac{1}{2}$)[7].

وكذلك العمل فيما أشبه ذلك.

وإن أردت أن تقسم المقسوم مفصّلا وتجمع الخارجات[8]، فلك ذلك.

---

[1] في (م) وفي (ط) وفي (ت) : سقطت "في سطر".
[2] في (م) : " لنزلنا الاثني عشر، أوّل منزلة من تحت ".
[3] في (أ) وفي (ط) وفي (ت) : "نجد ذلك".
[4] في (ب) : "هذه". وفي (ط) : "عدد".
[5] في (ب) : "فوقها"، وهذا خطأ.
[6] في (أ) وفي (ط) وفي (ت) : "فنجعله عوضًا منه".
[7] في (أ) وفي (ت) : (وصورته: $\frac{2}{6}\frac{1}{2}$).
[8] في (م) : "الخارج".

أو تحلّ المقسوم عليه إلى أعداده التي تركّب منها وتتخذها أئمّة وتقسم عليها المقسوم.
أو توفّق بين المقسوم والمقسوم عليه وتقسم وفق (المقسوم على وفق)[1] المقسوم عليه.

مثال الأوّل : لو أردنا قسمة أربعة وأربعين على أحد عشر رجلا[2]، مثلا. فإنّا نفصّل الأربعة والأربعين باثنين وعشرين (اثنين وعشرين)[3]. ولنا أن نفصّلها بغير ذلك بما[4] شئنا. فنقسم الاثنين والعشرين على أحد عشر. يخرج لنا اثنان. وكذلك من قسمته الثانية: نجمعها بأربعة، وهو الخارج من قسمة الأربعة والأربعين على أحد عشر.

ومثال الثاني : لو قيل لنا اقسم ستة وتسعين على اثني عشر. فلنا أن نحلّ الاثني عشر، المقسوم عليها، لما تركبت منه: إمّا ستة واثنين، وإمّا ثلاثة وأربعة، (كذلك سواء)[5]. فكأنا حللناها للأوّل: فإذا قسمنا الستة والتسعين على اثنين أوّلا، خرج ثمانية وأربعون، نقسمها على الستة الباقية[6] من الإمامين أيضًا، يخرج ثمانية، وهو المطلوب. ولو قسمنا المقسُوم أوّلا على الستة، ثمّ ما خرج على الاثنين، يخرج ثمانية. لا فرق. فاعلمه.

ومثال الثالث : لو قيل اقسم خمسة وثلاثين على خمسة عشر. (فتجد)[7] لكلّ واحد منهما خُمسًا، وهو الذي يتفقان به: أما[8] المقسوم فخُمسه سبعة وأما[9] المقسوم عليه فخُمسه ثلاثة. فنقسم سبعة على ثلاثة، يخرج اثنان وثلث، وهو مثل ما يخرج من قسمة الخمسة والثلاثين على الخمسة عشر. فاعلمه.

ومن القسمة، [28 ظ] نوع يختصّ باسم المحاصة[10].

[1] في (م) : سقطت العبارات "على وفق المقسوم"، وهو خطأ .
[2] في (م) : سقط لفظ "رجلا".
[3] في (م) : سقطت العبارة الثانية "اثنين وعشرين".
[4] في (أ) وفي (ط) وفي (ت) : "فيما".
[5] في (أ) : سقطت عبارة "كذلك سواء".
[6] في (أ) وفي (ط) وفي (ت) : "الثانية".
[7] في (ط) وفي (ت) : "فتأخذ".
[8] في (م) : "امام"، وهذا خطأ.
[9] في (م) : سقط لفظ "اما".
[10] يستعمل الشراح تارة "المحاصات" وتارة "المحاصة"، وليس هناك قاعدة واضحة.

ووجه العمل فيه أن تجمع أجزاء المحاصة وتتخذها إمامًا، ثمّ تضرب كلّ جزء من أجزاء المحاصة في[1] المقسوم، وتقسم الخارج على الإمام. يخرج المطلوب[2].

مثاله : رجل أفلس، فتصدّق عليه بعشرة دنانير، فأخذها. الغرماء يقسمونها على قدر أموالهم، وهم ثلاثة (مثلا)[3]. لأحدهم أربعة دنانير وللثاني خمسة وللثالث ستة. فنجمع هذه الأجزاء[4] التي عليه، وهي أجزاء المحاصّة. يكون مجموعها خمسة عشر دنانير. نجعلها إمامًا. ثمّ نضرب ما بيد الأوّل في العشرة المقسومة بأربعين. ونقسم له على الإمام، يخرج له ديناران وثلثا دينار، وهو ما يجب له من العشرة. ونضرب أيضًا ما بيد الثاني في العشرة بخمسين ونقسم له على الإمام. يخرج له ثلاثة دنانير وثلث دينار، وهو ما يجب له من العشرة. ثمّ نضرب أيضًا ما بيد الثالث في العشرة بستين، ونقسم له على الإمام، يخرج له أربعة دنانير، وهو ما يجب له من العشرة. فإذا جمعنا هذه الخارجات الثلاث كانت عشرة.

ولنا في العمل أربعة أوجه (أخرى)[5]:
أحدها، أن نسمّي ما بيد كلّ واحد منهم[6] من الإمام ونضرب ما خرج في المقسوم[7]، يكون المطلوب.
والثاني، أن نسمّي المقسُوم من الإمام، فما خرج (يسمّى)[8] جزء السّهم، نضربه فيما بيد كلّ واحد، يخرُجُ المطلوب.
والثالث، نقسم الإمام على ما بيد كلّ واحد ونقسم المقسوم على الخارج، يحصُلُ المطلوب.
والرّابع، نقسم الإمام على المقسُوم ونقسم ما بيد كلّ واحد على الخارج، يحصُلُ المطلوب.
ولها أوجه أخرى باعتبار تركيب النسبة[9] وتبديلها وغير ذلك [29 و] من أحوال النسبة على حسب ما يأتي بحول الله تعالى.

[1] في (أ) : "من"، وهو خطأ.
[2] في (م) : سقط لفظ "المطلوب".
[3] في (أ) : سقط لفظ: "مثلا".
[4] في (م) : سقط لفظ "الاجزاء".
[5] في (أ) : سقط لفظ "أخرى".
[6] في (أ) . سقط لفظ "منهم".
[7] في (أ) : زيادة عبارة: "ونقسم على الإمام".
[8] في (ب) : "وهو".
[9] في (أ) : "التسمية".

وفيما ذكرنا كفاية لمن (فهمه)[1].

وإن كان في أجزاء المحاصة كسور، فاضرب المسألة كلها في أقلّ عدد ينقسم على أئمّتها.
وإن كان بين الأجزاء كلّها اشتراك، فأزله بأن تأخذ عوض الأجزاء أوفاقها.

مثال منه : رجل أفلس وبيده اثنا عشر دينارا، وغُرماؤه ثلاثة. لأحدهم أربعة دنانير وثلث دينار وللثاني خمسة دنانير ورُبع دينار وللثالث ستة دنانير وسُدس دينار. فنضرب كلّ واحد من الأجزاء التي بيد كلّ واحد منهم في أقلّ عدد ينقسم على أيمّتها.

ومعرفة أقلّ عدد ينقسم على الأئمّة، في المثال وغيره بطريق الحلّ :

[نحلّ كلّ واحد من الأئمّة إلى أعداده التي تركبت منها، ونسقط من أعداد الثاني ما تكرّر فيها من أعداد الأوّل، ومن أعداد الثالث ما تكرّر فيها من أعداد الأوّل وباقي الثاني، ومن أعداد الرّابع ما تكرّر فيها من أعداد ما قبله كذلك، إلى آخرها، ثمّ نركب ما بقي بالضرب. وإن لم يتكرّر فيها عدد، فهي كلها متباينة[2]، نضرب بعضها في بعض. يكون أقلّ عدد ينقسم على الأئمّة][3].

فإذا أردنا عمل ذلك في المثال المذكور، فنجد مقام الثلث ثلاثة لا ينحلّ، ومقام الرّبع أربعة[4] تنحلّ اثنين واثنين، ولم تتكرّر منها شيء. فنحفظهما مع الثلاثة. ومقام السُّدُس ينحلّ إلى ثلاثة واثنين[5]، وقد تكرّرا، فنطرحهما ونركب ما بقي، وذلك ثلاثة في اثنين بستة، وستة[6] في اثنين باثني عشر. وهو أقلّ عدد ينقسم على الأئمّة. فنضرب المسألة كلّها فيه: نضربها أوّلا فيما بيد الأوّل، يخرج اثنان وخمسُون، نجعلها عوض ما كان |29 ظ| بيده. ثمّ نضربها أيضًا فيما بيد الثاني، يخرج ثلاثة وستون، نجعلها أيضًا[7] عوض ما كان بيده. ثمّ نضربها أيضًا فيما بيد الثالث، يخرج أربعة وسبعون، نجعلها أيضًا عوض ما كان بيده.

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1 في (أ) : "وفق".
2 في (أ) : "متناسبة" وهو خطأ.
3 الفقرة "تحل كل ... على الأئمة" منقولة حرفيا من **رفع الحجاب**، صفحة 266: 15-18.
4 في (م) : سقط لفظ "أربعة".
5 في (أ) وفي (ط) وفي (ت) : "اثنين وثلاثة".
6 في (م) : سقط لفظ "وستة".
7 في (ط) في (ت) : سقط لفظ "أيضا".

ثمّ ننظر إن كان بين هذه الأجزاء كلها اشتراك فنزيله ونأخذ من كلّ واحد وفقه، كما ذكر.

ومعرفة زوال الاشتراك في المثال وغيره :
[بأن نحلّ تلك الأعداد إلى ما تركبت منه ونسقط المُتكرّر في كلّ واحد منها من جميعها. فما بقي لكلّ واحد[1] منها نركبه بالضرب، فيكون وفقه. ومتى لم يبق لنا من عدد شيء فنجعل عوضه واحدا، لأنّ ضرب ذلك العدد الذي فني[2] لا يتضاعف، فيكون الواحد عوضه هو وفقه[3]. والاشتراك أبدًا[4] بين تلك الأعداد يكون بالجزء المُسمّى للعدد المُتكرّر المُسقط][5].
فإن كان اثنين فالاشتراك بالنصف، وإن كان خمسة فبالخُمس، وإن كان عشرة فبالعُشر، وإن كان أحد عشر فبالجزء من أحد عشر، وشبه ذلك.
فإذا اختبرنا الأجزاء المذكورة في المثال بهذا العمل نجدها مُتباينة، فنجمعها. فما كان فهو الإمام، وذلك تسعة[6] وثمانون ومائة. فنضرب ما بيد كلّ واحد منهم في الاثني عشر ونقسم على الإمام، فيخرج للأوّل ثلاثة دنانير وتسعا دينار وخمسة أسباع تسع دينار، وهو ما يجب له من الاثني عشر. وللثاني أربعة دنانير، وهو ما يجب له من الاثني عشر. وللثالث أربعة دنانير وستة أتساع دينار وسبعا تُسع دينار، وهو ما يجب له من الاثني عشر دينارا[7]. فلو جمعناها أيضا، يخرج منها اثنا عشر. وإن شئت عملت فيها بالأوجه المذكورة.

وأمّا التّسمية، فالعمل المشهور (العام)[8] فيها أن تحلّ المُسمّى منه إلى أعداده التي تركّب منها وتتّخذها |30 أ| أئمّة، تقسم عليها ما أردت تسميته، يخرج المطلوب.
ويُعرف قدره بنسبة[9] أجزائه إلى تلك الأئمّة المقسوم عليها.

مثاله : لو قيل سمّ أحد عشر من خمسة عشر.

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[1] في (م) : سقط لفظ "واحد".
[2] في **رفع الحجاب**، " فني في هذا الواحد ". وفي (ط) وفي (ت) : "فيه"، وهو لا يصح.
[3] في (ط) وفي (ت) : "وهو وجمعه".
[4] في (ت) : سقط لفظ "أبدا".
[5] الفقرات "فإذا أردنا عمل ذلك في المثال ... للعدد المتكرر المسقط."، منقولة من **رفع الحجاب**، صفحة 267: 1-5.
[6] في (أ) : "ستّة" وهو خطأ.
[7] في (م) : سقط لفظ "دينارا".
[8] في (ط) وفي (ت) : سقط لفظ "العام".
[9] في (أ) : "قدر نسبة" وهو خطأ.

(فنحل الخمسة عشر)[1]، المُسّمى منها، إلى أعدادها التي تركبّت منها، وذلك خمسة وثلاثة. فنجعلها تحت خطّ ونقسم الأحد عشر على الثلاثة أوّلا وما بقي نجعله فوقها. وما خرج نقسمه على الخمسة، وهو أقلّ منها، فنجعله فوقها. فيكون على هذه الصورة[2] :

$$\frac{2\ \ 3}{3\ \ 5}.$$

فننسب الثلاثة إلى الخمسة التي تحتها والإثنين إلى الثلاثة التي تحتها، ونضيف النّسبة إلى الخمسة. فما كان فهو قدر الأحد عشر من خمسة عشر، وذلك ثلاثة أخماس وثلثا خُمس. فاعلمه.

[وغير المشهور منها، هو أن تقسم المُسمّى منه على المُسمّى (وتُسمّي واحدا من الخارج أو تُسمّي واحدا من المُسمّى منه)[3] وتأخذ مثل تلك النّسبة من المُسمّى أو تضرب المُسمّى في عدد وتقسم الخارج على المُسمّى منه وما خرج على ذلك العدد المضروب فيه][4].

مثال من الأوّل : لو قيل سمّ أربعة من اثني عشر. فنقسم الاثنى عشر على الأربعة ونُسمّي واحدا من الخارج. فما كان فهو المطلوب. وذلك ثلث.

ومثال الثاني : لو قيل سمّ تسعة من خمسة عشر. فنُسمّي واحدا من الخمسة عشر، يكون ثلث خُمس. فنأخذ من التّسعة ثلث خُمسها، فما خرج فهو المطلوب. وذلك ثلاثة أخماس.

ومثال الثالث : لو قيل سمّ عشرة من ستّة عشر. فنضرب العشرة في أيّ عدد شئنا. فكأنّه ثمانية بثمانين. نقسمها على الستّة عشر. وما خرج على الثمانية يكون المطلوب، وذلك خمسة أثمان. فاعلمه.[30 ظ]

ولحلّ الأعداد مُقدّمة يجب حفظها. وهي :

[1] في (م) : سقطت الجملة " فنحمل الخمسة عشر".

[2] في (م) : غابت الصورة.

[3] في (م) : سقطت الفقرة: "وتسمي واحدا ... المال مسمى منه".

[4] في (س) : سقطت الفقرة: "وغير المشهور منها ... المضروب فيها"، وهي منقولة حرفيا من **رفع الحجاب** صفحة 267: 11-14.

كلّ عدد ليس في أوّله آحاد فالعُشر له، والخُمس، والنّصف الذي في طبيعة كلّ زوج.

مثاله : خمسون وشبهها. نصفُها خمسة وعشرون، وخُمسها عشرة، وعُشرها خمسة.

(وإن كان في أوّله خمسة، فالخُمس له.)[1]
وإن كان في أوّله آحاد، فإن كانت زوجا، فإنّه يُطرح بالطروح الثلاثة، يعني المذكورة في الطّرح.
فإن انطرح بتسعة، فله التُّسع والسُّدس والثُّلث.

مثاله : ستة وثلاثون. فإنّها مُنطرحة بتسعة، وتُسعها أربعة، وسُدسها ستّة، وثُلثلها اثنا عشر.

وإن بقي منه[2] ثلاثة أو ستّة، فالسُّدس له والثُّلث.

مثال ما بقي منه ثلاثة : ستّة وستّون وشبهها. سُدسها أحد عشر، وثُلثها اثنان وعشرون.
ومثال ما بقي منه ستّة : اثنان وأربعون وشبهها. سُدسها سبعة، وثُلثها أربعة عشر.

وإن بقي غير ذلك، فاطرحه ثمانية ثمانية. فإن انطرح فالثُّمن له، والرُّبع.

مثاله : أربعة وستّون. فإنّها مُنطرحة بثمانية، وثُمنها ثمانية، ورُبعها ستة عشر.

وإن بقي منه[3] أربعة، فالرُّبع له.

مثاله : ثمانية وستّون. رُبعها سبعة عشر.

وإن بقى غير ذلك، فاطرحه سبعة سبعة. فإن انطرح فالسُّبع له.

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[1] في (م) وفي (ط) : سقطت الجملة: " وإن كان في أوّله خمسة، فالخُمس له".
[2] في (أ) وفي (ط) : سقط لفظ "منه".
[3] في (أ) : سقط لفظ "منه".

مثاله : أربعة عشر. فإنّها مُنطرحة بسبعة، وسُبعها اثنان.

وإن لم ينطرح، فليس له إلا النّصف، ونصفه فرد يُطلب في الأجزاء الصمّ[1].

مثاله : ستة وعشرون. فإنها غير منطرحة بواحد من الطروحات الثلاث و ليس لها إلّا النّصف، ونصفُها ثلاثة عشر، عدد فرد. فاعلمه.

وإن كان فردا، فإنّه ينطرح بطرحين : تسعة وسبعة.
فإن انطرح بتسعة، فله [31 و] **التُّسع والثُّلث.**

مثاله : واحد وثمانون. فإنّها مُنطرحة بتسعة، وتُسعها تسعة، وثُلُثها سبعة وعشرون.

وإن بقي منه ثلاثة أو ستّة، فالثُّلث له.

مثال ما بقي منه ثلاثة : تسعة وثلاثون. ثُلثها ثلاثة عشر.
ومثال ما بقي منه ستّة : ثلاثة وعشرون ومائة. ثُلثها واحد وأربعون.

وإن بقي غير ذلك، فاطرحه سبعة سبعة. فإن انطرح، فالسُّبع له.

مثاله : سبعة وسبعون. فإنّها مُنطرحة بسبعة، وسُبعها أحد عشر.

وإن لم ينطرح، فاطلبه في الأجزاء الصمّ[2] بالقسمة عليها، ولا تزال تقسم المطلوب حلّه على الأجزاء الصّم[3] حتّى تجد العدد[4] الذي ينقسم عليه، أو تنتهي إلى عدد يكون مُربّعُه أعظم من عددك[5] المفروض، أو يكون الخارج من القسمة مثل المقسوم عليه أو أقلّ منه، وتبقى بعد[6] القسمة بقية، فتعلم حينئذ أنّه من الأجزاء الصمّ. فتكون التسميّة منه بالاشتقاق منه.

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[1] في (س) وفي (م) وفي (ط) : سقط لفظ "الصم".
[2] في (س) وفي (م) وفي (ط) : سقط لفظ "الصم".
[3] في (س) وفي (ت) : "الصحيحة".
[4] في (س) : سقط لفظ "العدد". وفي (ب): "الأعداد".
[5] في (ط) : "أجزاء عددك". لفظ "أجواء" زائد.
[6] في (أ) : "من".

مثاله : لو أردنا حلّ واحد وعشرين ومائتين. (فنجدها لا تنطرح بالطّرحين. فنحتاج أن نقسمها على الأجزاء الصمّ، كما ذكرنا.)[1] فنجدها لا تنقسم على الأحد عشر الذي هو أوّلها، ومُنقسمة على الثلاثة عشر، ويخرج من القسمة سبعة عشر[2]. فهي إذًا مُركّبة من ثلاثة عشر في سبعة عشر. ولو كان مُربّع الثلاثة عشر مثلا أكثر من المقسوم[3] لعلمنا أيضًا أنّه أصمّ[4]. ولو كان الخارج مثل الثلاثة عشر المقسوم عليها أو أقلّ وتبقى بقية من المقسوم، لعلمنا أيضا أنّه أصمّ. فنُسمّي منه بالاشتقاق. فنقول مثلا : ثلاثة أجزاء (من كذا أو مائة [31 ظ] وعشرون جزءا)[5] من كذا وكذا و شبه ذلك.

فصل في وُجدان الأجزاء الصمّ

والصّنعة في ذلك تُسمّى الغربال[6]. وهي أن تضع الأعداد الأفراد المُتوالية من ثلاثة، ثمّ تعدّ من كلّ عدد منها بقدر ما فيه من الآحاد على التّوالي[7]، فحيث ما نفد العدد فما بعده مُركّب ويعدّه ذلك العدد. ثمّ لا تزال تفعل كذلك حتّى تنتهي إلى عدد يكون مُربّعه أعظم من آخر عدد في الغربال، فتعلم أنّ العمل قد تمّ. وكُلّ عدد عليه علامة مُركّب وكُلّ عدد لا علامة عليه أصمّ.

مثاله : أن[8] نضع الأعداد الأفراد (من ثلاثة)[9] على التّوالي (كما ذكرنا في جدول على هذه الصورة)[10] :

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[1] في (ط) : سقطت الجملة "فنجدها ... كما ذكرنا".
[2] في (ط) : سقط لفظ "عشر".
[3] في (م) وفي (ت) وفي (ط) : "مربع المقسوم". وهو لا يصح.
[4] في (م) وفي (ط) : سقط لفظ "أصم".
[5] في النسخ الاخرى : سقطت العبارات "من كذا أو مائة وعشرون جزءا".
[6] في (ب) : زيادة في حاشية الورقة 134و : وهو نص ابن مجدي في تعريف صناعة الغربال.
[7] في (ب) وفي (س) : "الولاء".
[8] في النسخ الاخرى : سقطت "ان".
[9] في (أ) : سقطت "من ثلاثة".
[10] في (م) : " كما ذكرنا. كما في جدول على هذه الصورة".
وفي (أ) : " كما ذكر في جدول. وهذه صورة الجدول لاستخراج الأجزاء الصمّ. يتلوه الجدول ".
وفي (ط) : "كما ذكر في الجدول على هذه الصورة".
وفي (ت) : "كما ذكر في الجدول على هذه الصورة. وحيث هي موضوعة في قفا هذه الورقة في جدول على وجه الصحّة بعد أن اختبرتها وحرّرت عملها على الوجه المُرضي المعروف عند أهل هذا الفنّ. فعليك بحفظها لتعرف المُركب من الأصمّ بمجرد وقوفك على العدد. فإنها أصول عظيمة لمعرفة ذلك وقد نتبعها في الجدول بالامتحان واحدة واحدة".

*25	23	*21	19	17	*15	13	11	*9	7	5	3
*49	47	*45	43	41	*39	37	*35	*33	31	29	*27
73	71	*69	67	*65	*63	61	59	*57	*55	53	*51
97	*95	*93	91	89	*87	*85	83	*81	79	*77	*75
*121	119	*117	*115	113	*111	109	107	*105	103	101	*99
*145	143	*141	139	137	*135	133	131	*129	127	*125	*123

فإن أردنا أن نعلم ما هو منها[1] مُركب بثلاثة، فنعدّ بها من بيتها، ننفد في بيت السّبعة، [32 و] فالتّسعة التي بعدها مُركبة من الثلاثة، فنعلم عليها علامة. ثمّ نعُدّ بها أيضًا من بيت التسعة، ننفد في بيت الثلاثة عشر، فالخمسة عشر التي بعدها مُركبة من الثلاثة أيضًا، فنعلم عليها علامة. وكذلك إلى آخر الغربال. وكذلك نعمل بالخمسة، والسّبعة، ولا نعدّ بالتسعة لأنها مُركبة، (ولا بكُلّ عدد مُركب)[2]. فإذا انتهينا في مثالنا إلى أن نعدّ بالثلاثة عشر، فنعلم أنّ العمل قد تمّ، لأنّ مُربّعها تسعة وستون[3] ومائة، وهي أكبر من الخمسة والأربعين والمائة التي هي آخر العدد في الغربال المفروض.

ولو شئنا أن نعمل غربالا أكبر من هذا أو أصغر، لجاز لأنّ الطّريقة في ذلك كله واحدة.
فكُلّ عدد في هذا الغربال مُعلم عليه مُركب وكُلّ ما ليس عليه علامة أصمّ، كما قد رأيت. وهذه[4] الأجزاء الصمّ لا يعُدّها إلا الواحد كما (ذكرنا أوّل الكتاب)[5].
فإنا لا نجد عددا نضربه في عدد فيخرج منه مثلا ثلاثة عشر ولا واحد وخمسون ومائة وشبه ذلك. فإذا قيل لك من أيّ الأعداد تركبت ثلاثة عشر، فنقول من ثلاثة عشر في واحد. والضرب في واحد لا يتضاعف، كما تقدم. وكذلك الجواب في سائرها. فاعلمه.

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[1] في في النسخ الاخرى : سقط "منه".
[2] في (م) : سقطت العبارات "ولا بكُلّ عدد مُركب".
[3] في كل النسخ الاخرى : " تسعون "، وفهو خطأ.
[4] في (أ) : سقط لفظ: "هذه".
[5] في (أ) : " ذكر أوّلا "، وفي (ت) : "ذكر أوّل الكتاب".

الباب السادس : في الجبر والحطّ

(الجبر هو الإصلاح والحطّ ضدّه.
والمُراد بالجبر والحطّ)[1] معرفة ما يضرب في عدد ما فيأتي منه المطلوب.
ولا يكون الجبر إلّا من القليل إلى الكثير والحطّ على العكس (يعني من الكثير إلى القليل)[2].

مثال الجبر : لو قيل بماذا أجبر ثمانية، مثلا، حتى تكون تسعة عشر؟ ومثال الحط لو قال : بماذا أحط خمسين، مثلا، حتى تكون[3] ستة؟

والعمل في الجبر أن تقسم المجبور إليه على المجبور، يخرج المطلوب.

مثاله : لو قيل بماذا أجبر ثلاثة حتى تكون ستّة؟ فتقسم الستّة المجبور إليها على الثلاثة المجبورة يخرج اثنان، (فإذا ضربت ذلك في الثلاثة حصل ستّة)[4]، وهو المطلوب.

والعمل في الحط أن تُسمّي المحطوط إليه من المحطوط، فما خرج فهو الجواب.

مثاله : لو قيل بماذا أحط ثمانية حتى [32 ظ] تكون ثلاثة؟ فتُسمّي الثلاثة المحطوط إليها من الثمانية المحطوطة، يكون ذلك ثلاثة أثمان، (فإذا ضربت الثلاثة الأثمان في الثمانية حصل ثلاثة)[5]، وهو الجواب.
فقد كمل[6] القسم الأوّل بحمد الله (وحسن توفيقه)[7].

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[1] في (م) : سقطت العبارات "الجبر هو الإصلاح والحطّ ضدّه. والمُراد بالجبر والحطّ".
[2] في (س) : سقطت الجملة "يعني من الكثير إلى القليل".
[3] في (أ) وفي (ب) : "تصير".
[4] في (أ) : الجملة "فإذا ضربت ... ستة" زائدة.
[5] في (أ) : الجملة "فإذا ضربت ... ثلاثة" زائدة.
[6] في (أ) : "تمّ "
[7] في (أ) : "وعونه"، وفي (ط) : "وحسن عونه". وفي (ت) : سقطت العبارة.

القسم الثاني :

في الكسور

مخطوط المدينة (م) : 32 ظ – 40 ظ
مخطوط استانبول (أ) : 79 و – 85 و
مخطوط أكسفرد (ب) : 134 ظ – 141 ظ
مخطوط تهران (ط) : 32 ظ – 38 ظ
مخطوط تونس (ت) : 15 ظ – 19 ظ

الكسور[1] هي النّسبة التي بين عددين متى كانت جزءا أو أجزاء.
فالنّسبة التى بين الجزء وسميّه تسمّى كسرا.

مثال ذلك : ثلاثة وستّة[2]. فالإرتباط (الذي يحصل)[3] عند نسبة الأصغر (منها إلى[4] الأكبر)[5]، هو المُسمّى بالكسر، وليس باسم للثلاثة وحدها مع اعتبار الانفصال ولا إسمًا للستّة وحدها أيضًا ولا لمجموعهما لأنّه لمّا لم[6] تكن تلك النّسبة محسوسة وإنّما هي معنى معقول خاصة، يُسمّى بالكسر تشبيها بالأرض ذات الكسور أي ذات الصعود والهبوط. كما شبهوه أيضًا بالسّطح والجسم والخط. وليس في الكمّ المُنفصل شيء من ذلك إلّا مُجرّد الشبه[7].
ثمّ إنّ الكسر له أسماء، منها النّصف وغير ذلك ممّا سنذكره[8] إن شاء الله تعالى.

ويتعلّق بها (أي بالكسور)[9] بحسب مقصدنا ستّة أبواب.

---

[1] في (م) : سقط لفظ "الكسور".
[2] في (م) : سقط لفظ "ستة".
[3] في (أ) وفي (ت) : "بينهما".
[4] في (م) : سقط لفظ "إلى".
[5] في (أ) وفي (ت) : "منهما للأكبر".
[6] في (ب) : سقط لفظ "لم".
[7] في باقي النسخ : "التشبيه".
[8] في (أ) وفي (ت) : "بينهما".
[9] في (س) وفي (أ) : "من الأعمال". وفي (ب) وفي (ط) وفي (ت) : "يعني الكسور".

الباب الأوّل : في أسماء الكسور وبسطها.[1]

وللكسور عشرة أسماء بسائط. أوّلها النّصف، وهو أكبرُها، وصورتُه: $\frac{1}{2}$.

ثمّ الثلث، وصورتُه: $\frac{1}{3}$. ثمّ الرّبع، وصورتُه: $\frac{1}{4}$. ثمّ الخُمس، وصورتُه: $\frac{1}{5}$.

ثمّ السّدس، وصورتُه: $\frac{1}{6}$. ثمّ السّبع، وصورتُه: $\frac{1}{7}$. ثمّ الثّمن، وصورتُه: $\frac{1}{8}$.

ثمّ التّسع، وصورتُه: $\frac{1}{9}$. ثمّ العُشر، وصورتُه: $\frac{1}{10}$.

ثمّ الجزء [وهو جنس فيه أصناف كثيرة: تقول جزء من أحد عشر[2]، وصورته: $\frac{1}{11}$ وجزء من سبعة عشر[3]، وصورتُه: $\frac{1}{17}$، وشبه ذلك][4].

وتثنّى هذه الكسور وتجمع. وينتهي بجمع[5] كلّ كسر منها إلى أقلّ من سميّه بجزء.
(والسمّي[6] هو الأكبر من العددين المنسوب أحدُهُما [33 و] إلى الآخر)[7].

مثاله : نقول رُبع ورُبعان وثلاثة أرباع، ولا نقول أربعة أرباع. ونقول أيضًا سُبع وسُبعان وثلاثة أسباع (وأربعة أسباع)[8] وخمسة أسباع (وستّة أسباع)[9]، ولا نقول سبعة أسباع. فاعلمه.

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[1] في (م) : سقطت العناوين "الجزء الثاني ... القسم الأول".
[2] في (ت): زيادة لفظ "جزءا".
[3] في (ت): زيادة لفظ "جزءا".
[4] في (س) : سقطت الجملة "وهو جنس ... وشبه ذلك"، ويعتبرها ناسخ (ت) من شرح الهواري.
[5] وفي (ت) : "ينتهي جمعها بجمع".
[6] في (أ) في (ب) : "المسمى منه". وفي (ط) : سقطت العبارة "بجزء" وصارت الجملة : "يخرج السمي".
وفي (ت) : "المُسمّى".
[7] في (س) : غابت الجملة "والسمي ... إلى الآخر".
[8] في (م) : سقطت العبارة "وأربعة أسباع".
[9] في (م) : سقطت العبارة "وستة أسباع".

وتضاف هذه الأسماء البسائط بعضها إلى بعض، فيصير منها إسم مؤلّف من إسمين أو من أكثر من ذلك.

مثل أن نقول ثُمُنان وسُبع ثُمُن، وصورته[1]:

$$\frac{1\ 2}{7\ 8}.$$

(وما أشبه ذلك)[2].

ومثال منه آخر : ثمانية أتساع وأربعة أسباع التسع وسُدسا سُبع التسع وثلث سُدس سُبع التسع، وصورته[3] :

$$\frac{1\ 2\ 4\ 8}{3\ 6\ 7\ 9}.$$

وما أشبه ذلك.

والبسط هو أن تردّ[4] جميع ما فرض لنا في مسألة بعينها إلى أدق[5] كسر فيها.

[ومعرفة أدق[6] كسر (فيها)[7] هو الجزء المُسمّى بجميع أئمّة المسألة.
فإذا كان الكسر مُنتسبًا (مثل)[8] خمسة أسداس وأربعة أخماس سدس وثلثي خُمس سدس، وصورته[9]:

$$\frac{2\ 4\ 5}{3\ 5\ 6}$$

(بأن نضرب)[10] ما على أوّل إمام في الإمام الذى (يليه، وهو)[11] صرفه إلى المُسمّى منه، وهو أسداس، فيصير أخماس أسداس، لأنّ الإمام الثاني هو[12] عدّة

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[1] في (م) : الصورة خاطئة.
[2] في (ب) وفي (ط) وفي (ت) : سقط الجملة "و ما أشبه ذلك".
[3] في (م) : الصورة خاطئة.
[4] في (م) : "تردد".
[5] في (ط) : "أدنى".
[6] في **رفع الحجاب** : الصورة غائبة..
[7] في (ب) : سقط لفظ "فيها".
[8] في (ت) : "على".
[9] في (م) : الصورة خاطئة.
[10] في **رفع الحجاب** : "فان ضرب".
[11] في **رفع الحجاب** : "يليه هو".
[12] في (ط) وفي (ت) : سقط لفظ "هو".

ما في الواحد من الإمام الأوّل من الآحاد. فنجمعه مع الأربعة التي فوقه لأنّها أخماس أسداس، ثمّ نضرب ذلك في الثلاثة، الإمام الثالث، وهو عدّة ما في الواحد من الإمام الثاني من الآحاد[1] . فيكون الخارج أثلاث أخماس أسداس. فنجمعه مع الإثنين لأنّها ثلثا خُمس سُدس، يكون ذلك بسط المسألة، وهو ما فيها من أثلاث أخماس الأسداس، (وذلك تسعة وثمانون)[2]. فيكون ضرب الأئمّة بعضها في بعض، الذي هو (تسعون، هو ما في)[3] الواحد الصّحيح من تلك الأجزاء. و جزء واحد منها[4] هو[5] أدقّ كسر[6] في المسألة.

وإذا كان [33 ظ] الكسر[7] مُختلفا، مثل خمسة أسداس وأربعة أخماس[8]، وصورتها[9] :

$$\frac{4}{5}\,\frac{5}{6}.$$

فنضرب خمسة الأسداس في خمسة، إمام الأخماس، وذلك عدد ما في السّدس الواحد من أخماس، (فيصير أخماس أسداس)[10]. [ونضرب الأربعة الأخماس في ستّة، (إمام الأسداس، وذلك)[11] عدد ما في الخُمس الواحد من الأسداس][12]. فتصير أسداس أخماس. ([13]) فمجموع هذين البسطين[14] أجزاء من ثلاثين، وهي أسداس أخماس أو أخماس أسداس، (كلّ ذلك سواء)[15].

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[1] في (ب) : "الآحاد" وفي باقي المسخ : "الأثلاث".
[2] في **رفع الحجاب** : الجملة غائبة.
[3] في (أ) : سقطت الجملة. في (ط) وفي (ت) : "تُسع هو"، وهذا خطأ.
[4] في (ب) : سقط لفظ "منها".
[5] في (ب) : سقط لفظ "هو".
[6] في **رفع الحجاب** وفي (ط) وفي (ت) : "جزء"
[7] في (ط) وفي (ت) : " الجزء ".
[8] في (ط) وفي (ت) : زيادة "سدس"، وهذا خطأ.
[9] في (ط) وفي (ت) : الصورة خاطئة. وهي غائبة من **رفع الحجاب**.
[10] في (أ) وفي (ت) : سقطت الجملة "فيصير أخماس أسداس".
[11] في **رفع الحجاب** وفي (م) وفي (ب) : غابت " إمام الأسداس، وذلك ".
[12] في (ط) وفي (ت) : غابت الجملة " ونضرب الأربعة الأخماس ... الأسداس ".
[13] في **رفع الحجاب**، 273: 10-11 زيادة الجملة: "وقد علمت مما تقدم من القسمة أن سدس خمس هو بعينه خمس سدس، وهو جزء من ثلاثين."
[14] في (أ) وفي (ت) : "السطرين".
[15] في **رفع الحجاب** : سقطت الجملة.

وإذا كان الكسر[1] مُبعّضا، وهو أخذ الكسور بعضها من بعض، وهو إضافتها في اللفظ[2].

مثاله : ثلاثة أرباع خمسة أسداس وصورتها[3] :

$$\frac{5 \mid 3}{6 \mid 4}.$$

فنضرب الثلاثة في الخمسة، تكون خمسة عشر رُبع سُدس أو سُدس رُبع، وهي أجزاء من أربعة وعشرين جزءا[4] في الواحد، لأنّ خمسة الأسداس لما أردنا ثلاثة أرباعها، وذلك أمّا بأخذ رُبعها وهو خمسة أرباع سدس وضربه في ثلاثة بخمسة عشر رُبع سُدس، وأمّا بأخذ رُبع ثلاثة أمثالها. وثلاثة أمثالها هو ضربها في ثلاثة بخمسة عشر سُدسًا، ورُبع ذلك خمسة عشر رُبع سُدس، وهو خمسة عشر سُدس رُبع أيضًا. فثلاثة أرباع خمسة أسداس هي (خمسة أسداس)[5] ثلاثة أرباع، لأنّ ضرب ثلاثة في خمسة كضرب خمسة في ثلاثة.][6] فاعلم.

وهو مختلف باختلاف الكسور، يعني البسط.
وهي خمسة أنواع : مُفرد[7] ومُنتسب ومُختلف ومُبعّض ومُستثنى.

فبسط المُفرد[8] ما عليه[9].

مثل قولنا سبع[10]، بسطه واحد الذي هو فوق الخطّ. وكذلك لو كان مُضافًا مثل قولنا (ثُلث سُبع)[11]، بسطُه واحد الذي فوق الخطّ أيضًا.

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[1] في (ط) وفي (ت) : "البسط"، وهو خطأ.
[2] في **رفع الحجاب** : زيادة الجملة "وهو نسب مؤلفة أيضا". صفحة 273: 12-13.
[3] في (م) : الصورة خاطئة. وهي غائبة من **رفع الحجاب**.
[4] في (أ) وفي (ط) : سقط لفظ "جزءا".
[5] في (م) وفي (ط) : سقطت "خمسة أسداس".
[6] الفقرات: " ومعرفة أدق ... خمسة في ثلاثة " منقولة من **رفع الحجاب**، صفحة 272: 14 إلى 273: 19 ، ما عدا بعض التغييرات في بعض العبارات وكغياب صور الكسور.
[7] في (م) : "منفرد".
[8] في (م) : "منفرد".
[9] في (أ) : "ما على إمامه "، وفي (ج) : "ما عليه أي ما على إمامه".
[10] في (أ) : " تسبع $\frac{1}{9}$" . وصورة الكسر غائبة في باقي النسخ.
[11] في (أ) وفي (ت) : " ثلث ثلث سُبع".

وبسط المُنتسب ما على أوّل إمام مضروبا في الإمام الذى يليه [34 و] بالحمل إلى آخر السّطر، أو ما على أوّل إمام مضروبا فيما بعد إمامه (من الأئمة)[1]، وما على ثاني إمام (مضروبا أيضًا)[2] في ما بعد إمامه من الأئمة، وكذلك حتّى يتمّ السّطر، ويجمع الجميع.

مثاله : لو قيل ابسط خمسة أثمان وأربعة أسباع الثمن وثلاثة أخماس سُبع الثمن وثلثي خمس سبع الثمن وصورتها :

$$\frac{2\ 3\ 4\ 5}{3\ 5\ 7\ 8}.$$

فنضرب الخمسة التي على أوّل إمام في السّبعة، ثاني إمام، ونحمل الأربعة التي عليها بتسعة وثلاثين، نضربها أيضًا في الخمسة، الإمام الثالث، ونحمل الثلاثة التي عليها بثمانية وتسعين ومائة. نضربها أيضًا في الثلاثة، الإمام الرّابع، ونحمل الإثنين التي[3] عليها بستّة[4] وتسعين وخمسمائة، وهو أثمان أسباع أخماس أثلاث، وهو البسط، وصورته[5] : 596.

وبالوجه الثاني نضرب الخمسة التي على الإمام الأوّل في سائر الأئمّة سوى إمامها، كما ذُكر. يكون الخارج خمسة[6] وعشرين وخمسمائة، نحفظه.
ثمّ نضرب الأربعة التي على ثاني إمام فيما عدا إمامها من التي بعده. يكون الخارج ستّين، نحفظه أيضًا. ثمّ نضرب الثلاثة التي على ثالث إمام، في الثلاثة، الإمام الرّابع. ونحمل ما عليه، (لأنّه لم يبق بعد إمامه شيء[7]) [8]، بأحد عشر. نجمعها مع المحفوظين، يكون ذلك ستّة وتسعين وخمسمائة، وهو البسط (كما تقدم)[9].

---

[1] في (أ) وفي (ت) : سقطت "من الأئمة".
[2] في (أ) وفي (ب) : زيادة "مضروبا أيضًا".
[3] في (أ) وفي (ت) : سقط لفظ "التي".
[4] في (ط) وفي (ت) : "سبعة"، وهو لا يصح.
[5] في (ط) وفي (ت) : "597"، وهو لا يصح.
[6] في (م) : سقط لفظ "خمسة".
[7] في (م) : سقط لفظ "شيء".
[8] في (أ) وفي (ط) وفي (ت) : "لأنه ما بعد إمامه".
[9] في (ب) : زيادة.

وبسط المُختلف بضرب بسط[1] كلّ قسم في إمام غيره وتجمع الجميع.

مثاله : لو قيل ابسط خمسة أسباع ونصف سُبع وأربعة أسداس.
فننزلها تحت خطّين على هذه الصورة:

$$\frac{4}{6}\ \frac{1\ \ 5}{2\ \ 7}.$$

(فنجد القسم الأوّل من المُنتسب، فنأخذ)[2] بسطه كما تقدّم. يكون أحد عشر. نضربها في الستّة، إمام القسم الثاني، يكون الخارج ستّة وستّين. نحفظه. ونجد القسم الثاني من المفرد[3]، فبسطه أربعة التي هي[4] على الإمام. نضربها أيضًا في أئمّة القسم الأوّل (بستّة)[5] وخمسين. نجمعها مع المحفوظ. [34 ظ] يكون المجتمع إثنين وعشرين ومائة. وهو أسباع أنصاف أسداس، وهو البسط، وصورته[6]: 122.

وبسط المُبعّض بضرب ما فوق الخطّ بعضه في بعض.

مثاله : لو قيل ابسط سبعة أتساع[7] خمسة أسداس ثلاثة أعشار.
فننزلها في سطر ونفرّق بينها بعلامات أبدًا على هذه الصورة[8] :

$$\frac{3 \mid 5 \mid 7}{10 \mid 6 \mid 9}.$$

فنضرب السّبعة التي على الإمام الأوّل في الخمسة التي على الإمام الثاني وما اجتمع في الثلاثة التي على (الإمام الثالث)[9]، يكون البسط. وذلك خمسة ومائة أتساع أسداس أعشار، وصورته: 105.

---

[1] في (س) : سقط لفظ "بسط".
[2] في (ت) : "فتسطّح بسط الأول من المنتسب بأن تأخذ "
[3] في (م) وفي (ب) : "من المفرد".
[4] في (م) : سقط لفظ "هي".
[5] في (أ) : "بتسعة "، وهو خطأ.
[6] في (أ) وفي (ط) وفي (ت) : لا إشارة إلى الصورة.
[7] في (م) : " أسباع"، وهو خطأ.
[8] في (م) : $\frac{3 \mid 5 \mid 7}{10\ \ 6\ \ 4}$. وفي (ب) : $\frac{3 \mid 5 \mid 7}{10 \mid 6 \mid 4}$. وفي (ت) : $\frac{3\ \ \ 5\ \ 7}{10 \mid 6 \mid 4}$.
[9] في (أ) وفي (ت) : "ثالث إمام".

وبسط المُستثنى، أمّا المُنقطع فكالمُختلف. ويُطرح الأقلّ من الأكثر.

والمُنقطع هو الذي يكون ما (بعد 'الا')[1] ليس مأخوذا ممّا قبلها، إنّما هو مأخوذ من الواحد، وحينئذ إستثني.
مثل قولنا : نصف إلّا ثلثا، معناه إلّا ثلث واحد، فبعد أخذ الثلث من الواحد استثني من النّصف.
وقوله "فكالمُختلف" يعني أنّك تصيّر[2] ما (قبل 'الا')[3] مع ما بعدها على قسمين مُختلفين، فنضرب بسط كلّ قسم في إمام غيره، وهنا بطرح الأقلّ من الأكثر.

مثاله : لو قيل ابسط ستّة أثمان إلّا تسع واحد[4]. فننزلها في سطر على هذه الصورة:

$$\frac{6}{8} \text{الا} \frac{1}{9}.$$

فنضرب السّتّة، بسط ما قبل 'الا'، في التسعة، إمام ما بعدها، بأربعة وخمسين. نسقط منها ضرب الواحد، بسط ما بعد 'إلا'، في الثمانية، إمام ما قبلها، وذلك ثمانية. الباقي ستّة وأربعون أثمان أتساع ، وهو البسط. وكذلك العمل فيما أشبه هذا، (على ما ذكر)[5].

وأمّا المُتّصل، فيضرب بسط المُستثنى منه في بسط المُستثنى، | 35 و | **ويضرب أيضًا في أئمّته ويطرح الأقلّ من الأكثر.**

والمُتّصل هو الذى يكون ما (بعد 'الا')[6] مأخوذا ممّا قبلها بغير واسطة[7].
مثل قولنا : نصف إلّا ثلثه[8]، وثلث النّصف سُدس. فكأنّه قال ثلثا[9] أو نصفا إلّا سدسًا، فصار[1] مُنقطعًا.

---

[1] في (ت) : "ما بعد الاول"، ولا يصح هذا التعبير.
[2] في (أ) : "تضرب".
[3] في (ط) وفي (ت) : "ما قبلها".
[4] في (أ) : سقط لفظ "واحد".
[5] في (أ) وفي (ط) : سقطت الجملة "على ما ذكر".
[6] في (ط) وفي (ت) : "بعدها".
[7] في (م) : "وساطة".
[8] في (ب) و (ت) : "ثلثا".
[9] في (م) : "ثلثان"، وهوخطأ.

فلو قيل : ابسط ستّة أسباع ونصف سُبع إلّا ثلثها[2].
فننزلها في سطر على هذه الصورة[3]:

$\frac{6}{7}$ $\frac{1}{2}$ الا $\frac{1}{3}$.

فنأخذ بسط[4] ما قبل[5] 'الا'، وهو المُستثنى منه، فنضربه في أئمّة ما (بعد 'الا')[6]، وهو المُستثنى، يكون ذلك تسعة وثلاثين. نحفظه. ثمّ نضرب بسط المُستثنى منه أيضًا في بسط المستثنى، يكون (تكملة أيضًا)[7] ثلاثة عشر. فنسقطه[8] من المحفوظ. الباقي ستّة وعشرون أسباع أنصاف أثلاث، وهو البسط.

تكملة :
[وإذا تكررت الاستثناءات وكانت كلّها بواو العطف على الأوّل، وكانت مع الأوّل كلّها إمّا مُتّصلة بالمُستثنى منه وإمّا مُنفصلة، فهي كلّها كسور مُختلفة مُستثناة من المُستثنى منه][9].

مثال الأوّل : لو قيل ابسط خمسة وثلثا إلّا (رُبعها وإلّا سُبعها وإلّا خُمسها)[10].
فننزلها في سطر على هذه الصورة[11]:

$5\ \frac{1}{3}$ الا $\frac{1}{4}$ والا $\frac{1}{7}$ والا $\frac{1}{5}$.

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[1] في (م) : سقط لفظ "فصار".
[2] في (م) وفي (ت) وفي (ط) : "ثلثا".
[3] في (م) : الصورة خاطئة، وفي (ط) : " $\frac{6}{7}$ $\frac{1}{2}$ الا $\frac{1}{3}$ ".
[4] في (أ) : سقط لفظ "بسط".
[5] في (م) : سقط لفظ "قبل". وفي (ط) : "ثالث امام".
[6] في (أ) وفي (ط) وفي (ت) : "ما بعدها".
[7] في باقي النسخ : "ذلك".
[8] في باقي النسخ : "تسقط ذلك".
[9] في (س) : التكملة غائبة وهي منقولة من **رفع الحجاب**، صفحة 275: 8-9، ويعتبرها ناسخ (ت) تابعة لشرح الهواري.
[10] في (أ) وفي (ت) : "رُبعا وإلّا سُبعا وإلّا خُمسا".
[11] في (أ) : 5 و $\frac{1}{3}$ الا $\frac{1}{4}$ الا $\frac{1}{7}$ الا $\frac{1}{5}$. وفي (ط) : 5 $\frac{1}{3}$ الا $\frac{1}{4}$ الا $\frac{1}{7}$ الا $\frac{1}{5}$

فالرّبع والسّبع والخُمس كُسور مُختلفة مُستثناة من الخمسة والثلث، المستثنى منها، وهي كلّها مُتّصلة بها. فنُزيل حرف الاستثناء الثاني والثالث. فيصير وضغ المسألة هكذا[1]:

$$5\frac{1}{3}\text{الا}\frac{1}{4}\ \frac{1}{7}\ \frac{1}{5}\ .$$

فنعمل على ما تقدّم. يخرج البسط إثنا[2] عشر وتسعمائة أثلاث أرباع أسباع أخماس، وصورته[3]: 912.

ومثال الثاني : لو قيل في المسألة بعينها إلّا رُبع واحد وإلّا سُبع واحد وإلّا خُمس واحد. فهي إذًا كسور مُختلفة مُستثناة [35 ظ] من المُستثنى منه، وهي كلّها مُنفصلة. فيصير وضعها أيضًا كالمُتّصلة. ويعمل فيها ما تقدّم. يخرج البسط واحدا وتسعين وتسعمائة وألفا أثلاث أرباع أسباع أخماس، وصورته: 1991 .

[وإن تكررت الاستثناءات بغير حرف عطف بحيث أن يكون كلّ مُستثنى مُستثنى[4] ممّا يليه مما قبله، وكلّها مُنفصلة[5] أو مُتّصلة.
فنأخذ المُستثنى والمستثنى منه من آخر[6] المسألة، فنعمل فيهما ما ذكر، (مُتّصلين كانا أو مُنفصلين[7]). فما كان بسطهما، فهو مُستثني ممّا يليه قبله. فنعمل (فيهما كذلك)[8]. فما حصل[9] بسط ذلك، فهو مُستثنى ممّا يليه قبله. وكذلك إلى أوّلها.][10]

وإن شئنا عملنا في المُنقطع منها بالوجهين الآخرين المتقدّمين.

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[1] في (أ) وفي (ط) وفي (ت) : "على هذه الصورة".
[2] في (م) : "ستة"، وهو خطأ.
[3] في (م) : "916"، وهو خطأ.
[4] في (م) : سقط لفظ "مستثنى".
[5] في (ب) وفي (ط) وفي (ت) : "منقطعة ".
[6] في (ط) في (ت) : "أجزاء".
[7] في (م) : "منقطعين".
[8] في (ب) وفي (ط) وفي (ت) : "على ما ذُكر".
[9] في (أ) : "حصل من". وفي (ط) و في (ت) : " فضل "
[10] الفقرة : " وإن تكررت الاستثناءات بغير حرف عطف ... يكون وضع ما بعد "إلا" مبعضا منفصلا." من **رفع الحجاب**، صفحة 275: 14-15. لاكن تصرف الهواري في نقله.

[ومتى كان بعضُها[1] مُنقطعًا وبعضُها[2] مُتّصلا، فلا بُدّ من ردّ[3] المُتّصل على صورة المُنقطع في الوضع. فتصير كلّها مُنقطعة[4].
مثل خمسة أسداس (إلّا ثلاثة أرباعها، هوخمسة أسداس إلّا ثلاثة أرباع خمسة أسداس)[5]، فيكون وضع ما بعد 'إلّا' مُبعضًا مُنقطعا[6].][7]

والصّحيح إن كان مع[8] هذه الكسور في مسألة من أوّلها ضرب في الأئمّة وجمع مع البسط لتصير كسورا[9].

مثاله : لو قيل ابسط خمسة وخمسة أسداس وثلاثة أرباع سُدس. فننزلها في سطر هكذا[10]:

$$.\frac{3}{4}\frac{5}{6}5$$

فنضرب الخمسة الصّحيحة في الستّة، الإمام الأوّل، وما اجتمع في الأربعة، الإمام الثاني، فيكون الخارج: (عشرين ومائة)[11]. (نجمعه مع بسط الكسور، وذلك ثلاثة وعشرون. فيكون المجتمع ثلاثة وأربعين)[12] ومائة أسداس[13] أرباع، وصورته: 143. وهو البسط.

وإن كان[14] في آخرها ضُرب فيه البسط، (لأنّ تلك الكسور مبعضة منه)[15].

[1] في **رفع الحجاب** : "بعضُ الاستثناءات". وفي (أ) : "بعضًا".
[2] في (أ) : "بعضًا".
[3] في (أ) : سقط لفظ "ردّ".
[4] في (أ) : سقط لفظ "منقطعة".
[5] في (ت) : سقطت الجملة " إلّا ثلاثة ... خمسة أسداس".
[6] في (ت) : سقط لفظ "منقطعا".
[7] الفقرة : "ومتى كان بعض ... يكون وضع ما بعد "إلا" مبعضا منقطعا." منقولة من **رفع الحجاب**، صفحة 275: 3-5.
[8] وفي (ط) وفي (ت) : "في".
[9] في (س) : سقط العبارة "لتصير كسورا".
[10] في (ت) : الصورة : $\frac{3}{4}\frac{5}{7}5$ ، وهي خاطئة.
[11] في النسخ الاخرى : "مائة وعشرين".
[12] في (م) : سقطت الجملة "نجمعه مع ... ثلاثة وأربعين".
[13] في (ت) : "أسباع"، وهو خطأ.
[14] في (أ) : "كان الصحيح".
[15] في (س) : سقطت الجملة. وفي (أ) وفي (ط) : "لأنّه تكون تلك الكسور مبعضة"، ويعتبر ناسخ (ت) هذه الجملة تابعة لشرح الهواري.

مثاله[1] : لو قيل ابسط أربعة أسباع وستّة أثمان عشرة. فننزلها في سطر هكذا:

$$.10\frac{6}{8}\ \frac{4}{7}$$

[36 و] فيكون بسط الكسور على ما تقدّم أربعة وسبعين. نضربها في العشرة الصّحيحة، تكون أربعين وسبعمائة أسباع أثمان، وهو البسط. وصورته[2]: 740.

وإن كان في وسطها، فبإضافته إلى ما قبله يكون مؤخرا، وبإضافته إلى ما بعده يكون مقدّما – فتبسُطه على إحدى الإضافتين – ومع الباقي كالمُختلف في التأخير، وفي التّقديم يضرب في مبسوط[3] الباقي.

[ومعنى الإضافة إلى ما قبله أن يكون الكسر[4] الأوّل مأخوذا من الصّحيح وحده ويكون معه قسما والكسر الباقي قسما مثل المُختلف، فتضرب بسط كلّ قسم في إمام غيره وتجمع الجميع][5].

مثاله : لو قيل ابسط أربعة أتساع خمسة وثلاثة أسداس. فننزل المسألة في سطر هكذا:

$$.\frac{3}{6}5\frac{4}{9}$$

فنضرب الأربعة التي على التّسعة في الخمسة الصّحيحة بعشرين، وهي بسط القسم الأوّل، فنضربها في السّتّة، إمام القسم الثاني، (بمائة وعشرين. نحفظها. ثمّ نضرب الثلاثة، بسط القسم الثاني)[6]، في التّسعة، إمام القسم الأوّل، بسبعة وعشرين. نجمعها مع المحفوظ. يكون سبعة وأربعين ومائة أسداس أتساع، وهو البسط. وصورته[7]: 147.

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[1] في (أ) وفي (ط) : " مثال منه "
[2] في (أ) وفي (ط) وفي (ت): لا إشارة إلى الصورة.
[3] في (أ) : "يكون الكسر". وفي (ت) : "يكون المختلف"
[4] في (ط) وفي (ت) : " المختلف "
[5] في (س) : كلّ الفقرة "ومعنى الإضافة ... تجمع الجميع" غائبة، ويعتبرها (ت) من شرح الهواري.
[6] في (م) : سقطت الجملة " بمائة ... القسم الثاني".
[7] في (أ) وفي (ط) وفي (ت): لا إشارة إلى الصورة.

[والإضافة إلى ما بعده أن يكون الكسر الأوّل مأخوذا من الصّحيح والكسر الذي بعده. فالصّحيح مُضاف إلى ما بعده وهو مُقدّم. فتبسطه معه وتضرب ذلك في مبسوط الباقي، وهو الكسر الأوّل، لأنّه مُبعّض منه][1].

مثاله : لو قيل ابسط ثلثي سبعة وأربعة أسباع. فننزلها هكذا[2]:

$$.\frac{4}{7}\,7\,\frac{2}{3}$$

فنضرب السّبعة الصّحيحة في السّبعة، إمام الكسر، ونحمل (الأربعة التي فوقها)[3] ونضرب ذلك في إثنين، بسط الباقي (الذي هو الكسر الأوّل. يكون)[4] المجتمع ستّة ومائة أثلاث أسباع، |36 ظ| وهو البسط: وصورته[5]: 106.

[ويُفهم[6] من ذلك أنّ كلّ كسر، وإن كان أكثر[7] من كسر واحد يكون مأخوذا (من الصّحيح وحده)[8]، فإنّ الصّحيح مؤخر عنه، فنبسطه معه كقسم واحد ومع سائر الكسور التي لا تؤخذ منه كالمُختلفة. وإنّ كلّ كسر، وإن كان أكثر[9] من كسر واحد يكون مجموعا مع الصّحيح، فإنّ الصّحيح مقدّم عليه، فنبسطه معه كقسم واحد، ويُضرب بسطه في بسط الكسور المأخوذة من ذلك[10] الصّحيح وما معه (من الكسور. فإن بقي (في المسألة كسور)[11] غير مأخوذة من الصّحيح وما معه)[12]، فهي[13] كسور مُختلفة ويصير الصّحيح وما معه، والمأخوذ من ذلك قسما

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[1] في (س) : كلّ الفقرة "والإضافة ... مبعض منه" غائبة، ويعتبرها (ت) من شرح الهواري.

[2] في (ت) : $\frac{4}{7}\ \frac{2}{7\ 3}$ ، وهذه الصورة خاطئة.

[3] في (م) : سقطت الجملة "الأربعة التي فوقها".

[4] في (م) : سقطت الجملة "الذي هو الكسر الأوّل. يكون".

[5] في (أ) : سقط لفظ "وصورته". وفي (ط) وفي (ت): لا إشارة إلى الصورة.

[6] في (م) : سقط لفظ "ويفهم".

[7] في (أ) : "أكبر".

[8] في (م) : سقطت العبارة "من الصّحيح وحده".

[9] في (أ) : "أكبر".

[10] في (أ) : سقط لفظ "ذلك".

[11] في (م) : سقطت الجملة "في المسألة كسور".

[12] في (أ) : سقطت الجملة " من الكسور. فإن بقي في المسألة كسور غير مأخوذة من الصّحيح وما معه".

[13] في (أ) : "فهو".

واحدا، وكلّ كسر من تلك الكسور المُختلفة قسما. فيضرب بسط كلّ كسر[1] في إمام غيره ونجمع الجميع.][2]

وينبغي أن يزال الإشتراك بين البسط والأئمّة.

وقد ذكرنا وجه العمل في ذلك بالحلّ في باب القسمة.

[وتختصّ الكُسُور المُبعّضة أن (يزال)[3] الإشتراك فيها قبل البسط، لأنّ ما فوق الخطّ[4] هي الأعداد التي ينحلّ إليها البسط وما تحت الخطّ هي الأعداد التي تنحلّ إليها الأئمّة، فنسقط المُتكرّر فيها من كلّ واحد منهما.][5]

---

1 في **رفع الحجاب** : " قسم "

2 الفقرة : " ويفهم من ذلك ... ويجمع الجميع." منقولة من **رفع الحجاب**، صفحة 276: 16 إلى 277: 3.

3 في (أ) : "لا يزال"، وهو خطأ.

4 في (ط) : زيادة "و ما تحت الخط"، وهو خطأ.

5 الفقرة منقولة من **رفع الحجاب**، صفحة 277: 4-6.

الباب الثاني في جمع الكسور و طرحها

والعمل في الجمع أن تضرب[1] بسط كلّ سطر في أئمّة الآخر وتقسم[2] المجموع على الأئمّة. وفي الطرح تسقط الأقلّ من الأكثر قبل القسمة على الأئمّة.

مثال من الجمع : لو قيل اجمع ثلاثة وأربعة أخماس وستّة أثمان إلى أربعة أعشار وثلاثة أثمان عشر ونصف ثمن العشر.
فننزل المجموع في سطر والمجموع إليه في سطر آخر تحته على هذه الصورة[3]:

$$\frac{6}{8}\frac{4}{5}3$$
$$.\ \frac{1\quad 3\quad 4}{2\quad 8\quad 10}$$

[37 و] فنبسط الأعلى كما تقدّم. يكون بسطُهُ إثنين وثمانين ومائة، نضربه في أئمّة الأسفل، يكون ذلك عشرين ومائة وتسعة وعشرين ألفا. فنحفظه. ثمّ نضرب بسط الأسفل، وذلك أحد وسبعون، في أئمّة الأعلى، يكون الخارج أربعين وثمانمائة وألفين. نجمعه مع المحفوظ، يكون ستّين وتسعمائة وأحد وثلاثين ألفا، وصورته[4]: 31960. فنقسمه على أئمّة السّطرين[5]، فما خرج فهو المطلوب. وذلك أربعة وتسعة أعشار وسبعة أثمان العُشر ونصف ثمن العشر وصورته:

$$.\ \frac{1\quad 7\quad 9}{2\quad 8\quad 10}4$$

وجوابها بخمسة.

ومثال آخر من الطرح : لو قيل اطرح سبعة أعشار إثنين إلّا ثلث واحد من أربعة وثلاثة أرباع خمسة أسداس.

---

[1] في (س) : "يضرب".
[2] في (س) : "يقسم".
[3] في (ط) وفي (ت) : $\frac{\frac{6}{8}\frac{4}{5}3}{\frac{1\ 3\ 4}{2\ 8\ 10}}$ ، وهذه الصورة خاطئة.
[4] في النسخ الاخرى : لا إشارة إلى الصورة.
[5] في (أ) : "السّطر".

فننزل المطروح منه في سطر والمطروح في سطر آخر تحته، كالمجموع، على هذه الصورة[1] :

$$\begin{array}{c} 4\,\frac{3\ |\ 5}{4\ |\ 6} \\ \frac{7}{10}\,2\ \text{الا}\ \frac{1}{3} \end{array}.$$

فنبسط الأعلى، كما تقدّم. يكون بسطُهُ أحد عشر ومائة. نضربها في أئمّة الأسفل، يكون الخارج ثلاثين وثلاثمائة وثلاثة آلاف. فنحفظه. ثمّ نضرب بسط الأسفل، وذلك إثنان وثلاثون في أئمّة الأعلى. يكون الخارج ثمانية وستّين وسبعمائة. نسقطه من المحفوظ. فما بقي فهو المطلوب، وذلك إثنان وستّون وخمسمائة وألفان، (وصورته[2]: 2562). فنقسمه على أئمّة السّطرين، فما خرج فهو الجواب[3]، وذلك: ثلاثة وخمسة أعشار وثلاثة أسداس العُشر ونصف سُدس العُشر. وصورته:

$$3\,\frac{5\ \ 3\ \ 1}{10\ \ 6\ \ 2}\ .$$

(وجوابها يُطرح)[4].

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[1] في (أ) : $\begin{array}{c} 4\,\frac{3\ \ 5}{4\ \ 6} \\ \frac{7}{10\ \ 2}\ \text{الا}\ \frac{1}{3} \end{array}$ ، في (ت) : $\begin{array}{c} 4\,\frac{3}{4}\,\frac{5}{6} \\ \frac{7}{10\ \ 2}\ \text{الا}\ \frac{1}{3} \end{array}$ ، والصورتان خاطئتان.

[2] في (أ) وفي (ط) وفي (ت) : لا إشارة إلى الصورة.

[3] في (أ) وفي (ت) : "المطلوب".

[4] في (أ) : "وجوابها طرح". وفي (ب) : سقطت الجملة "وجوابها بطرح". وفي (ت) : "وجوابها يطرح".

الباب الثالث في [37 ظ] ضرب الكسور

وهو تبعيض أحد الكسرين[1] بقدر الآخر[على عكس الصّحيح. (أو تضعيف الكسر بقدر الصّحيح)[2].
فإن كان (المضروب صحيحا والمضروب فيه كسرا)[3] أو بالعكس، فإمّا أن يخرج الصّحيح بقدر الكسر أو يضعف الكسر بقدر الصّحيح][4].
والعمل في ذلك، (أي في ضرب الكسور)[5]، أن تضرب (بسط كلّ سطر في بسط الآخر وتقسم المجتمع[6])[7] على الأئمّة.

مثال منه : لو قيل اضرب ثلاثة أرباع وثلثا في ثلاثة أتساع وأربعة أسداس التسع وخُمس سُدس التّسع.
فننزل المضروب في سطر والمضروب فيه تحته[8] في سطر آخر، على هذه الصورة :

$$\frac{1\ 3}{3\ 4}$$
$$.\frac{1\ 4\ 3}{5\ 6\ 9}$$

(فنضرب بسط الأعلى، وذلك ثلاثة عشر، في بسط الأسفل)[9]، وذلك أحد عشر ومائة، فيكون الخارج ثلاثة وأربعين وأربعمائة وألفًا، وصورته[10] : 1443. نقسمه على الأئمّة، يخرج المطلوب. وذلك أربعة أتساع ورُبع خُمس سُدس تسع، وصورته :

$$.\frac{1\ 0\ 0\ 4}{4\ 5\ 6\ 9}$$

1 في (س) : "المضروبين".
2 في (م) وفي (ط) وفي (ت) : سقطت الجملة "أو تضعيف الكسر بقدر الصّحيح".
3 في (ب) : "المضروب فيه صحيحا والمضروب كسرا".
4 في (س) : سقطت الفقرة "على عكس الصّحيح ... بقدر الصّحيح". ويعتبرها (ت) من شرح الهواري.
5 في (س) : سقطت الجملة "أي في ضرب الكسور". ويعتبرها (ت) من شرح الهواري.
6 في (أ) وفي (ط) وفي (ت) : "الجميع". وفي (ب) : "المجموع".
7 في (س) : "مبسوط أحد السطرين في مبسوط الآخر ويقسم الخارج"
8 في (ط) وفي (ت) : سقط لفظ "تحته".
9 في (س) : سقطت الجملة "فتضرب بسط الأعلى، وذلك ثلاثة عشر، في بسط الأسفل". ويعتبرها (ت) من شرح الهواري.
10 في (ط) وفي (ت) : لا إشارة إلى الصورة.

(وجوابها بواحد)[1].

ومثال منه آخر : لو قيل اضرب ثلث أربعة وثمن في خُمس ثلثي عشرة. فننزل المسألة في سطرين كما تقدم، على هذه الصورة[2] :

$$\begin{array}{ccc} \frac{1}{8} & 4 & \frac{1}{3} \\ 10\,\frac{2}{3} & | & \frac{1}{5} \end{array}.$$

(فنضرب بسط الأعلى، وذلك ثلاثة وثلاثون)[3] في بسط الأسفل، وذلك عشرون، يكون الخارج ستين وستمائة، وصورته[4]: 660 . نقسمه على الأئمّة. يخرج المطلوب. وذلك واحد وخمسة أسداس، وصورته: $1\frac{5}{6}$. وجوابها بإثنين.

[1] في (ب). سقطت الجملة "وجوابها بواحد".

[2] في (م) : $\begin{array}{ccc} \frac{1}{8} & 4 & \frac{1}{3} \\ \hline 10 & 2 & 1 \\ & 3 & 5 \end{array}$. وفي (ت) : $\begin{array}{ccc} \frac{1}{8} & 4 & \frac{1}{3} \\ 10\,\frac{2}{3} & & \frac{1}{5} \end{array}$. والصورتان خاطئتان.

[3] في (م) : "فتضربه".

[4] في (أ) وفي (ط) وفي (ت) : لا إشارة إلى الصورة.

والعمل فيهما أن تضرب مبسُوط[1] كلّ سطر في أئمّة الآخر وتقسم[2] خارج المقسوم على خارج المقسوم عليه أو تسمّي[3].

مثال من القسمة : لو قيل اقسم ستة وثلثا على أربعة أخماس سبعة أثمان ثلاثة. فننزل المقسوم 38 و في سطر والمقسوم عليه في سطر آخر تحته، على هذه الصورة[4] :

$$\frac{1}{3}\,6$$

$$.\;3\,\frac{7 \mid 4}{8 \mid 5}$$

فنضرب (بسط المقسوم)[5]، وذلك تسعة عشر، في أئمّة المقسوم عليه. يكون الخارج ستين وسبعمائة، وصورته[6] : 760، وهو خارج المقسوم. فنحفظه. ثمّ نضرب بسط المقسوم عليه، وذلك أربعة وثمانون، في أئمّة[7] المقسوم. يكون الخارج إثنين وخمسين ومائتين، وهذه صورته[8] : 252، وهو خارج المقسوم عليه. فنقسم عليه المحفوظ، فما خرج فهو المطلوب. وذلك ثلاثة وسُبع تُسع، وصورته:

$$.\,\frac{1\ \ 0}{7\ \ 9}\,3$$

وجوابها بأربعة.

ومثال آخر من التّسمية : لو قيل سمّ ثلاثة ورُبعًا إلّا تُسعيها من ستّة وثُمُنين وثلاثة أخماس.

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[1] في (س) : "بسط". وفي (م) : "مبسوط بسط".
[2] في (س) : "يقسم"
[3] في (أ) وفي (ب) : "تسمّي منه".
[4] في (م) وفي (ت) : $\frac{1}{3}6$ / $3\frac{7\ \ 4}{8\ \ 5}$. والصورة خاطئة.
[5] في (م) : "المقسوم". وفي (ب) : "بسط الأول".
[6] في (أ) وفي (ط) وفي (ت) : لا إشارة إلى الصورة.
[7] في (م) : "إمام".
[8] في (أ) وفي (ط) وفي (ت) : لا إشارة إلى الصورة.

فننزل المسألة على هذه الصورة :

$$\begin{array}{c} \frac{2}{9}\ \text{الا}\ \frac{1}{4}\,3 \\ \frac{3}{5}\ \frac{2}{8}\,6 \end{array}.$$

فنضرب بسط المُسمّى، وذلك واحد وتسعون، في أئمّة المُسمّى منه. يكون الخارج أربعين وستمائة وثلاثة آلاف، وصورته[1] : 3640. نحفظه. ثمّ نضرب بسط المُسمّى منه، وذلك أربعة وسبعون ومائتان، في أئمّة المُسمّى. يكون الخارج أربعة وستّين وثمانمائة وتسعة آلاف، وصورته[2] : 9864. ونُسمّي منه المحفوظ. فما كان فهو المطلوب. وذلك خمسون جزءا من سبعة وثلاثين ومائة جزء وخمسة أتساع الجزء من سبعة وثلاثين ومائة جزء[3]، وصورته :

$$.\frac{5\quad 50}{9\quad 137}$$

وجوابها بطرح.

ومتى استوت أئمّة السّطرين فتقسم البسط على البسط أو تسمّي منه[4] من غير ضرب في الأئمّة.

مثاله بالقسمة : لو قيل اقسم ثمانية وتسعة[5] أعشار وثلثي عُشر على خمسة أعشار |38 ظ| وثلث[6] عُشر.
فننزل المسألة هكذا[7] :

$$\begin{array}{c} \frac{2\quad 9}{3\quad 10}\,8 \\ \frac{1\quad 5}{3\quad 10} \end{array}.$$

فنقسم بسط المقسوم، وذلك تسعة[8] وستّون ومائتان، على بسط المقسوم عليه، وذلك ستّة عشر.

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[1] في (أ) وفي (ط) وفي (ت) : لا إشارة إلى الصورة.
[2] في (أ) وفي (ط) وفي (ت) : لا إشارة إلى الصورة.
[3] في (ت) : سقط لفظ ""جزء".
[4] في (س) : سقط لفظ ""منه".
[5] في (م) وفي (ب) : "سبعة". أخطأ ناسخ (م) هنا وكذلك في الصورة الموالية للقسمة. أما ناسخ (ب) فمثاله صحيح.
[6] في (ط) : "ثلثي".
[7] في (ب) : $\begin{array}{c} \frac{2\quad 7}{3\quad 10}\,8 \\ \frac{1\quad 5}{3\quad 10} \end{array}$ وفي (ط) : $\begin{array}{c} \frac{2}{3}\ \frac{9}{10}\,8 \\ \frac{1\quad 5}{3\quad 10} \end{array}$. والصورتان خاطئتان.
[8] في (ب) : "ثلاثة".

(فما خرج فهو المطلوب وذلك)[1] ستّة عشر وستة[2] أثمان ونصف ثُمن، على هذه الصورة[3]:

$$\frac{1}{2}\,\frac{6}{8}\,16$$.

ومثال آخر بالتسمية : لو قيل سمّ إثنين وثلثا من ستّة وثلثين.
فننزل المسألة هكذا :

$$\frac{1}{3}\,2$$
$$\frac{2}{3}\,6$$.

ونسمّي بسط المُسمّى، وذلك سبعة، من بسط المسمّى منه، وذلك عشرون. فما خرج فهو المطلوب، وذلك ثلاثة أعشار ونصف عشر، وصورته :

$$\frac{1}{2}\,\frac{3}{10}$$.

ومتى استوى البسطان، فتقسم أئمّة المقسوم عليه على أئمّة المقسوم، أو تُسمّي من غير ضرب في البسط. (لأنا إذا ضربنا في الأئمّة صار المقسوم مُركّبًا من بسطه وأئمّة المقسوم عليه وصار المقسوم عليه مُركّبًا من بسطه وأئمّة المقسوم، فيذهب البسطان عند زوال الاشتراك. وهي علّة ما قبله أيضًا)[4].

[مثال من ذلك : لو قيل اقسم خمسة على خمسة أسداس.
فنقسم الستّة، إمام المقسوم عليه، على واحد، بستّة[5]. لأنّه متى كان المقسوم أو المقسوم عليه صحيحًا بسطه عينه[6] وإمامه واحد.][7]
وكذلك : لو قيل سمّ خمسة أسداس من خمسة، لسميّت واحدا من ستّة بسُدس. فاعلمه.

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[1] في (ب) : "يخرج المطلوب". في (ط) : "يكون الخارج".
[2] في (ب) : "ثلاثة"
[3] في (ب) : $\frac{1}{2}\frac{3}{8}16$. وفي (ط) : $\frac{1}{2}\frac{6}{8}16$.
[4] الجملة: "انا إذا ضربنا" إلى "ما قبله أيضا" غائبة في (س)، هي موجودة في ابن غازي 137: 13-20. ويعتبرها (ت) من شرح الهواري.
[5] في (أ) وفي (ط) وفي (ت) : "يكون الخارج ستة".
[6] في (ب) : "عينه". في (ت) : "بسطه فكسره".
[7] الفقرة منقولة من **رفع الحجاب**، صفحة 279: 15-16.

والعمل فيهما[1] أن تقسم المجبور إليه على المجبور أو تسمّي المحطوط إليه من المحطوط، يخرج[2] المطلوب.[3]

ومعنى قوله : "المطلوب" ما يضرب في المجبور فينجبر، أو في المحطوط فينحطّ.

أما الجبر ففيه ستّ مسائل. أحدها جبر الكسر إلى الكسر.

مثاله : لو قيل بكم نجبر نصفًا حتّى يكون[4] [39 و] تسعة أعشار؟
فنقسم تسعة الأعشار، المجبور إليها، على النّصف، المجبور، فما خرج فهو المطلوب. وذلك واحد وثمانية أعشار.

والثانية جبر الكسر إلى الصحيح والكسر.

مثاله : لو قيل بكم نجبر سُبعين ونصف سُبع حتّى تكون خمسة ونصفًا؟
فنعمل فيها كما تقدّم، يخرج المطلوب. وذلك خمسة عشر وأربعة أعشار.

والثالثة جبر الكسر إلى الصحيح.

مثاله : لو قيل بكم نجبر ثلثي خمسة أسباع حتّى تكون عشرة؟
فنعمل كما تقدّم. يخرج (المطلوب وذلك)[5] واحد وعشرون.

والرابعة : جبر الصحيح إلى الصحيح والكسر.

مثاله : لو قيل بكم نجبر خمسة حتّى تكون عشرة وأربعة أسداس؟

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[1] في (ت) : "في ذلك".
[2] في (ت) : "فما خرج فهو".
[3] يلي هذه القاعدة تفصيل الجبر إلى ستة مسائل وكذلك مسائل الحط، وهي غائبة من **التلخيص** ويتبع كل قاعدة مثال. يعتبر ناسخ (أ) القواعد من كلام ابن البنا والأمثلة من شرح الهواري. وأما ناسخ (ت) فهو يعتبر الكل من شرح الهواري.
[4] في (ت) : "تصير".
[5] في (أ) وفي (ط) وفي (ت) : سقطت الجملة "المطلوب، وذلك".

فنعمل كما تقدّم، يخرج المطلوب. وذلك إثنان وعُشر وثلث عشر.

والخامسة : جبر صحيح وكسر إلى صحيح.

مثاله : لو قيل بكم نجبر أربعة وثلاثة أعشار ونصف عُشر حتّى تكون ثمانية؟
فنعمل كما تقدّم، يخرج المطلوب. وذلك واحد وثلاثة وسبعون جزءا من سبعة وثمانين جزءا.

والسادسة : جبر صحيح وكسر إلى صحيح و كسر.

مثاله : لو قيل بكم نجبر ثلاثة وثلاثة أخماس إلّا ثلث واحد حتّى تكون إثني عشر وثلاثة أخماس؟
فنعمل كما تقدّم، يخرج المطلوب. وذلك ثلاثة وستة أسباع. فافهم تصب إن شاء الله تعالى.

وأمّا الحطّ، أيضًا، ففيه ستّة مسائل.

أحدها : حطّ كسر إلى كسر.

مثاله : لو قيل بكم نحطّ سبعة أعشار حتّى تصير[1] ثلثا؟
فنعمل كما ذُكر، تُسمّي الثلث، المحطوطة إليه، من السبعة الأعشار المحطوطة، |39 ظ| فما خرج فهو المطلوب. وذلك ثلاثة أسباع وثلث سُبع.

والثانية : حطّ صحيح إلى صحيح وكسر.

مثاله : لو قيل[2] بكم نحطّ ثمانية حتّى تصير إثنين ونصفا؟
فنعمل كما تقدّم، يخرج المطلوب. وذلك ثمنان ونصف[3] ثمن.

والثالثة : حطّ صحيح إلى كسر.

[1] في (ت) : "تكون".
[2] في (م) وفي (ط) وفي (ت) : سقطت الجملة "لو قيل".
[3] في (ط) في (ت) : "ونصف" ، وهذا غير صحيح.

مثاله : لو قيل بكم نحطّ عشرة حتّى تصير ثلاثة أرباع؟
(فنعمل كما تقدّم، يخرج المطلوب. وذلك ثلاثة أرباع)[1] عُشر.

والرّابعة : حطّ صحيح و كسر إلى صحيح وكسر.

مثاله : لو قيل بكم نحطّ سبعة وربعًا حتّى تكون[2] ثلاثة وأربعة أسداس؟
فنعمل كما تقدّم، يخرج المطلوب. وذلك أربعة عشر جزءا من تسعة وعشرين جزءا وأربعة أسداس الجزء من تسعة[3] وعشرين جزءا.

والخامسة : حطّ صحيح وكسر إلى صحيح.

مثاله : لو قيل بكم نحطّ أحد عشر وتسعة أعشار وأربعة أسباع العُشر حتّى تصير[4] خمسة.
فنعمل كما تقدّم، يخرج المطلوب. وذلك ثمانية وثلاثون جزءا من ثلاثة وتسعين جزءا وثمانية أتساع الجزء من ثلاثة وتسعين جزءا.

والسادسة : حطّ صحيح وكسر إلى كسر.

مثاله : (لو قيل)[5] بكم نحطّ إثنين وثلثا حتّى تصير تسعًا؟
فنعمل كما تقدّم، يخرج المطلوب. وذلك ثلاثة أسباع التسع. فاعلمه (وتدبره)[6].

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[1] في (أ) : سقطت الجملة "فتعمل كما تقدّم، يخرج المطلوب. وذلك ثلاثة أرباع".
[2] في (أ) وفي (ط) وفي (ت) : "تصير".
[3] في (م) : "سبعة"، وهو خطأ.
[4] في (ت) : "لتكون".
[5] في (ط) وفي (ت) : سقطت الجملة "لو قيل".
[6] في (أ) وفي (ط) وفي (ت) : سقطت الجملة "وتدبره".

الباب السادس في التصريف[1]

[وهذا الباب على نوعين. نوع يقصد منه الإسم فقط][2].

[مثل ما يقال خمسة أسداس وثلاثة أرباع كم عُشرًا هي[3]؟
وهو يريد بذلك تسميّة هذين الكسرين بإسم كسر هو العُشر. فنعمل على[4] ما ذُكر، يخرج واحد وخمسة أعشار وخمسة أسداس عُشر، وذلك هو (ما يجتمع)[5] في تينك الكسرين بعد صرفهما إلى إسم العُشر. فقد انتقلت المسألة من إسم الأسداس والأرباع | 40 و | إلى إسم الأعشار وكسورها. وهو نقل نوع من الكسر إلى نوع آخر.[[6]] وهذا النوع هو المقصود في الكتاب.

والنوع الثانى يقصد منه : كم في الجملة من آحاد ذلك الإسم؟
والعمل في هذا النوع[7] كالعمل في الصّحيح إذا أردنا صرفه كما ذكرناه[8] في القسم الثاني[9] من الضرب[10].

فلو قيل : خمسة أسداس وثلاثة أرباع، كم عشرا فيهما؟
فنضربهما في عشرة صحيحة، يخرج (لك واحد و)[11] خمسة أعشار وخمسة أسداس العُشر[12]. وهو الجواب.
وهو مقدار ما سئل عنه من الأعشار فهي خمسة عَشَرَ عُشرًا وخمسة أسداس عُشر][13].

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[1] في (أ) وفي (ب) : "الصرف"
[2] الجملة "وهذا الباب على نوعين: نوع يقصد منه الاسم فقط" منقولة أيضا من **رفع الحجاب**، صفحة 279: 18. يعتبر ناسخ (أ) القواعد من كلام ابن البنا والامثلة من شرح الهواري. وأما ناسخ (ت) فهو يعتبر الكل من شرح الهواري.
[3] في (ط) وفي (ت) : "فيها".
[4] في (أ) وفي (ت) : "كما".
[5] في (ب) : "المجتمع في". وفي (ط) وفي (ت) : "ما يجتمع من ذينك".
[6] في **رفع الحجاب** صفحة 280: 5-7 : زيادة "وهذا النوع ينتفع به في جميع الكسور المختلفة كما يفعله أهل العمل بالرومي بصرف بعضها إلى بعض، و حينئذ تجمع أو تطرح كما يفعل في الصحيح وذلك بيّن".
[7] في (م) وفي (ط) : سقط لفظ "النوع".
[8] في (م) : "أردناه".
[9] في (ط) وفي (ت) : سقط لفظ "الثاني".
[10] في (م) : "الصرف".
[11] في (م) : سقطت العبارة " لك واحد و ". وهذا خطأ.
[12] في (ط) : سقط لفظ " عُشر". وهذا خطأ.
[13] الفقرات "مثل ما يقال .... وخمسة أسداس أعشار" منقولة أيضا من **رفع الحجاب**، صفحة 280: 1-11.

[وليس يحتاج هذا النوع إلى القسمة (على إمام المصروف إليه)[1] كما احتاجه النوع الأوّل، لأنّ هذا بمنزلة لو قال : (خمسة دراهم، كم عُشرًا[2] فيها؟)[3]، فنضرب الخمسة في العشرة، يكون الخارج خمسين، وهو الجواب[4].

وكذلك العمل في الكسور سواء.][5]

ولمّا كان هذا النوع من باب الضرب لم يذكره المؤلّف[6] في الكتاب، وذكر النوع الخاص بالباب.

والعمل فيه، (يعني في النوع المقصود)[7]، أن تضرب مبسوط المصروف في إمام المصروف إليه ويقسم المجتمع على أئمّة المصروف أوّلا، وما خرج على إمام المصروف إليه آخرا. (والأحسن عندهم في هذا النوع أن يصرف الكسر إلى كسر أدقّ منه)[8].

مثاله : لو قيل ستّة أثمان وأربعة أعشار، كم تُسعًا يكون[9] فيها؟
فننزل المصروف في سطر وإمام المصروف إليه في سطر تحته على هذه الصورة :

$$\frac{\frac{4}{10}\,\frac{6}{8}}{9}.$$

فنضرب بسط المصروف، وذلك إثنان وتسعون، في (التسعة، مقام)[1] المصروف إليه. يكون الخارج ثمانية وعشرين وثمانمائة وصورته[2] : 828.

يعتبر ناسخ (أ) القواعد من كلام ابن البنا والامثلة من شرح الهواري. وأما ناسخ (ت) فهو يعتبر الكل من شرح الهواري.
[1] في (ت) : سقطت الجملة "على الإمام المصروف إليه".
[2] في (م) : سقط لفظ "عشرا".
[3] في (ت) : سقطت الجملة "خمسة دراهم، كم عُشرًا فيها؟".
[4] في (أ) : "المطلوب".
[5] الفقرات "وليس يحتاج ... وكذلك العمل في الكسور سواء " منقولة من **رفع الحجاب**، صفحة 280: 16-18.
يعتبر ناسخ (أ) القواعد من كلام ابن البنا والامثلة من شرح الهواري. وأما ناسخ (ت) فهو يعتبر الكل من شرح الهواري.
[6] في (أ) وفي (ط) وفي (ت) : "المُصنّف رضي الله عنه".
[7] في (س) : سقطت الجملة "يعني في النوع المقصود". ويعتبرها (ت) من شرح الهواري.
[8] في (س) : سقطت الجملة "والأحسن ... كسر أدقّ منه ". ويعتبرها (ت) من شرح الهواري.
[9] في (ب) : سقط لفظ "يكون".

نقسمها أوّلا على أئمّة المصروف، (ثمّ ما بقي على إمام المصروف)[3] إليه. يخرج المطلوب. وذلك واحد وتُسع وثلاثة أعشار [40 ظ] التسع وأربعة أثمان عُشر التسع، (وصورتها[4]:

$$(1\frac{4\quad 3\quad 1}{8\quad 10\quad 9}).$$

وكذلك العمل فيما أشبهها.

(كمل القسم، بحمد الله وعونه)[5].

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1 في (أ) وفي (ط) وفي (ت) : "إمام". وفي (ب) : "مقام".
2 في (أ) وفي (ت) : لا إشارة إلى الصورة.
3 في (م) : سقطت الجملة "ثمّ ما بقي على إمام المصروف".
4 في (أ) وفي (ت) : لا إشارة إلى الصورة. وفي (ط) الصورة خاطئة.
5 في (أ) وفي (ط) وفي (ت) : "تمّ القسم الثاني والحمد لله".

القسم الثالث :

في الجذور

ويتعلق بها من الأعمال في ما قصدناه[1] أربعة أبواب.

مخطوط المدينة (م) : 40 ظ – 50 و
مخطوط استانبول (أ) : 85 و – 93 و
مخطوط أكسفرد (ب) : 141 ظ – 149 ظ
مخطوط تهران (ط) : 38 ظ – 46 ظ
مخطوط تونس (ت) : 19 ظ – 24 ظ

[1] في (س) : "في مقصدنا".

الباب الأوّل في أخذ جذر العدد الصحيح وجذر الكسور.

وهو ينقسم قسمين مُنطق وغير مُنطق.
[(والمُنطق هو كلّ عدد معلوم النّسبة إلى الواحد من صحيح، أو كسر، أو صحيح وكسر، وغير المُنطق ما لا يُعلم له نسبة إلى الواحد.) [1]

مثل جذر عشرة، أو جذر نصف، أو جذر عشرة ونصف.

وغير المُنطق على قسمين : قسم (يلفظ به بالجذر)[2] مرّة واحدة، مثل ما ذكرنا ويُسمّى[3] المُنطق في القُوّة. وقسم يُلفظ فيه بالجذر أكثر من مرّة واحدة، مثل (جذر جذر)[4] عشرة. وهذا القسم يُسمّى المُوسّط**]**[5].

والجذر عبارة عن كلّ عدد يُضرب[6] في مثله فيأتي منه المطلوب جذره.
وقد تقدّم مثاله.

وفي اللّغة: ما كان أصلا لكلّ شيء.
(قيل له)[7] جذر بالذال المعجمة (وبفتح الجيم وكسرها)[8]، والفتح أفصح فيما أخبرني به الأستاذ شيخُنا **أبو العباس**، رضي الله عنه.

ثمّ إنّ الصّحيح مرتبة مجذورة ومرتبة غير مجذورة كذلك على توالي العدد[9].
[لأنّه وُجد في الاستقراء في الآحاد والعشرات وكانت المؤون مجذورة لأنّها من ضرب العشرات في نفسها[10]، وكانت الآلاف غير مجذورة لأنّها مع المئين

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[1] في (س) : سقطت الجملة " والمُنطق هو كلّ عدد معلوم النّسبة إلى الواحد من صحيح، أو كسر، أو صحيح وكسر، وغير المُنطق ما لا يُعلم له نسبة إلى الواحد."
[2] وفي (أ) وفي (ب) : "لا يلفظ به بالجذر أكثر من".
[3] في **رفع الحجاب**: "يُسمّى هذا القسم".
[4] في (ت) : "جذر"، وهو خطأ.
[5] الفقرات: "والمُنطق ... الموسّط"، منقولة من **رفع الحجاب**، صفحة 283: 3-7.
[6] في (أ) وفي (ط) وفي (ت) : "تضربه".
[7] في (ب) : "فيه".
[8] في (أ) وفي (ت) : "وفتح الجيم وكسرها". في (ب): "وتُفتح الجيم وتُكسر".
[9] في (ب) : "الأعداد".
[10] في (ب): " مثلها ".

بمنزلة (العشرات مع الآحاد)[1]، وكذلك ما بعد ذلك. وقيل للمنزلة[2] مجذورة لأنّها

يقع فيها عدد مجذور][3].

[وللعدد علامات يُعلم[4] بها أنّه غير مجذور. فإن خلى منها احتمل أن يكون مجذورا.
وهي (كلّ عدد في أوّله)[5] إثنان أو ثلاثة أو سبعة أو ثمانية، فهو غير مجذور.
وكلّ عدد في أوّله واحد ونصف عشراته (مخالف لعدد)[6] المئين بالزوجية والفردية، فهو غير مجذور.
مثل واحد وأربعين وثلاثمائة، 41 و ومثل واحد وستّين وأربعمائة، وما أشبههما.
وكلّ عدد في أوّله خمسة وعشراته غير العشرين، فهو غير مجذور.
مثل خمسة وسبعين، ومثل خمسة وثمانين ومائة، وشبه ذلك.
وكلّ عدد في أوّله ستّة وعشراته زوج، فهو غير مجذور.
مثل ستّة وأربعين، ومثل ستّة وعشرين وثلاثمائة، وشبه ذلك.
وكلّ عدد في أوّله غير الستّة[7] وعشراته فرد، فهو غير مجذور][8].
يعني من دلائل غير[9] المُربّع، وما عداها قد تقدّم، أنّه بنفس وجودها عُلم أنّ العدد غير مجذور.
ودلائل المُربّع خمسة أعداد[10]: الواحد، وقد اشترط فيه تنصيف[1] عشراته. والخمسة، وقد اشترط فيها أن تكون عشراتها عشرين. (والستّة وقد اشترط فيها

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1 في جميع نسخ شرح الهواري : "الآحاد مع العشرات"، وهذا خاطأ.

2 في (أ) وفي (ط) وفي (ت) : "مرتبة".

3 الفقرات: "لأنّه وُجد ... يقع فيها عدد مجذور"، منقولة من **رفع الحجاب**، صفحة 283: 8-11.
ونلاحظ أنّ ابن البنا أوضح محتوى هذه الهقرة في كتاب **المقالات الاربع**، قائلا:
"واعلم أنّ مراتب العدد منها مرتبة مجذورة، ومرتبة غير مجذورة. وذلك أنّ مرتبة الآحاد هي أوّل مرتبة من مراتب الأعداد، وأوّلها الواحد، وله جذر بواحد. والمرتبة الثانية مرتبة العشرات، أوّلها العشرة، وليس لها جذر. والمرتبة الثالثة مرتبة المئين، أوّلها مائة، ولها جذر بعشرة. والمرتبة الرابعة مرتبة الآلاف، أوّلها ألف، وليس لها جذر. والمرتبة الخامسة مرتبة عشرات الآلاف، أوّلها عشرة آلاف، ولها جذر بمائة. وعلى هذا الترتيب أبدًا: مرتبة مجذورة، ومرتبة غير مجذورة. فافهم." (**المقالات**: صفحة 187)

4 في (أ) : " يُفهم ".

5 في كلّ القواعد الآتية ذكرها من بعد يعوّض ناسخ (ب) الجملة "كلّ عدد في أوّله" بالجملة "كلّ عدد أوّله".

6 في (أ) : "يختلف بعدد".

7 في (ب) : "غير الستّة وغير الأربعة".

8 القواعد المنصوص عليها في الفقرات: "وللعدد علامات ... وعشراته فرد، فهو غير مجذور. "، منقولة من **رفع الحجاب**، صفحة 284: 14 إلى 285: 1. أما كلّ الأمثلة تبدوا مستنبطة.

9 في (م) وفي (ط) وفي (ت) : سقط لفظ "غير".

10 يكتب ابن البنا في كتابه: "**المقالات في علم الحساب**"، تحقيق أحمد سعيدان:

أن تكون عشراتها فردا)[2]. فلم يبق منها إلّا تسعة وأربعة. فإذا كانت أوّل العدد، كما ذُكر، وعشراته فرد، فهو غير مجذور. وإن كانت زوجا، احتمل[3].

[وكلّ عدد أوّله[4] أصفار عدّتها فرد، فهو غير مجذور.
مثل العشرة، ومثل (واحد وعشرين ألف)[5]، ومثل ثلاثة آلاف، وشبه ذلك.
وكلّ عدد في أوّله أصفار عدّتها زوج بحيث (لو لم تكن)[6] لم يكن العدد مجذورا، فهو غير مجذور.
مثل خمسمائة[7]، ومثل ثلاثين ألفا، وشبه ذلك.
وكلّ عدد يُطرح بتسعة، فلا يفنى ولا يبقى منه واحد، ولا أربعة، ولا سبعة، فهو غير مجذور.
مثل خمسة وعشرين وأربعمائة، وشبهها.
وكلّ عدد يُطرح بثمانية، فلا يفنى ولا يبقى منه واحد، ولا أربعة، فهو غير مجذور.
مثل ستّة وستّين ومائتين، وشبه ذلك.
وكلّ عدد يُطرح بسبعة، فلا يفنى ولا يبقى منه واحد، ولا إثنان، ولا أربعة، فهو غير مجذور.
مثل تسعة وأربعين وثلثمائة، وشبه ذلك][8].

"... دليل المربّع ، وهي الواحد والأربعة والخمسة والستّة والتّسعة، فإن ذلك العدد المفؤوض يمكن أن يكون مربّعًا، فاطلبه. ... وإن كان في أوّل مرتبة غير هذه الأعداد الخمسة، فإن عددك المفروض غير مجذور البتّة. فتأخذ جذره، إن أردت بالتّقريب". (**المقالات**: صفحة 187-188)

[1] في (م) وفي (ط) وفي (ت) : "نصف".

[2] في (م) وفي (أ) وفي (ط) وفي (ت) : سقطت الجملة "والستّة وقد اشترط فيها أن تكون عشراتها فرد"، وهو خطأ.

[3] في (ت) : "احتمل ذلك".

[4] في (أ) وفي (ط) وفي (ت) : "في أوّله".

[5] في (ت) : " الواحد والعشرين والألف" ، وهذا العدد غير وارد في هذه الحالة.

[6] في (م) : "لو لم تكن" غائبة. وفي (ط) وفي (ت) : "لو".

[7] في (م) : "خمسة"، وهو خطأ.

[8] القواعد المنصوص عليها في الفقرات: "وكلّ عدد في أوّله أصفار ... مثل تسعة وأربعين وثلثمائة، وشبه ذلك "، منقولة من **رفع الحجاب**، صفحة 285: 2-6. أما كلّ الأمثلة تبدوا مستنبطة.

والعمل في أخذ جذر العدد[1] الصّحيح بأن تعدّ[2] مراتبه بجذر، (لا جذر،)[3] إلى آخر السّطر. ثمّ [41 ظ] تأتي إلى آخر المجذورة[4] وتضع تحتها عددا تضربه في نفسه فيفنى به[5] ما عليه أو يبقى[6] ما لا يمكن في الصّحيح أقلّ منه. ثمّ تقهقر مُضاعفًا تحت منزلة لا جذر فتطلب عددا تضعه تحت المجذورة قبلها فتضربه في المقهقر المضعف ثمّ في نفسه، فيفنى به ما على رأسهما أو يبقى ما لا يمكن أقلّ منه. ثمّ لا تزال تفعل كذلك من تضعيف المُقهقر والنّقل حتّى تأتي على جميع السّطر. فما خرج في السّطر الثاني قبل التّضعيف فهو الجذر. وإن بقي شيء فسمّه من ضعف الجذر الصّحيح إن كان مثل الجذر أو أقلّ. وإن كان أكثر من الجذر، فزد فيه واحدًا وفي الجذر المُضعف إثنين أبدًا[7]. ثمّ (تسمّيه منه وتزيد التّسمية على الجذر الصّحيح، فما كان فهو الجذر الذي يُضرب في نفسه)[8]، فيأتي منه المطلوب جذره بتقريب.

مثال من ذلك : لو قيل كم جذر خمسة وعشرين وستمائة؟
فننزلها في سطر هكذا : 625. فالمنزلة الأولى مجذورة والثانية غير مجذورة والثالثة مجذورة، كما تقدّم. فنطلب[9] عددا نضعه تحت السّتّة، التي في المنزلة المجذورة، نضربه في نفسه، تفنى به السّتّة أو يبقى ما لا يمكن في الصّحيح أقلّ منه. نجده إثنين. نضربها في نفسها بأربعة. نسقطها من السّتّة، الباقي منها إثنان. نثبتها مكانها. ثمّ نقهقر الإثنين مُضاعفة تحت منزلة لا جذر، وذلك تحت الإثنين. ثمّ ننظر عددا نضعه تحت المجذورة التي قبل تلك المقهقر (تحتها، وذلك تحت)[10] الخمسة. نجده خمسة، ولا يجوز أن يكون ثمّ غيرها.

[ولو[11] كانت ستّة (لم تضع تحتها إلّا (أربعة أو ستّة،
ولو كان واحدًا لم تضع تحته إلّا واحدا أو تسعة،
ولو كانت)[1] أربعة لم تضع تحتها إلّا إثنين أو ثمانية.

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[1] في (ب) : سقط لفظ "العدد".
[2] في (م) : "بان تقدر". وفي (أ) : "بأن تُعلمَ".
[3] في (م) : سقطت عبارة "لا جذر"، وهو خطأ.
[4] في (س) و (ت) : "المجذورة فيه".
[5] في (ط) وفي (ت) : سقط لفظ "به".
[6] في (أ) : "لا يبقى"، وهو خطأ.
[7] في (س) : سقط لفظ "أبدا".
[8] في (ط) وفي (ت) : سقطت الجملة " تسمّيه منه ... في نفسه".
[9] في النسخ الباقية : "فتنظر".
[10] في **بغية الطلاب** لابن غازي، صفحة 146 : "إليها، وهي".
[11] في (م) : سقطت "لو". وبذلك يختل المفهوم.

ولو كانت تسعة، لم تضع تحتها [42 و] إلّا ثلاثة أو سبعة][2]. فاعلم.

فنضرب الخمسة في الأربعة المُضعفة بعشرين، وعلى رأسها إثنان وعشرون، تفنى العشرون، وتبقى الإثنان مكانها. ثمّ (نضربها أيضًا)[3] في نفسها بخمسة وعشرين، وعلى رأسها خمسة وعشرون. تفنى بها. (فمعنا خمسة وبعدها إثنان المضعفة، فهي إذًا خمسة وعشرون)[4]، وهو المطلوب.

وكذلك تعمل فيما إذا كانت المنازل أكثر.

مثال منه آخر : لو قيل كم جذر عشرين؟
فنعلم أنّه ليس لها جذر مُنطق لأنّ في أوّلها صفرا واحدا، لكن نعمل كما[5] تقدّم. وما بقي نقسمه[6] على حسب ما تقدّم ذكره. فنطلب عددًا نضربه في نفسه يفنى به[7] (أو يبقى منه)[8] ما لا يمكن أقلّ منه. نجده أربعة، نضربه في نفسه بستّة عشر. الباقي أربعة، مثل الجذر، نُسميّها من ضعفه. تكون نصفا[9]، نحملها على الجذر الصّحيح، يكون أربعة ونصفا. وهو جذر العشرين بتقريب.

مثال منه[10] آخر : لو قيل كم جذر[11] أربعة وخمسين؟
فنعمل كما تقدّم. يخرج الجذر الصّحيح سبعة، وتبقى خمسة، أقلّ من الجذر[12]، نسمّيها من ضعفها. تكون سُبعين ونصف سُبع. (نحملها على الجذر، يكون المُجتمع سبعة وسُبعين ونصف سُبع)[13]. وهو الجذر المطلوب بتقريب.

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[1] في (أ) وفي (ط) وفي (ت): سقطت الجملة "أربعة أو ستّة، ... ولو كانت"، وعوضت بالعبارة "أو كانت"، وهو خطأ.
[2] في (ب): "ولو كانت ثلاثة أو أربعة أو سبعة أو ثمانية لم تضع تحتها إلّا إثنين. ولو كانت تسعة، لم تضع تحتها إلّا ثلاثة". وهذه الفرضية غائبة من كل النسخ الأخرى وغائبة أيضًا من **بغية الطلاب** لابن غازي.
[3] في (ط) وفي (ت) : "تضرب الخمسة".
[4] في (أ) : سقطت الجملة "فمعنا خمسة ... فهي إذًا خمسة وعشرون".
[5] في النسخ الأخرى : "على ما".
[6] في النسخ الأخرى : "نسمّيه".
[7] في (أ) : "أبدًا".
[8] في (م) : سقطت العبارة "أو يبقى منه"، وهو خطأ.
[9] في (أ) وفي (ت) : "نصف الجملة".
[10] في (أ) : سقط لفظ "منه".
[11] في (أ) : سقط لفظ "جذر".
[12] في (أ) : "المجذور"، وهو خطأ.
[13] في (ت) : سقطت الجملة " نحملها على الجذر ... ونصف سُبع".

(مثال آخر منه)[1] ولو قيل كم جذر إثنين وتسعين؟
فنعمل كما تقدّم. يخرج الجذر الصّحيح[2] تسعة، ويبقى أحد عشر، أكثر من الجذر. فنزيد فيها واحدا، وفي ضعف التّسعة، الجذر، إثنين. ونُسمّي الأقلّ من الأكثر ونحمله على الجذر. فما كان، فهو الجذر[3] المطلوب. وذلك تسعة وثلاثة أخماس. فإذا ضربنا التّسعة والثلاثة أخماس، الجذر المُقرّب، في نفسه، خرج منه إثنان وتسعون | 42 ظ | وأربعة أخماس الخُمس. فقد وقع التّقريب بهذا الكسر الزّائد.

وإن أردت تدقيق التقريب، فسمّيه من ضعف الجذر، فما خرج فاسقطه من الجذر، يبقى جذر، مربّعه أقرب إلى العدد المطلوب جذره من المُربّع الأوّل[4].

وقوله فسمّيه[5] يعني به الكسر الزائد على المُربّع، وهو الذي وقع به التقريب. لأنّ أخذ الجذر بالتقريب (يكون من قبل المُربّع القريب الأصغر، كما تقدّم)[6]. ويكون من قبل المُربّع القريب الأكبر، وهو المقصود في قوله "فسمّيه إلى آخره"، وصفة العمل فيه أن نسقط العدد المطلوب جذره من المُربّع ونُسمّي ما بقي من ضعف جذر المُربّع وننقص الخارج من جذر المُربّع أيضًا، يبقى جذر العدد بتقريب.

فإذا أردنا أخذ جذر الإثنين والتّسعين من قبل المُربّع الأكبر، فنجعل المُربّع الأكبر الإثنين والتّسعين والأربعة أخماس الخُمس. فإذا أسقطنا منه العدد بقي الكسر الذي وقع به التّقريب. فنُسمّيه من ضعف الجذر، (كما ذكر)[7]. يكون نصف سُدس عُشر، نسقطه (من الجذر)[8]، يبقى تسعة وخمسة أعشار وخمسة أسداس عُشر ونصف سُدس عُشر. فمُربّع هذا[9] الباقي أقرب من المُربّع الأوّل. فاعلمه.

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[1] في (أ) وفي (ط) وفي (ت) : سقطت "مثال آخر منه".
[2] في (ب): "المطلوب".
[3] في (م) وفي (ط) : سقط لفظ "الجذر".
[4] في (أ) : سقط لفظ "الأول".
[5] في (أ) : "تسمّيه".
[6] في النسخ الأخرى : سقطت الجملة "يكون من قبل المُربّع القريب الأصغر،كما تقدّم".
[7] في (ب) زيادة : "كما ذكر،أعني من ضعف تسعة وثلاثة أخماس".
[8] في (ب) زيادة : "أعني من تسعة وثلاثة أخماس".
[9] في النسخ الأخرى : "جذر".

وفي التّقريب وجه آخر. وهو أن تضرب العدد المطلوب جذره في عدد مُربّع أعظم منه، ويؤخذ جذر المُجتمع بتقريب، ويقسم على جذر المُربّع المضروب فيه. فما خرج، فهو الجذر المقرّب.

مثاله : لو قيل كم جذر إثني عشر؟
فنضربها مثلا في ستّة عشر. يكون الخارج إثنين وتسعين ومائة. نأخذ جذرها. يكون ثلاثة عشر وستّة أسباع. نقسمه على جذر السّتّة عشر. [43 و] فما خرج، فهو جذر الإثني عشر بتقريب. وذلك ثلاثة وثلاثة أسباع ورُبع سُبع.

[وإنّما شرط ما لا يُمكن في الصّحيح أقلّ منه، لأنّه إذا عمل بالكسر على غير العمل[1] المشهور كان الباقي أقلّ ممّا (يبقى بالصّحيح)[2].][3]

[(ولو قيل كم جذر خمسة وعشرين وستّمائة؟)[4]
مثاله في المسألة المتقدّمة : نجعل تحت السّتّة إثنين ونصفا. فيكون مُربّعها ستّة ورُبعا[5]. فقد زاد. فتذهب السّتّة بالسّتّة، ويذهب الرُّبع بالخمسة والعشرين، رُبع المائة. فيفنى العدد كلّه ويبقى قبل الآخر صفران. نأخذ أحدهما. فتكون الإثنان والنّصف عشرات، وذلك خمسة وعشرون. ونضعّف الإثنين والنّصف بخمسة[6]، وننقل تحت العشرات. ونطلب ما نضرب في المُضاعف، فلا نجد شيئا لأنّ ما فوقه أصفار. فنجعل صفرا وننصّف ما ضاعفنا[7]. فيكون نصف خمسين][8].

[مثال منه آخر : لو قيل كم جذر تسعة وعشرين وسبعمائة؟
لوعملنا بالصّحيح لبقي من السّبعة، التي هي سبعمائة، ثلاثة في مرتبتها. ولو عملنا بالكسر، لبقي أقلّ. فلو جعلنا تحتها إثنين ونصفا، لبقي ثلاثة أرباع واحد في تلك المرتبة وثلاثة أرباع مائة بخمسة وسبعين[9]. نضيفها إلى التّسعة

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[1] في (أ) : سقط لفظ "العمل".
[2] في النسخ الأخرى : "بقي في الصحيح".
[3] الجملة "وإنما ... بالصحيح" منقولة من **رفع الحجاب**، صفحة 283: 11-12.
[4] في جميع نسخ كتاب الهواري، نص المسألة : " ولو قيل كم جذر خمسة وعشرين وستّمائة؟"غائب، وهو موجود في **رفع الحجاب**، صفحة 284: 4..
[5] في (م) : "ربع ستة وعشر وربعًا"، وهي زيادة خاطئة.
[6] في (ت) : "ونصف الإثنين والنصف خمسة"، وهذا غير صحيح.
[7] في (أ) : "ضُعف".
[8] هذا المثال منقول من **رفع الحجاب**، صفحة 284: 4-9.
[9] في (ت) : "ومثال سبعين"، وهذا خطأ.

والعشرين،التي معنا[1]، تكون أربعة ومائة. ثمّ نقهقر الإثنين والنّصف مُضاعفًا، وذلك خمسة، ونطلب عددًا نضربه في الخمسة وفي نفسه، نجد إثنين، ولا يبقى من عددنا شيء. فننصّف ما ضاعفنا، وهو خمسون. نصفُها خمسة وعشرون. فجملة الجذر سبعة وعشرون. أو نُضاعف الإثنين، يصير ذلك ||43 ظ| أربعة وخمسين. فنأخذ نصفها.][2]

مثال آخر : لو قيل كم[3] جذر مائة؟
فالقانون فيها وفي أمثالها أن نأخذ نصف عدد الأصفار أبدًا[4]، (ونحمله على جذر العدد الباقي، يكون الجذر. فالمائة يتقدّمها صفران، نأخذ أحدهما)[5]،
ونحمله على جذر الواحد الباقي. يكون عشرة، وهو جذر المائة. فاعلمه.

وأمّا تجذير الكسور، فهو أن تضرب البسط في الإمام وتقسم جذر الخارج على الإمام.
وإن كان[6] للبسط جذر منطق وللإمام مثله، فاقسم جذر البسط على جذر الإمام[7].
[وهي بالنّسبة إلى التجذير، أعني الكسور، على أربعة أضرب.
أحدها أن يكون للبسط جذر منطق وللإمام مثله. فالعمل فيه ما ذُكر][8].

مثاله : لو قيل كم جذر أربعة أسداس وسُدس سُدس (وصورتها[9]:

$$\frac{1}{6}\frac{4}{6}) ؟$$

فنأخذ جذر البسط بخمسة. نقسمه على جذر الإمام، وذلك ستّة، فما خرج فهو الجذر المطلوب. وذلك خمسة أسداس.

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[1] في (ط) وفي (ت): "الذي معها" والعبارة غائبة في (أ).
[2] هذا المثال منقول من **رفع الحجاب**، صفحة 283: 13 إلى 284: 3.
[3] في (م) : سقط لفظ "كم".
[4] في (أ) : سقط لفظ "أبداً".
[5] في (ت) : سقطت الجملة "ونحمله على جذ العدد الباقي ... نأخذ أحدهما".
[6] في (أ) : سقط لفظ "كان".
[7] في (ب): سقط لفظ "الامام".
[8] في (س) : لا توجد هذه القواعد الأربعة في **التلخيص**. لكن ناسخ (أ) يعتبرها من كلام ابن البنا ويعتبرها ناسخ (ت) من شرح الهواري.
[9] في (ط) وفي (ت) : لا إشارة إلى الصورة.

مثال منه آخر : لو قيل كم[1] جذر إثنى عشر ورُبع؟.
فنأخذ جذر المقام بإثنين. نقسم عليها جذر البسط، وهو سبعة، فما خرج فهو المطلوب. وذلك ثلاثة ونصف.
وإن شئنا، عملنا فيها بالوجه الأوّل لأنّه عام، وهذا خاص. فاعلمه.

والثاني، أن لا يكون لواحد منهما جذر مُنطق. والعمل فيه بالوجه الأوّل.

مثاله : لو قيل كم جذر أربعة أتساع وثلاثة أسداس تسع؟[2]، وصورتها :

$$\frac{3\ \ 4}{6\ \ 9}.$$

فتضرب البسط في الإمام، يكون الخارج ثمانية وخمسين وأربعمائة وألفًا. نأخذ جذره، وذلك ثمانية وثلاثون وثلاثة أجزاء من تسعة عشر جزءا ونصفا لجزء من تسعة عشر جزءا. نقسمه على الإمام، فما خرج [44 و] فهو المطلوب. وذلك ثلاثة عشر جزءا من تسعة عشر جزءا وثلاثة أتساع الجزء من تسعة عشر جزءا وخمسة أسداس تُسع الجزء من تسعة عشر جزءا ونصف سُدس تُسع الجزء من تسعة عشر جزءا، وصورته :

$$\frac{1\ \ 5\ \ 3\ \ 13}{2\ \ 6\ \ 9\ \ 19}.$$

والثالث، أن يكون للإمام جذر مُنطق وليس للبسط جذر مُنطق. فهذا الضرب[3] إن شئنا عملنا فيه بالوجه الأوّل أو بالثاني.

مثال منه : لو قيل كم جذر عشرة وسبعة أثمان ونصف ثُمن؟ وصورتها :

$$\frac{1\ \ 7}{2\ \ 8}10.$$

[1] في (م) وفي (ط) : سقط لفظ "كم".
[2] في (ت) : سقط لفظ "تسع"، وهو خطأ.
[3] في (أ): سقط لفظ "الضرب".

فإن أردنا، عملنا بالوجه الأوّل : ضربنا البسط في الإمام. فيكون الخارج ثمانمائة وألفين. نأخذ جذره، وذلك إثنان وخمسون وثمانية وأربعون جزءا من ثلاثة وخمسين جزءا ونصف الجزء من ثلاثة وخمسين جزءا. نقسمه على الإمام، يخرج المطلوب. وذلك ثلاثة وستّة عشر جزءا (من ثلاثة وخمسين جزءا) وثُمنا[1] الجزء ورُبع ثُمن الجزء ومن ثلاثة وخمسين جزءا. وصورته[2] :

$$.\frac{1 \quad 2 \quad 16}{4 \quad 8 \quad 53}\,3$$

وبالوجه الثاني، نأخذ جذر البسط وذلك ثلاثة عشر وثلاثة أجزاء من ثلاثة عشر جزءا. نقسمه على جذر المقام، فما خرج فهو المطلوب. (وذلك ثلاثة وأربعة أجزاء من ثلاثة عشر جزءا)[3]، وصورته :

$$.\frac{4}{13}\,3$$

وهذا الوجه أقرب من الأوّل.

والرّابع، أن يكون للبسط جذر منطق وليس للإمام جذر منطق. فالعمل فيه بالوجه الأوّل.

مثاله : لو قيل كم جذر أربعة أسباع ونصف سُبع؟ وصورته[4] :

$$.\frac{1 \quad 4}{2 \quad 7}$$

فنضرب البسط في الإمام ونأخذ جذر الخارج. وذلك أحد عشر وجزآن من أحد عشر جزءا ونصف الجزء من أحد عشر جزءا. نقسمه على الإمام. فما خرج فهو المطلوب. وذلك ثمانية أجزاء من أحد عشر جزءا وخمسة [44 ظ] أسباع الجزء (وثلاثة أرباع سُبع الجزء)[5] من أحد عشر جزءا.

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[1] في (أ) : "ثمن" وهو خطأ.

[2] في (أ) وفي (ط) : الصورة خاطئة. (انظر الملاحظة السابقة).

[3] في (أ) : سقط لفظ "جزءا". وفي (ط) : " وذلك ثلاثة عشر وثلاثة أجزاء من ثلاثة عشر جزءا ". وهذا لا يصح.

[4] في (ط) : الصورة خاطئة.

[5] في (ط) : " ثلاثة أسباع الجزء ". وهذا لا يصح والصورة خاطئة.

وصورته :

$$\frac{3 \quad 5 \quad 8}{4 \quad 7 \quad 11}.$$

وهذه الأضرب الأربعة[1]، جذر الأوّل منها بالتّحقيق[2] والثلاث الأخيرة بالتّقريب.
ولو شئنا لقرّبنا[3] الجذور فيها كالصّحيح، والعمل فيها واحد.

وأمّا تجذير ذوات الأسماء والمُنفصلات، فهو أن تسقط رُبع مُربّع أصغر الإسمين من رُبع مُربّعٍ[4] أكبرهما وتأخذ جذر الباقي وتحمله على نصف أكبر الإسمين وتنقصه أيضًا من نصف أكبر الإسمين وتوقّع[5] الجذر على كلّ واحد منهما.
فإن كان المطلوب جذره ذا الإسمين، فجذره مجموع هذين الجذرين.
وإن كان منفصلا، فجذره فضل ما بين هذين الجذرين.

قلت نحتاج أن نقدّم ههنا مقدّمة في بيان ذوات الأسماء ومُنفصلاتها[6] وايجادها، وحينئذ نمثّل أخذ جذورها، إن شاء الله تعالى.

فنقول: [إنّ ذوات الأسماء ستّة ومنفصلاتها ستّة[7].
وذو الإسمين هو عدد وجذر عدد (أو جذر عدد وجذر عدد)[8]، لا يجتمعان إلّا بحرف العطف.
مثل: خمسة وجذر ثلاثة، أو جذر خمسة وجذر ثلاثة[9].

والمُنفصل هو ذو الإسمين إذا فُصل الإسم[10] الأصغر من الأكبر بحرف الاستثناء.
مثل خمسة إلّا جذر ثلاثة، أو جذر خمسة إلّا جذر[11] ثلاثة.

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[1] في (ط) وفي (ت) : "الأربعة الجذور".
[2] في (ب) : سقط لفظ "بالتحقيق".
[3] في (أ) : "ضربنا".
[4] في (ب) : سقط لفظ "مربع"، وهو خطأ.
[5] في (أ) : "وتضع".
[6] في (أ) : "المنفصلات".
[7] في (ب) : سقط لفظ "ستّة".
[8] في (ط) وفي (ت) : سقطت الجملة "أو جذر عدد وجذر عدد".
[9] في (ط) : سقطت العبارة "وجذر ثلاثة". وفي (ت) : سقط لفظ "ثلاثة".
[10] في (ب) : سقط لفظ "الاسم".
[11] في (ط) وفي (ت) : سقط لفظ "جذر".

والثلاثة الأولى من ذوات الأسماء أو من المنفصلات، جذورها أقرب إلى المُنطق في المرتبة من جذور الثلاثة الأخيرة. وتتميّز[1] الثلاثة الأولى عن الأخيرة بأن يضرب فضل ما بين مُربّعي الإسمين في مُربّع الأكبر منهما، فإن[2] خرج مُربّع، فهو من الثلاثة الأولى. وإن كان غير مُربّع، فهو من الثلاثة الثانية.

ثمّ إنّ الإسم الأكبر مُنطق في الأوّل، وفي الرّابع.
(مثال الأوّل : خمسة وجذر واحد وعشرين، ومثال الرّابع : إثنان وجذر إثنين)[3].

والأصغر مُنطق في الثاني، [45 و] وفي الخامس[4].
(مثال الثاني : خمسة وجذر خمسة وأربعين، ومثال الخامس : خمسة وجذر إثنين وسبعين)[5].

وليس واحد منهما مُنطقا في الثالث، وفي السّادس.
(مثال الثالث[6] : جذر عشرة وجذر ثمانية عشر، ومثال السّادس : جذر سبعة وجذر ثمانية)[7].

ويلزم ممّا ذُكر من خواصّها[8] أنّا إذا أردنا إيجادها، فإنّا ننقص مُربّعا من مُربّع ولا يكون الباقي مُربّعا ونصل جذر الباقي بجذر المُربّع الأكبر، يكون ذا الإسمين الأوّل.

(ونجعل أيّ عدد شئنا الإسم الأكبر، والأصغر جذر مُسطّح طرفيه في النّسبة العددية بشرط أن لا يكون[9] منطقا وهو قليل)[10].

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[1] في (ط) وفي (ت) : "وتتبيّن".
[2] في (ط) وفي (ت) : "فما".
[3] في **رفع الحجاب** : المثالان غائبان.
[4] في (أ) : "الرّابع"، وهو خطأ.
[5] في **رفع الحجاب** : المثالان غائبان.
[6] في (أ) : "السّادس"، وهو خطأ.
[7] في **رفع الحجاب** : المثالان غائبان.
[8] في (ط) وفي (ت) : "خروجها"، وهو خطأ .
[9] في (ط) وفي (ت) : "أن يكون"، وهو خطأ .
[10] في **رفع الحجاب** : الجملة غائبة.

وننقص عددا غير مُربّع من مُربّع، ولا يكون الباقي مُربّعا، ونصل جذر الباقي بجذر المُربّع يكون ذا الإسمين الرّابع.
ونضرب مُربّعين في فضل[1] ما بينهما، ولا يكون مُربّعا، ونصل جذر أكبر الخارجين بجذر فضل ما بينهما (يكون ذا الإسمين الثاني.
ونضرب مُربّعين في غير فضل ما بينهما)[2]، (ولا يكون مربّعًا، ونصل جذر أكبر الخارجين بجذر فضل ما بينهما)[3] يكون ذا الإسمين الثالث.
ونزيد مُربّعا على مُربّع ولا يكون مجمُوعهما مُربّعا، ونصل جذر المُجتمع بجذر أحد المُربّعين يكون ذا الإسمين الخامس.
ونزيد عددا غير مُربّع على مُربّع ولا يكون مجمُوعهما مُربّعا ونصل جذر المجمُوع بجذر العدد المزيد يكون ذا الإسمين السّادس][4].

(فهذه ذوات الأسماء الستّة)[5]. فلنرجع الآن إلى الأمثلة منها بالجذر.

مثال منها : لو قيل ثمانية وجذر ستّين، كم جذرهما؟
فالعمل فيه ما ذكر. وذلك أن نسقط رُبع مُربّع جذر الستّين، لأنه الأصغر، وذلك خمسة عشر، من رُبع مُربّع الثمانية، لأنّه الأكبر، وذلك ستّة عشر. الباقي واحد، نأخذ جذره بواحد، نحمله على نصف الثمانية، لأنّها الأكبر بخمسة، ونسقطه أيضًا من [45 ظ] نصفها. الباقي ثلاثة. فنوقّع[6] الجذر على الخمسة والثلاثة. يكون ذلك جذر خمسة وجذر ثلاثة، وهو الجذر المطلوب.
وكذلك القياس في سائرها.

ولو قيل ثمانية إلّا جذر ستّين، كم جذرها؟
فالعمل كما[7] تقدّم. ونسقط جذر الثلاثة من جذر الخمسة. وجذر فضل ما بينهما هو الجذر المطلوب. وذلك جذر الخمسة إلّا جذر الثلاثة.

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[1] في (ت) : "في غير فضل".
[2] في (م) : سقطت الجملتان "يكون ذا الإسمين الثاني. ونضرب مُربّعين في غير فضل ما بينهما".
[3] في (أ) : سقطت الجملة "ولا يكون مربّعًا ... فضل ما بينهما".
[4] الفقرة: "وذوات الأسماء ستّة ... يكون ذا الإسمين السادس" منقولة من **رفع الحجاب**، صفحة 287: 19 إلى 288: 19.
[5] في النسخ الاخرى : سقطت الجملة "فهذه ذوات الأسماء الستّة".
[6] في (أ) : "فنضع".
[7] في (أ) وفي (ب) : "كما".

وفيه وجه آخر : وهو أن نسقط مُربّع أصغر الإسمين (من مُربّع أكبرهما ونأخذ جذر الباقي ونحمله على أكبر الإسمين ونأخذ جذر نصف المُجتمع وننقصه أيضًا من أكبر الإسمين ونأخذ جذر نصف المُجتمع)[1]. فإن كان المطلوب جذره ذا الإسمين، فجذره مجموع[2] هذين الجذرين. وإن كان منفصلا، فجذره فضل ما بين هذين[3] الجذرين.

فإن قيل ثمانية وجذر خمسة وخمسين، كم جذرها؟
فهذا هو ذو الإسمين الأوّل. فنأخذ جذره كما تقدّم. يكون جذر خمسة ونصف وجذر إثنين ونصف. ويُسمّى أحد الإسميات، ومنفصله هو المنفصل الأوّل، ومنفصل جذره هو جذر[4] منفصله، ويُسمّى منفصلا من الستّة.

وإن قيل سبعة وجذر إثني عشر ومائة، كم جذرها؟
فهذا هو ذو الإسمين الثاني. فنأخذ جذره كما تقدّم، فيكون، بعد الجمع والطرح، جذر جذر خمسة وثمانين وثلاثة أرباع وجذر جذر واحد وثلاثة أرباع. ويُسمّى ذا المُوسّطين[5] الأوّل. ومنفصله هو المنفصل الثاني، ومنفصل جذره هو جذر[6] منفصله، ويُسمّى منفصل المُوسّط[7] الأوّل.

وإن قيل جذر إثنين وثلاثين وجذر أربعة عشر، كم جذره؟
فهذا ذو الإسمين الثالث. فنأخذ جذره كما تقدّم. فيكون، بعد الجمع والطرح، جذر جذر أربعة وعشرين ونصف وجذر جذر نصف، [46 و] ويُسمّى ذا الإسمين المُوسّطين[8] الثانى. ومنفصله هو المنفصل الثالث، ومنفصل جذره هو جذر منفصله، ويُسمّى منفصل المُوسّط[9] الثانى.

وإن قيل[10] سبعة وجذر ثلاثين، كم جذره؟

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[1] في (أ) : سقطت الجملة "من مُربّع أكبرهما ... جذر نصف المُجتمع". وفي (ت) : إختلطت الجمل، فأصبحت خاطئة.
[2] في (أ) : سقط لفظ "مجموع".
[3] في (أ) وفي (ت) : سقط لفظ "هذين".
[4] في (أ) : سقط لفظ "جذر".
[5] في (م) : "الوسطين".
[6] في (ت) : سقط لفظ "جذر".
[7] في (م) : "الوسط".
[8] في (م) : "الوسطين".
[9] في (م) : "الوسط".
[10] في (م) : سقط "قيل".

فهذا هو ذو الإسمين الرّابع. فنأخذ جذره كما تقدّم. فيكون ثلاثة ونصفا وجذر أربعة وثلاثة أرباع مأخوذًا جذره وثلاثة ونصفا إلّا جذر أربعة وثلاثة أرباع مأخوذًا جذره، ويُسمّى الأعظم. ومنفصله هو المنفصل الرّابع، ومنفصل جذره وهو جذر منفصله، ويُسمّى الأصغر.

وإن قيل لك[1] ثلاثة وجذر عشرين، كم جذره؟
فهذا هو ذو الإسمين الخامس. فنأخذ جذره كما تقدّم. فيكون جذر خمسة وجذر (الاثنين ونصف وربع)[2] مأخوذًا جذره [وجذر خمسة إلّا جذر (الاثنين ونصف وربع)[3] مأخوذًا جذره][4]، ويُسمّى القوّي على مُنطق ومُوسّط. ومُنفصله هو المنفصل الخامس، ومنفصل جذره هو جذر منفصله، ويُسمّى المتصل بمُنطق يصير الكلّ مُوسّطًا.

وإن قيل جذر عشرة وجذر أحد عشر، كم جذره؟
فهذا هو ذو الإسمين السّادس. فنأخذ جذره كما تقدّم. فيكون نصفا وجذر إثنين وثلاثة أرباع مأخوذًا جذره وجذر إثنين وثلاثة أرباع إلّا نصفا مأخوذًا جذره، ويُسمّى القويّ على مُوسّطين. ومُنفصله هو المُنفصل السّادس، ومُنفصل جذره هو جذر منفصله، ويُسمّى المُتّصل بموسط يصير الكلّ موسّطًا.

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[1] في (م) : سقط "لك".
[2] في النسخ الاخرى : "أربعة ونصف"، وهو خطأ.
[3] في (م) وفي (ط) وفي (ت) : "أربعة ونصف"، وهو خطأ.
[4] وفي (أ) سقطت الجملة "وجذر خمسة إلّا جذر الاثنين ونصف وربع مأخوذًا جذره".

الباب الثاني في جمع جذور الأعداد و طرحها

تضرب[1] العددين اللذين تريد جمع جذريهما أو طرحهما أحدهما في الآخر.
فإن خرج مُربّعًا فإنّ جذري العددين يجتمعان [46 ظ] وينطرحان.
وإن لم يكن مُربّعا فإنّهما لا يجتمعان ولا ينطرحان.
فإذا علمت أنّهما يجتمعان، فخذ جذري الخارج وزد عليه[2] مجموع العددين، فما اجتمع فخذ جذره، يكن المطلوب.

مثال من ذلك : لو قيل اجمع جذر ثلاثة إلى جذر سبعة وعشرين.
فنضرب الثلاثة في السّبعة وعشرين بواحد وثمانين، وهو مُربّع. فنأخذ جذريه[3] بثمانية عشر، نحملها على مجموع العددين. وجذر المجتمع هو المطلوب. وذلك جذر ثمانية وأربعين.

وفيه وجه آخر : وهو أن نقسم أحد المجمُوعين على الآخر[4] ونحمل واحدًا على الخارج فنضرب المُجتمع في المقسوم عليها منهما. فما خرج، فهو مجمُوعهما. فكأنّا قسّمنا جذر السبعة والعشرين على جذر الثلاثة. يخرج ثلاثة، نحمل عليها واحدًا، ونضرب المُجتمع في جذر الثلاثة، المقسوم عليها[5]، على ما تبيّن في ضرب الجذور، يكون الخارج جذر ثمانية وأربعين، وهو المطلوب، كما تقدّم.

ومثال منه آخر : لو قيل اجمع جذر إثنين إلى جذر ثمانية.
فنضرب الإثنين في الثمانية بستّة عشر ونأخذ جذريهما بثمانية، نحملها على مجموع العددين. وجذر المُجتمع هو المطلوب[6]. وذلك[7] جذر ثمانية عشر.
ولو شئنا لعملنا فيها بالوجه الثاني، يخرج المطلوب.

ومثال منه آخر : لو قيل اجمع نصف جذر عشرين إلى جذري[8] خمسة.

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[1] في (أ) وفي (ت) : "تصرب أحد"، لفظ "أحد" زائدة.
[2] في (س) : "وزد على" و في (أ) : "وزده على".
[3] في (ب): "جذره مرّتين".
[4] في (ت): "مجموع العددين"، وهو خطأ.
[5] في (أ) : "تخرج الثلاثة، تحملها عليها".
[6] في (أ) : "المطلوب، كما تقدّم".
[7] في (أ) : " مثال منه آخر : وذلك ". هذه العبارات زائدة.
[8] في (ط) : "جذر". ولا يتطابق مع حل المسألة.

فنصف جذر عشرين أقلّ من جذر واحد، (فنردّه إلى جذر واحد)[1]، كما ذكر في باب القسمة، فيكون على ما تبيّن (في عمل)[2] الضرب، جذر خمسة. (وجذرا خمسة)[3] أكثر من جذر واحد، فنردّه إلى جذر واحد. فيكون أيضًا جذر عشرين. فكأنّه قيل اجمع جذر خمسة [47 و] إلى جذر عشرين. فنعمل (على ما)[4] تقدّم، يخرج المطلوب، وذلك جذر خمسة وأربعين.

وكذلك لو كان باختلاف مرتبة الجذور لرددناه إلى مرتبة واحدة.

مثل أن يكون المجمُوع جذر عدد مُنطق في القوّة والمجمُوع إليه جذر جذر[5] عدد (أي) مُوسّط. فنردّ المُنطق في القوّة مُوسّطا من نسبة صاحبه، حينئذ يجمع[6].

[ومثال منه آخر : لو قيل اجمع جذر ثلاثة إلى جذر خمسة عشر.
فنجد مُسطّحهما ليس بمُربّع. فهما مُتباينان، فنجمعُهما بحرف العطف، وذلك[7] جذر ثلاثة وجذر خمسة عشر.

وكلّ ما كان من أمثال هذا ممّا لا يُجتمع إلّا بحرف العطف هو المُسمّى بذي الاسمين.][8]

ومثال آخر منه : لو قيل اجمع نصف جذر جذر ثمانين إلى[9] ثلث رُبع جذر أربعة وثمانين وستمائة.
فقد عُلم ممّا تبيّن في باب الضرب أنّ نصف جذر جذر ثمانين هو جذر جذر خمسة وأنّ ثلث رُبع جذر أربعة وثمانين وستمائة هو جذر أربعة وثلاثة أرباع. (فكأنّه قيل اجمع جذر جذر خمسة إلى جذر أربعة وثلاثة أرباع)[10]. فنردّهما إلى مرتبة واحدة، كما ذُكر قبل. فتصير المسألة كأنّه قيل اجمع جذر جذر خمسة إلى

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[1] في (ط) : سقطت الجملة " فنردّه إلى جذر واحد ".
[2] في (ط) : " من ".
[3] في (م) وفي (أ) : سقطت "وجذرا خمسة". وفي (ط) : "وجذري عشرين".
[4] في (ط) وفي (ت) : "كما".
[5] في (ط) : سقط لفظ "جذر" الثاني.
[6] في (أ) : سقط لفظ "يجمع".
[7] في (ط) وفي (ت) : "كذلك".
[8] في (م) : سقطت الفقرة "ومثال منه آخر : لو قيل اجمع جذر ثلاثة ... هو المُسمّى بذي الإسمين".
[9] في (ت) : "إلا" وهو خطأ.
[10] في (ت) : سقطت الجملة " فكأنّه قيل اجمع ... وثلاثة أرباع ".

جذر جذر (إثنين وعشرين)[1] وأربعة أثمان ونصف ثمن. ومُسطّحُهما أيضًا ليس بمُربّع، فنجمعُهما بحرف العطف. وذلك جذر جذر خمسة وجذر جذر (إثنين وعشرين)[2] وأربعة أثمان ونصف ثمن. فاعلمه.

وفي الطرح تطرح جذري الخارج في[3] ضرب العددين (من مجموع العددين)[4] وتأخذ جذر الباقي، (يكون المطلوب.

مثال منه : لو قيل اطرح جذر[5])[6] ثمانية من جذر[7] إثنين وثلاثين.
فنضرب الثمانية في الإثنين والثلاثين بستّة وخمسين ومائتين. فنسقط جذريه، وذلك إثنان وثلاثون، من مجمُوع العددين ونأخذ جذر الباقي. يكون المطلوب، وذلك جذر ثمانية.

(وفيه وجه آخر : وهو أن يقسم أحد المطروحين على الآخر، ثمّ يؤخذ فضل ما بين الخارج والواحد ويضرب في المقسوم عليه، منهما يخرج المطلوب)[8].
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مثاله : لو قيل اطرح جذر إثني عشر من جذر سبعة وعشرين.
فنقسم جذر السّبعة والعشرين على جذر الإثني عشر، يخرج واحد ونصف. نأخذ فضل ما بينها وبين الواحد، وذلك نصف، نضربه في جذر الإثني عشر، المقسوم عليها، يخرج جذر[9] ثلاثة، وهو المطلوب[10].

ولو سمّينا[11] (جذر الإثني عشر من)[1] جذر سبعة وعشرين، يخرج ثلثان. نأخذ فضل ما بينهما وبين الواحد، وذلك ثلث، نضربه في جذر السّبعة والعشرين، المُسمّى منها، فما خرج فهو المطلوب، وذلك جذر ثلاثة[2]. فاعلمه.

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[1] في النسخ الاخرى : "عشرين"، وهذا خطأ.
[2] في النسخ الاخرى : "عشرين"، وهذا خطأ.
[3] في (س) وفي (ط) وفي (ت): "من".
[4] في (م) : سقط "من مجموع العددين".
[5] في (أ) : سقط لفظ "جذر".
[6] في (م) : سقط " يكن المطلوب أطرح جذر".
[7] في (أ) : سقط لفظ "جذر".
[8] في (س) : غابت الجملة "وفيه وجه آخر ... المطلوب". ويعتبرها ناسخ (ت) من شرح الهواري.
[9] في (أ) : سقط لفظ "جذر".
[10] في (ت) : سقط لفظ "جذر".
[11] في (ط) وفي (ت) : "شئنا".

(ولو كان المطروح أو المطروح منه أكثر[3] من جذر واحد، أو أقلّ، أو باختلاف مرتبة الجذور فيهما، فلا بدّ من ردّهما إلى جذر واحد، ومرتبة واحدة، كالجمع سواء)[4].

ومثال منه آخر : لو قيل اطرح جذر ثمانية من جذر عشرة.
فنجد مُسطحهما غير مُربّع. فهما مُتباينان. فطرحهما بحرف الاستثناء، وذلك جذر عشرة إلّا جذر ثمانية.
وكذلك العمل فيما أشبهه. فاعلمه.

وكلّ ما كان[5] من مثل هذا أيضًا ممّا لا ينطرح[6] إلّا بحرف الاستثناء هو المُسمّى بالمنفصل. فافهمه[7].

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[1] في (م) : سقط "جذر الإثني عشر من".
[2] في (أ) : "ثمانية" ، وهو خطأ.
[3] في (أ) وفي (ب) : "أكبر".
[4] في (س) : غابت الجملة "ولو كان المطروح ... كالجمع سواء". ويعتبرها ناسخ (ت) من شرح الهواري.
[5] في (ت) : سقط لفظ "كان".
[6] في (أ) : "يطرح".
[7] في النسخ الاخرى : "فاعلمه".

الباب الثالث في ضرب الجذور

والعمل في ذلك أن تضرب أحد العددين في الثاني وتأخذ جذر الخارج، فما كان فهو الخارج من ضرب جذر أحدهما في جذر الآخر.

مثال منه : لو قيل اضرب جذر ثمانية في جذر تسعة.
فنضرب الثمانية في التّسعة وجذر المُجتمع هو المطلوب. (وذاك جذر إثنين وسبعين)[1].

مثال منه آخر : لو قيل اضرب (جذر جذر)[2] خمسة في جذر جذر سبعة.
فنضرب الخمسة في السبعة ونوقّع جذر الجذر، كما كان في المضروبين، على الخارج، يكون المطلوب. وذلك جذر جذر خمسة وثلاثين.
|48 و|
وكذلك القياس في سائر المُوسّطات. (وإن بعدت عن المنطق ما بعدت)[3].

ومثال آخر منها : لو قيل اضرب (جذر جذر جذر)[4] ثلاثة في جذر جذر ثمانية.
فنضرب الثمانية في الثلاثة ونُوقّع جذر جذر الجذر على الخارج، يكون المطلوب. وذلك جذر جذر جذر أربعة وعشرين.

ومثال منه آخر : لو قيل اضرب ثلاثة في جذر سبعة إلّا إثنين.
فالعمل فيها أن نضرب الثلاثة في جذر السّبعة ونُنقّص من المُجتمع ضربها أيضًا في الإثنين المُستثناة، فما بقي فهو المطلوب. وذلك جذر ثلاثة وستّين إلّا ستّة.

وأصله ضرب الزّائد والنّاقص على ما يأتي في الجبر، بحول الله تعلى.

وإن أردت ضرب عدد في جذر عدد، فربّع العدد واصنع بالعددين[5] كما ذُكر.

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[1] في (ط) وفي (ت) : غابت الجملة : "وذاك جذر إثنين وسبعين".
[2] في (ت) : "جذر"، وهو خطأ.
[3] في (أ) : غابت الجملة : "وإن بعدت عن المنطق ما بعدت".
[4] في (أ) وفي (ط) : "جذر جذر"، وهو خطأ.
[5] في النسخ الأخرى : "بالمُربّعين".

مثاله : لو قيل اضرب ثلاثة في جذر سبعة.
(فنضرب الثلاثة في نفسها بتسعة. فكأنّه قيل: اضرب جذر تسعة في جذر سبعة[1])[2]. فنعمل كما تقدّم. يخرج المطلوب، وذلك جذر ثلاثة وستّين.

ومثال منه آخر : لو قيل اضرب إثنين في جذر جذر ثلاثة.
فنضرب الإثنين في مثلها، وما اجتمع في مثله، فما كان ضُرب في الثلاثة. (وجذر جذر)[3] المُجتمع هو المطلوب، وذلك جذر جذرثمانية وأربعين.

وكذلك القياس فيما هو أكثر من ذلك. فاعلمه.

ومن هذا الأصل، أعني قوله: "وإن أردت ضرب عدد في جذر عدد، إلى آخره"، يُعرف العمل في ردّ المسألة إلى جذر واحد إذا كان اللفظ بأكثر من جذر واحد، (أو بأقلّ من جذر واحد، وتضعيف جذر[4] عدد أو تجزئته.

ومثال جذر[5] ما هو بأكثر من جذر واحد)[6] : لو قيل اضرب إثنين في جذري سبعة.
| 48 ظ | لوجب أن ننظر لأيّ عدديهما جذر السّبعة جذر. وذلك أن نضرب الإثنين، التي هي عدد[7] الجذر، في نفسها، وما اجتمع في السّبعة[8]، ونأخذ جذر الخارج، يكون المطلوب. وذلك جذر ثمانية وعشرين. فكأنّه قيل: اضرب إثنين في جذر ثمانية وعشرين. فنعمل كما[9] تقدّم. يخرج المطلوب، وذلك جذر إثني عشر ومائة.

ومثال منه آخر : لو قيل اضرب خمسة في ثلاثة أجذار جذر إثنين.
فننظر أيضًا لأيّ عدد تكون ثلاثة أجذار جذر[10] إثنين جذر جذر، وذلك بأن نضرب الثلاثة، عدد الأجذار، في نفسها، (وما اجتمع في نفسه)[1]، وما اجتمع

[1] في (أ) : "تسعة" ، وهو خطأ.
[2] في (ط) : سقطت الجملة "فنضرب ... في جذر سبعة".
[3] في (ط) : "جذر جذر جذر"، وهو خطأ.
[4] في (أ) : سقط لفظ "جذر".
[5] في (م) وفي (ب) : سقط لفظ "جذر".
[6] في (ط) وفي (ت) : سقطت الجملة "أو بأقلّ من جذر واحد ... بأكثر من جذر واحد".
[7] في (م) : سقط لفظ "عدد".
[8] في (م) : "الستّة"، وهو خطأ.
[9] في (ط) وفي (ت) : "على ما".
[10] في (ط) وفي (ت) : سقط لفظ "جذر".

في الإثنين، ونأخذ (جذر جذر)[2] الخارج. يكون المطلوب، (وذلك جذر جذر إثنين وستّين ومائة. فكأنّه قيل: اضرب خمسة في جذر جذر إثنين وستّين ومائة. فنعمل كما[3] تقدّم. يخرج المطلوب)[4]، وذلك جذر جذر خمسين ومائتين وألف ومائة ألف.

وكذلك العمل فيما هو أكثر من ذلك.

ومثال ما هو بأقلّ من جذر واحد : لو قيل اضرب ثلثين[5] في نصف جذر عشرين.
لنظرنا نصف جذر عشرين وصرّفناه إلى ما يكون جذرا له، على ما تقدّم. وذلك بأن نضرب النّصف في جذر العشرين، فيكون، على ما بيّناه، جذر الخمسة. فكأنّه قيل: اضرب ثلثين[6] في جذر خمسة. فنعمل فيها أيضًا على ما تقدّم. فما خرج فهو المطلوب، وذلك جذر إثنين وتسعين.

ومثال منه آخر : لو قيل اضرب جذر خمسة في نصف جذر جذر أربعين.
فننظر نصف جذر جذر أربعين ونصرّفه إلى ما يكون له (جذر جذر)[7]، على ما تقدّم، وذلك بأن نضرب النّصف في مثله، وما اجتمع في مثله، والمُجتمع في الأربعين. فيكون، على ما بيّناه، جذر جذر إثنين ونصف. فكأنّه قيل: اضرب جذر خمسة في (جذر جذر)[8] إثنين ونصف[9]. (فنعمل فيه أيضًا كما تقدّم. فما خرج فهو المطلوب، |49 و| وذلك جذر جذر اثنين وستّين ونصف)[10].
وكذلك العمل فيما أشبه ذلك.

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[1] في (ط) : سقطت الجملة "و ما اجتمع في نفسه"، وهو خطأ.
[2] في (م) : "جذر"، وهو خطأ.
[3] في (ت) : "على ما".
[4] في (م) : سقطت الجملة "وذلك جذر جذر ... المطلوب".
[5] في (أ) : "إثنين"، وهو خطأ.
[6] في (أ) : "إثنين"، وهو خطأ.
[7] في (أ) : "جذر حذر جذر"، وهو خطأ.
[8] في (م) : "جذر"، وهو خطأ.
[9] في (أ) : سقطت لفظ "نصف".
[10] في (ت) : سقطت الجملة "فنعمل فيه ... اثنين وستّين ونصف ".

ومثال تضعيف الجذور : لو قيل اضعف جذر ثلاثة مرّتين.
فكأنّه قيل: اضرب إثنين في جذر ثلاثة. فنعمل كما[1] تقدّم، يخرج المطلوب. وذلك جذر إثني عشر.

ومثال منه آخر : لو قيل اضعف جذر سبعة خمس مرّات.
فكأنّه قيل: اضرب خمسة في جذر سبعة. فنعمل كما تقدّم، يخرج المطلوب. وذلك جذر خمسة وسبعين[2] ومائة.

وكذلك العمل في ما أشبه ذلك.

ومثال تجزئة الجذور : لو قيل كم نصف جذر عشرة؟
(فكأنّه قيل: اضرب نصفا في جذر عشرة)[3]. فتعمل كما تقدّم، يخرج المطلوب. وذلك جذر إثنين ونصف.

ومثال منه آخر : (لو قيل)[4] كم ثلث أربعة أثمان [(جذر جذر)[5] ستّين؟
فكأنّه قيل اضرب ثلث أربعة أثمان][6] في جذر جذر ستّين. فنعمل كما[7] تقدّم، يخرج المطلوب[8]. وذلك جذر جذر سُدسي تسع ونصف[9] سُدس تسع. فاعلمه[10].

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[1] في (ت) : "على ما".
[2] في (م) : "تسعين"، وهو خطأ.
[3] في (أ) : سقطت الجملة "فكأنّه قيل: اضرب نصفا في جذر عشرة".
[4] في (ت) : سقطت الجملة "لو قيل".
[5] في (م) : "جذر"، وهو خطأ.
[6] في (أ) : سقطت الجملة "جذر جذر ستّين؟ فكأنّه قيل اضرب ثلث أربعة أثمان".
[7] في (ت) : "على ما".
[8] في (أ) : سقطت لفظ "فكأنّه قيل: اضرب نصفا في جذر عشرة".
[9] في (أ) : سقطت لفظ "ونصف"، وهو خاطأ.
[10] في (ب) : "فاعلم ذلك".

الباب الرّابع في قسمة جذور الأعداد و تسميتها

تقسم العدد على العدد (أو تسمّيه منه)[1] وتأخذ جذر الخارج، فما كان فهو الخارج من قسمة (جذر المقسوم على)[2] جذر المقسوم عليه.

مثال منه : لو قيل اقسم جذر عشرين على جذر ثلاثة.
فنقسم العشرين على الثلاثة ونوقّع الجذر على الخارج، يكون المطلوب. وذلك جذر ستّة وثلثين[3].

ومثال منه آخر : لو قيل اقسم جذر ثلاثة على جذر ثمانية.
فنسمّي[4] الثلاثة من الثمانية ونُوقّع الجذر على الخارج، يكون المطلوب. وذلك جذر ثلاثة أثمان.

وكذلك العمل في المُوسّطات.

مثال منها : لو قيل اقسم جذر جذر ستّة على (جذر جذر)[5] إثنين. [49 ظ]
فنقسم الستّة على الإثنين ونُوقّع جذر الجذر على الخارج، يكون المطلوب. وذلك جذر جذر ثلاثة.

ومثال منه آخر : لو قيل اقسم جذر جذر ثمانية عشر على جذر جذر إثنين وثلاثين.
فنُسمّي الثمانية عشر من الإثنين والثلاثين ونُوقّع جذر الجذر على الخارج، يكون المطلوب. وذلك جذر جذر أربعة أثمان ونصف ثمن.

فهذا هو القياس في سائر المُوسّطات[6] وإن بعدت عن المُنطق ما بعدت.

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[1] في (م) : "أو تسمّيه". وفي (ب) وفي (ط) : سقطت الجملة "أو تسمّيه منه".
[2] في (م) : سقطت الجملة "جذر المقسوم على".
[3] في (ت) : "ثلاثين"، وهو خطأ.
[4] في (أ) : "تقسم".
[5] في (م) : "جذر"، وهو خطأ.
[6] في (أ) : "المُتوسّطات".

ومتى ورد اللفظ في هذه الأبواب الثلاثة، (يعني الجمع والضرب والقسمة)[1]، بأكثر من جذر واحد، أو بأقلّ من جذر واحد، أو اختلاف مراتب الجذور، فردّ ذلك إلى جذر واحد ومرتبة واحدة.

قلت وقد تقدّم في بابي الجمع والضرب أمثلة منه وكيفية العمل فيها وأمثالها. ولنورد من ذلك أيضًا في هذا[2] الباب أمثلة :

فمن ذلك، لو قيل اقسم جذر جذر أربعة عشر على جذر إثنين.
فالمقسُوم عليه مُنطق في القوّة والمقسُوم مُتوسّط. (فنردّ جذر)[3] الإثنين حتّى يكون[4] مُتوسّطا مثل المقسُوم، وحينئذ نقسم. فيكون جذر جذر أربعة، فإذا قسّمنا عليها المقسُوم[5] وأوقعنا جذر الجذر، كما هو في المقسُومين على الخارج، كان المطلوب. وذلك جذر جذر ثلاثة ونصف.

مثال منه آخر : لو قيل اقسم جذري خمسة عشر على إثنين.
فقد عُلم أنّ جذري الخمسة عشر مثل جذر ستّين وأنّ الإثنين هو جذر أربعة، فكأنّه قيل: اقسم جذر ستّين على جذر أربعة. فنعمل كما تقدّم، يخرج المطلوب. وذلك جذر خمسة عشر.

ومثال منه آخر : لو قيل اقسم نصف جذر أربعة وعشرين على جذر إثنين.
فقد عُلم أنّ نصف جذر أربعة وعشرين مثل جذر ستّة. فكأنّه قيل: اقسم جذر ستّة على جذر اثنين. فنعمل كما تقدّم، (يخرج المطلوب)[6]، وذلك جذر ثلاثة.

وأما القسمة على ذوات الأسماء والمُنفصلات، فهو أن تضرب المقسُوم والمقسُوم عليه في مُنفصل المقسُوم عليه إن كان مُتّصلا[7] من إسمين، أو في مُتّصله[8] إن كان مُنفصلا[9]، ثمّ تقسم الخارج من المقسوم على الخارج من المقسوم عليه.

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[1] في (س) : سقط الجملة "يعني الجمع والضرب والقسمة".
[2] في (م) : سقط لفظ "هذا".
[3] كل النّسخ خاطئة.
[4] في (ت) : "يكون الجذر".
[5] في (ط) : سقط لفظ "المقسوم". وهذا خطأ.
[6] في (ط) : سقطت الجملة "يخرج المطلوب".
[7] في (ط) : سقط لفظ "متصلا".
[8] في (ط) وفي (ت) : "منفصله"، وهو خطأ.
[9] في (أ) "مُتّصلا"، وهو خطأ.

مثال من ذلك : لو قيل اقسم إثني [50 و] عشر على خمسة وجذر ثلاثة.
فنضرب الإثني عشر، المقسُومة[1]، في الخمسة إلّا جذر الثلاثة، مُنفصل المقسوم عليه، ونعتبر الزّائد والنّاقص على ما تبيّن في الجبر، يكون ذلك ستّين إلّا جذر إثنين وثلاثين وأربعمائة، وهو الخارج من المقسوم.
فنقسمه على الخارج من ضرب الخمسة وجذر الثلاثة، المقسوم عليها، في الخمسة إلّا جذر الثلاثة، مُنفصلها، وذلك إثنان وعشرون – لأنّ كلّ ذي إسمين يُضرب في مُنفصله، أو بالعكس، فإنّ الخارج فضل ما بين مُربّعي الإسمين –. فيكون الخارج من القسمة هو المطلوب، وذلك إثنان وثمانية أجزاء من أحد عشر جزءا[2] إلّا جذر تسعة أجزاء من أحد عشر وتسعة أجزاء من أحد عشر (في الجزء)[3] من أحد عشر.

ومثال منه آخر : لو قيل اقسم عشرة على ثلاثة إلّا جذر سبعة.
فنضرب العشرة، المقسومة، في الثلاثة وجذر السبعة، مُتّصل المقسوم عليه، يكون الخارج ثلاثين وجذر سبعمائة. نقسمه على الخارج من ضرب الثلاثة إلّا جذر السبعة، المقسوم عليها، في مُتّصلها، وذلك إثنان. فيكون الخارج من القسمة هو المطلوب. وذلك خمسة عشر وجذر خمسة وسبعين ومائة. فاعلمه.

وقد كمل القسم الثالث، بحمد الله (وحسن عونه)[4].

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[1] في (ط) وفي (ت): سقط لفظ "المقسُومة".
[2] في (م) وفي (ط) : سقط لفظ "جزءا".
[3] في (أ) وفي (ت) : "جزءا".
[4] في (ب) : "وعونه" وفي (ت): " تعالى ". وفي (ط) : "بحمد الله".

[الجزء الثاني :

في القوانين التي يمكن بها الوصول إلى معرفة المجهول المطلوب من المعلوم المفروض

وهو ينقسم قسمين : قسم[1] في العمل بالنّسبة (وقسم في الجبر)[2] والمقابلة][3]

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[1] في (ت) : "الأول".
[2] في (ت) : "الثاني بالجبر".
[3] في (م) وفي (أ) : سقطت العناوين والجملة: " الجزء الثاني" إلى "وقسم في الجبر والمقابلة".

القسم الأول : في العمل بالنّسبة

وهي على ضربين : بالأربعة الأعداد المتناسبة[1] وبالكفّات.

مخطوط المدينة المنورة : 50 و – 56 و
مخطوط استانبول : 93 و – 97 ظ.
مخطوط أكسفرد (ب) : 149ظ – 154 ظ
مخطوط تهران (ط) : 46 زظ – 51 و
مخطوط تونس (ت) : 24 ظ – 28 و

[1] في (أ) وفي (ب) : سقط لفظ "المتناسبة". وزاد ناسخ (ب) في حاشية الورقة (150و) : "أي الأربعة التي ذكرها المصنف وهي الهندسية".

<الضرب الأول : العمل بالأربعة الأعداد المتناسبة >

قلت : والنّسبة على أنواع، منها العددية والتأليفية والمؤلّفة ونسبة المساواة (المذكورة هنا)[1].
أمّا العددية، فقد تقدّم حكمها، وأمّا الثلاث الأخرى فلم يذكرها واستغنى عنها[2] بالمذكورة، لأنّها أصل لتلك وهي قاعدة الحساب، وترجع الثلاث إليها ولا ترجع هي إليهن، على ما تبين في **رفع الحجاب**.

والأربعة الأعداد (المتناسبة هي التي)[3] نسبة الأوّل منها للثاني[4] كنسبة الثالث للرّابع[5]، وضرب الأوّل في الرّابع كضرب الثاني في الثالث.
ومتى ضربت[6] الأوّل في الرّابع وقسمت على الثاني [50 ظ] **خرج الثالث، أو على الثالث خرج الثاني.**
ومتى ضربت الثاني في الثالث وقسمت على الأوّل خرج الرّابع أو على الرّابع خرج الأوّل.
فأيّهما يكون مجهولا يخرج بهذا العمل من الثلاثة الباقية المعلومة.

ووجه العمل في ذلك أن تضرب العدد المُنفرد[7] المُخالف لجنس الآخرين في العدد المجهول نسبته وتقسم على العدد[8] الثالث يخرج المجهول.

مثال ذلك : نسبة ثلاثة إلى ستّة كنسبة أربعة إلى ثمانية. فإنّ الثلاثة من الستّة نصف، والأربعة من الثمانية نصف. فضرب الأوّل، وهو الثلاثة، في الرّابع، وهو الثمانية، كضرب الستّة، وهو الثاني، في الأربعة، وهو الثالث.

فلو ضربنا الثلاثة في الثمانية وقسمناها على الستّة خرجت الأربعة، أوعلى الأربعة خرجت الستّة.

---

[1] في (ت) : "المذكور هنا الأوّل".
[2] في النسخ الاخرى : سقط لفظ "عنها".
[3] في (ط) وفي (ت): سقط لفظ "المتناسبة"، وفي (أ) : سقط لفظ "التي".
[4] في (ت) : " إلى الثاني ".
[5] في (ت) : " إلى الرّابع ".
[6] في كل النسخ ما عدا (م) : "متى ضُرب ... وقُسم ...".
[7] في (ط) وفي (ت) : "المفرد".
[8] في (س) : سقط لفظ "العدد".

ولو ضربنا أيضًا الستّة في الأربعة وقسمنا على الثلاثة خرجت الثمانية، أو على الثمانية خرجت الثلاثة.

فلو كان الرّابع (منها مثلا)[1] مجهولا، وهي الثمانية، وأردنا استخراجها، فنعمل كما ذكرنا ونضرب (العدد المُنفرد)[2] المخالف لجنس الآخرين – (أي لصنفهما)[3]، وهو المُراد به[4]، حيث وقع ذكره في هذا الكتاب – وهو في المثال ستّة، لأنّها منسوبة إليها (والباقيات منسوبات في العدد)[5] المجهول نسبته، وهو الأربعة، لأنّ نسبة الرّابع مجهولة، (بأربعة وعشرين)[6]. نقسمها على الثالث من الثلاثة الباقية، وهو الأوّل، يخرج ثمانية، وهو الرّابع المجهول.

ولو كان الأوّل مجهولا، لقسمنا الأربعة والعشرين على الثمانية، يخرج الثلاثة.

ولو جُهل الثاني، وهي الستّة، لضربنا الثمانية، وهي العدد المنفرد أيضًا، لأنّها منسوبة إليها وحدها، (والباقيات منسوبات)[7] في العدد المجهول نسبته، وهو الأوّل، لأنّ الثاني نسبته مجهولة، بأربعة وعشرين. نقسمها على الأربعة[8]، وهو الثالث من الثلاثة الباقية، يخرج ستّة، وهو الثاني المجهول.

ولو كان الثالث مجهولا، وهو الأربعة، [|51 و|] لقسمنا الأربعة والعشرين على الستّة تخرج الأربعة. فاعلمه.

والأربعة الأعداد المتناسبة [إذا بدلت فكانت نسبة الأوّل للثالث والثاني للرّابع، أو خولف فيها فكانت نسبة الثاني للأوّل والرّابع للثالث، أو رُكّبت فكانت نسبة مجموع الأوّل والثاني إلى أحدهما[9] كنسبة مجموع الثالث والرّابع إلى أحدهما،

---

[1] في (ط) وفي (ت) : سقط لفظ "منها" وفي (ب) : سقط لفظ "مثلا".
[2] في (ت) : سقط لفظ "العدد"، وفي (ط) وفي (ت) : "المبعود".
[3] في (أ) : " إلى نصفها " ، وهذا خطأ.
[4] في (ط) وفي (ت) : سقط لفظ "به".
[5] في (ب) : "الباقيان منسوبان في العدد". وفي (ط) وفي (ت) : " الباقيات منسوبات فالعدد ".
[6] في (ب) : "فأربعة وعشرون".
[7] في (ب) وفي (ط) : "الباقيان منسوبان".
[8] في (أ) : " الثالث ".
[9] في **رفع الحجاب**، النص أوضح حيث يكتب: "إلى الأوّل أو إلى الثاني"، وكذلك في الحالات المشابهة التابعة في نفس الفقرة.

(أو فُصّلت فكانت نسبة فضل ما بين الأوّل والثاني إلى أحدهما)[1] كنسبة فضل ما بين الثالث والرّابع إلى أحدهما، أو رُكّب تبديلها، (أو فضل تبديلها)[2] أو بدل تركيبها، أو بدل تفضيلها، أو خولف في ذلك كلّه، بقيت متناسبة][3].

ولاستخراج المجهول منها أربعة أوجه غير المذكورة.

أحدها : لو جُهل الرّابع مثلا، نقسم الثاني على الأوّل ونضرب الخارج في الثالث، يكون الرّابع.

والثاني : نقسم الثالث على الأوّل ونضرب الخارج في الثانى، يكون الرّابع.

والثالث : نقسم الأوّل على الثاني وما خرج يقسم عليه الثالث، يكون الرّابع.

والرّابع : نقسم الأوّل على الثالث وما خرج يقسم عليه الثاني، يكون الرّابع.

وإنّما لم يذكرها المؤلف[4]، لأنّ الذي ذكره هو الأصل لها وترجع هي إليه ولا يرجع هُو إليها.

وحكى الفقيه **أبو محمد عبد الحق بن طاهر**[5]**:** "إنّ الوجه الأوّل يُسمّى عملا والأربعة الأخرى تُسمّى أقيسة".

---

[1] في (م) وفي (ت) : الجملة : " أو فُصّلت ... أحدهما " غائبة.

[2] في (ط) وفي (ت) : سقطت الجملة " أو فضل تبديلها ".

[3] الفقرة : " إذا بدلت ... بقيت متناسبة." منقولة من **رفع الحجاب**، صفحة 294: 3-9.
ولكن يبدو أن الهواري لخّص النصّ الأصلي حسب ما ورد في تحقيق أبلاغ وهو: " إذا بدلت فكانت نسبة الأوّل للثالث والثانى للرّابع، أو خولف بها فكانت نسبة الثاني للأوّل ونسبة الرّابع للثالث، أو رُكّبت فكانت نسبة مجموع الأوّل والثاني إلى الأوّل أو إلى الثاني أحدهما كنسبة مجموع الثالث والرّابع إلى الثالث أو إلى الرّابع، أو فُصّلت فكانت نسبة فضل ما بين الأوّل والثاني إلى أحدهما كنسبة فضل ما بين الثالث والرّابع إلى أحدهما، أو رُكّب تبديلهما، أو فُصّل تبديلهما أو بُدل تركيبها، أو بدّل تفصيلها، أو رُكّب تبديل تركيبها، أو رُكّب تبديل تفصيلها، أو فُصّل تبديل تركيبها، أو فُصّل تبديل تفصيلها، أو خولف في ذلك كلّه، بقيت متناسبة".

[4] زائد في (أ) و (ت): "المؤلف عليه الرحمة".

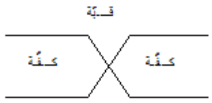
[5] لم نتمكن بعدُ بتثبيت هوية هذا الفقيه.

<الضرب الثاني : العمل بالكفات>

وأمّا الكفات، فهي من الصّناعة الهندسيّة.

وإنّما كانت من الصّناعة الهندسيّة [لأنّ نسبة خطأ كل كفّة إلى فضل ما بين كفّته والعدد المجهول كنسبة العدد المفروض إلى المجهول][1]. وهو تبديل تفصيل، تبديل نسبة الجزء إلى كفّته كنسبة العدد المعلوم إلى المجهول، على ما تبيّن في **رفع الحجاب**.

وصُورتها أن تصوّر ميزانا على هذه الصُورة :



وتضع المعلوم المفروض على قبته. | 51 ظ | وتتّخذ إحدى الكفتين من أيّ عدد شئت وتفعل (في ذلك ما فُرض)[2] **من الجمع أو الحطّ أو غير ذلك من الأعمال، ثمّ تقابل (بها ما)**[3] **على القبّة.**
فإن أصبت، فتلك الكفّة هي العدد المجهول.
وإن أخطأت، فارسم الخطأ فوق الكفّة إن كان زائدًا أو تحتها إن كان ناقصًا.
ثمّ اتّخذ الكفّة الأخرى من أيّ عدد شئت غير الأوّل واصنع بها كما صنعت بالأول. ثمّ اضرب خطأ كلّ كفّة في صحيح الأخرى. ثمّ انظر: فإن كان الخطآن زائدين أو ناقصين فانقص أقلّهما من أكثرهما وأقلّ الضربين[4] **من أكثرهما. واقسم الباقي من الضربين على الباقي من الخطأين.**
وإن كان أحدهما زائدًا والآخر ناقصًا قسمت مجموع الضربين على مجموع الخطأين. (يعني[5] **يخرج المطلوب)**[6]**.**

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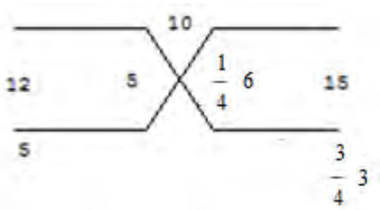
[1] الجملة : "نسبة ... المجهول"، منقولة من **رفع الحجاب**، صفحة 297: 17-18.
[2] في (أ) : "ما فُرض في ذلك".
[3] في (س) . لفظ "بها" غائب، وفي (ط) وفي (ت) : "بما".
[4] في (أ) : يستعمل الناسخ لفظ "المضروبين" عوض "الضربين" هنا وفي باقي الجملة.
[5] في (أ) وفي (ط) وفي (ت) : سقط لفظ "يعني".
[6] في (س) : الجملة "يجرج المطاوب" غائبة.

مثال من ذلك : لو قيل[1] مال ذهب ثُلثه ورُبعه، فبقي منه عشرة. كم المال؟
فنُصوّر ميزانا كما وُصف ونضع العشرة المعلومة على القبّة. فكأنّا أخذنا الكفّة الأولى من خمسة عشر، فنضعها ما بين خطي الميزان، ثمّ نأخذ ثُلثها ورُبعها، وذلك ثمانية وثلاثة أرباع، (ونسقطه منها، الباقي)[2] ستّه ورُبع، وهو الجزء الذي نقابل به ما على القبّة. فنضعه في داخل الميزان أيضًا مُوازيًا للكفّة. ويبقى من العشرة التي على القبّة، بعد المقابلة، ثلاثة وثلاثة أرباع. وهو خطأ الكفّة، وهو ناقص، نضعه تحت الكفّة كما ذُكر.
فلو كان الجزء المقابل به، مثلا عشرة، لكانت الكفّة هي المجهول.
وكأنّا أخذنا الكفّة الثانية من إثني عشر. فنضعها أيضًا في داخل الكفة[3] من الناحية الثانية، ونأخذ ثُلثها ورُبعها بسبعة، نطرحه منها، تبقى خمسة، وهو الجزء الذى نقابل به أيضًا. نضعه في داخل الميزان مُوازيًا للكفّة. ويبقى من العشرة، أيضًا بعد المُقابلة، 52 و خمسة، وهو خطأ الكفّة الثانية، وهو ناقص. نضعه تحت الكفّة، فيكون على هذه الصورة :



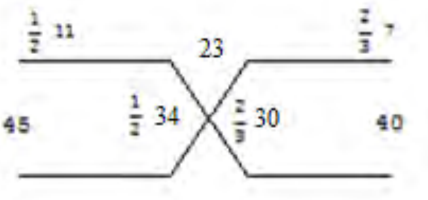
ثمّ نضرب الثلاثة والثلاثة الأرباع، خطأ الأولى، في الإثني عشر، صحيح الثانية، يكون ذلك خمسة وأربعون، وهو أحد الضربين. ثمّ نضرب الخمسة، خطأ الثانية، في الخمسة عشر، صحيح الأولى، يكون ذلك خمسة وسبعون، وهو الضرب الثاني. فنسقط منه الضرب الأوّل، الأصل، لأنّ الخطئين ناقصأن. الباقي ثلاثون. نحفظها. ثمّ نسقط الخطأ الأوّل، لأنّه الأقلّ من الخطأ الثاني، لأنّه[4] الأكثر. الباقي واحد ورُبع. نقسم عليه المحفوظ، فما خرج، (فهو المال المجهول. وذلك أربعة وعشرون)[5].

---

[1] في (م) : سقطت عبارة "لو قيل".
[2] في (أ) : "نسقط منها الباقي"، وهو لا يصحّ.
[3] في (أ) وفي (ب) : "الميزان".
[4] في (م) : سقط لفظ "لانه".
[5] في (أ) : الجملة "فهو ... عشرون" غائبة.

مثال منه آخر : لو قيل مال (أخذنا جمع)[1] ثُلثه وخمسه[2]، وحملنا عليه نصف ما بقي[3]، فكان ثلاثة وعشرين. كم المال؟

فنُصوّر الميزان أيضا ونضع الثلاثة والعشرين فوق القبّة. وكأنّا أخذنا إحدى الكفّتين من أربعين. فنأخذ ثُلثها وخُمسها، ونحمل عليه نصف ما بقي، كما ذُكر، يكون ثلاثين وثُلثين[4]. وهو الجزء الذي نُقابل به ما على القبّة. فنخطأ بسبعة وثُلثين، زائدة. وكأنّا أخذنا الكفّة الثانية من خمسة وأربعين. فنأخذ ثُلثها وخُمسها ونصف ما بقي. يكون أربعة وثلاثين ونصفا، وهو الجزء الذي نقابل به أيضًا. فنخطأ بأحد عشر ونصف، زائدة، (فنضعه والأوّل فوق كفّتيها. فيكون)[5] على هذه الصورة :



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ثمّ نضرب السّبعة والثُلثين، خطأ الكفّة[6] الأولى في الخمسة والأربعين، صحيح الثانية، | 52 ظ | يكون ذلك خمسة وأربعين[7] وثلاثمائة، وهو الضرب الأوّل. وبالعكس، ستّون وأربعمائة، وهو الضرب الثاني. فنسقط منه الضرب الأوّل الأقلّ، لأنّ الخطأين زائدان. يبقى خمسة عشر ومائة. نقسمها على فضل ما بين الخطأين، وذلك ثلاثة وخمسة أسداس. فما خرج، فهو المال المجهول، وذلك ثلاثون.

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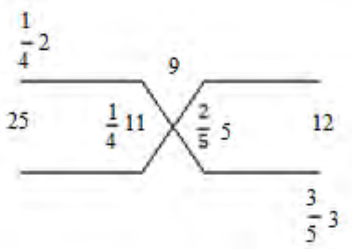
1 في (أ) وفي (ب) : "جُمع" وفي (ت) : "أخذنا".
2 في (م) : "خمسة"، وهذا خطأ.
3 في (ت) : "نصفها".
4 في (ط) في (ت) : سقط لفظ "ثلثين".
5 في (أ) : "فنضع الأوّل فوق كفّتيها"، وهذا لا يصح.
6 في (أ) وفي (ب) : سقط لفظ "الكفة".
7 في (أ) : "وعشرين"، وهذا لا يصح.

ومثال منه آخر : لو قيل مال حملنا[1] على فضل ما بين رُبعه (وثلاثة أخماسه عُشرُه)[2]، فكان تسعة. كم المال؟

فنصوّر الميزان أيضًا ونتّخذ احدى الكفّتين من إثني عشر ونسقط رُبعها من ثلاثة أخماسها. الباقي أربعة وخُمس. نحمل عليها عُشر الكفّة، يكون الجميع خمسة وخُمسين، وهو الجزء[3] الذي[4] نقابل به ما على القبّة. فنخطأ بثلاثة وثلاثة أخماس، ناقصة. نثبتها تحت الكفّة ونتّخذ الكفّة الثانية من خمسة وعشرين. ونسقط رُبعها أيضًا من ثلاثة أخماسها. الباقي ثمانية وثلاثة أرباع. نحمل عليها عُشر الكفّة، يكون الجميع أحد عشر و رُبعا، وهو الجزء الذي نقابل به أيضًا. فنخطأ بإثنين ورُبع، زائدة. نثبته فوق الكفّة، فيكون على هذه الصورة[5] :



ثّم نضرب خطأ الأولى في صحيح الثانية بتسعين، وبالعكس بسبعة وعشرين. فنجمع هذين الضربين، لأنّ أحد الخطأين زائد والآخر ناقص، (يكون الجميع سبعة[6] عشر ومائة)[7]. نقسمها على مجموع الخطأين، وذلك خمسة وأربعة أخماس ورُبع خُمس. فما خرج، فهو المال المجهول، وذلك عشرون. فاعلمه.

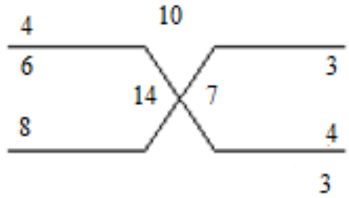
وإن شئت، فاتّخذ الكفّة الثانية من العدد الأوّل أو من غيره، واخرج جزء و 53 **هذا الذي تقابل به ما على القبّة، واضربه في صحيح الأولى، واضرب خطأ الأولى في صحيح الثانية. ثمّ إن كان خطأ الأولى ناقصًا جمعت الضربين**[8]

[1] في (أ) : "حملته"، وهذا خطأ.
[2] في (أ) : "وثلاثة أخماس عُشره"، وهذا لا يصح.
[3] في (أ) وفي (ب) : "الجزء الأول".
[4] في (أ) : سقط لفظ "الذي".
[5] في (أ) : الصورة غائبة.
[6] في (م) : "تسعة"، وهذا خطأ.
[7] في (أ) : سقطت الجملة "يكون ... مائة ".
[8] في (ت) : "المضروبين".

وإن كان زائدًا أخذت فضل[1] ما بينهما. فما كان قسّمته على جزء الكفّة الثانية، يخرج المطلوب.

وهذا الوجه الثاني[2] لا يُعمل به[3] إلّا في ما فيه تناسب.

مثال من ذلك : لو قيل عشرة قسّمتها قسمين، فكان ثُلث أحدهما رُبع الآخر[4]. فنُصوّر الميزان أيضًا ونضع العشرة على قبّته، ثمّ نتّخذ كفّته من عددين، يكون[5] ثُلث أحدُهما رُبع الآخر. وهُما، مثلا[6]، ثلاثة وأربعة. فنقابل بمجموعهما، لأنّه جزء العشرة. فنخطأ بثلاثة ناقصة، ثمّ نتّخذ كفّة ثانية كذلك، وكأنّها ستّة وثمانية. فنخرج جزءها، وهو مجموع العددين، وذلك أربعة عشر. فتكون على هذه الصورة[7] :



فبأيّ قسم أردنا استخراجه، ضربنا خطأ الأولى في مقامه من الثانية وجزء الثانية في مقامه من الأولى.
(ونتمثّل)[8] العمل. فما خرج فهو القسم المطلوب وباقي العشرة القسم الآخر.
فكأنّا أردنا، مثلا، القسم الأصغر. فنضرب الثلاثة، خطأ الأولى، في السّتة، (مقامه في الثانية)[9]، بثمانية عشر. ثمّ نضرب الأربعة عشر، جزء الثانية، في الثلاثة، مقامه من الأولى، باِثنين وأربعين. نجمعه مع الضرب الأوّل لأنّ الخطأ ناقص، فيكون ذلك ستّين، نقسمها على الأربعة عشر، جزء الثانية. فما خرج فهو المطلوب، وذلك أربعة وسُبعان. وباقي العشرة للقسم الأكبر، وذلك خمسة وخمسة[10] أسباع.

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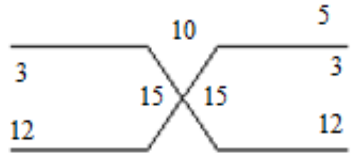
[1] في (ط) : سقط لفظ "فضل".
[2] في (ط) وفي (ت) : سقط لفظ "الثاني".
[3] في (أ) : سقط لفظ "به".
[4] في (م) : الجملة "وهُما، مثلا، ثلاثة" زائدة هنا.
[5] في (أ) وفي (ب) : سقط لفظ "يكون".
[6] في (أ) : سقط لفظ "مثلا".
[7] في (أ) : غابت الصورة.
[8] في (أ) : "وبه تم" وفي (ب): "وتكمل"، وفي (ت) : "مثل".
[9] في (ب) : "مقام الثانية" وفي (أ) : "مقام السّتّة" ، وهو خطأ.
[10] في (م) : "أربعة"، وهذا خطأ.

ومثال منه آخر : لو قيل عشرة قسّمتها قسمين، فقسّمت الأكبر على الأصغر، فخرج |53 ظ| أربعة. كم أحدهما؟[1]
فأحد القسمين، لا محالة، أربعة أمثال الآخر. فنُصوّر الميزان أيضًا ونضع العشرة على قبّته، ثمّ نتّخذ كفّة من عددين أحدهما (أربعة أمثال)[2] الآخر. وهما، مثلا، ثلاثة وإثنى عشر. ونُقابل بمجمُوعهما العشرة، لأنّه الجزء. فنخطأ بخمسة زائدة. ثمّ نتّخذ كفّة ثانية. فكأنّا أخذنا الأولى بنفسها. فيخرج جزءها وذلك خمسة عشر. فيكون على هذه الصورة[3] :



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فنعمل كما ذُكر في المثال قبل.
يخرج أحد القسمين إثنين والآخر باقي العشرة، وذلك ثمانية. فاعلمه.

ولو شئنا في المسألتين[4]، لعملناهما بالوجه الأوّل.

ونتبع بهذا الوجه الثاني [في كلّ مسألة يكون المفروض فيها كأنّه احدى الكفّتين وخطأها.

مثل ما[5] يقال : مالٌ طرحنا ثُلثه ورُبعه من ثُلث ستّين ورُبعها، فبقي أربعة عشر. كم المال؟
فالستّون هي احدى الكفّتين والأربعة عشر خطأها، وهو زائد. فنتّخذ كفّة أخرى من أيّ عدد شئنا ونخرج ثُلثها ورُبعها، وهو جزءها الذي نقابل به ما على القبّة لو كان ثمّ عدد. فنعمل على ما ذُكر، يخرج المال ستّة وثلاثين.

[1] في (ط) وفي (ت) : "المال".
[2] في (م) وفي (ت) وفي (ط) : " رُبع ".
[3] في (أ) : غابت الصورة.
[4] في (ت) : "المقامين".
[5] في (أ) وفي (ب) : "أن".

فقد صار عمل الكفّات راجعًا إلى الأربعة الأعداد المُتناسبة. فلها أوجه من العمل تُعرف من تناسب الخطأين وفضلي الكفّتين تركيبا و تفصيلا.][1]

فمن ذلك ما أملى[2] عليّ شيخُنا الفقيه العلامة **أبو العباس** أبقى الله بركته، حالة قراءتي عليه، وذلك في يوم الأربعاء الثامن[3] والعشرين لرجب الفرد من عام تاريخه :

" ثلاثة أوجه،

أحدها : تضرب فضل ما بين الكفّتين في أحد الخطأين، فإن كان الخطآن زائدين 54 و أو ناقصين، قسمت الضرب[4] على ما بينهما. وإن كان أحدُهما زئدًا[5] والآخر ناقصًا، قسمت الضرب على مجمُوعهما، فما خرج من ذلك تزيده على الكفّة التي ضربت في خطئها إن كان ناقصًا وتنقصُه منها إن كان زائدًا. يحصل المطلوب.

والوجه الثاني : تضرب ما بين الكفّتين في مجموع الخطأين إن كانا زائدين أو ناقصين، وتقسم على ما بينهما. وإن كان أحدُهُما زائدًا والآخر ناقصًا، تضرب ما بين الكفّتين في فضل ما بين الخطأين وتقسم على مجموع الخطأين، فما خرج من ذلك تحفظه. فإن شئت زدت المحفوظ على ما بين الكفتين وأخذت نصف المجتمع، تزيده على الكفّة التي خطأها أكبر[6] إن كان ناقصًا وتنقصه منها إن كان زائدًا، يحصل المطلوب.

وإن شئت فخذ المحفوظ وما بين الكفّتين وانقص أقلّهما من أكثرهما وخذ نصف الباقي، تزيده على الكفّة التي خطأها أقلّ إن كان ناقصًا وتنقصه منها إن كان زائدًا، يحصل المطلوب.

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[1] الجملة : " في كلّ مسألة ... تركيبا وتفصيلا." منقولة من **رفع الحجاب**، صفحة 298: 13-19.
[2] في (أ) : "كما أملى". وفي (ب) : "أملأ". وفي (ط) وفي (ت) : "أملاه".
[3] في (ط) وفي (ت) : "الثاني".
[4] في (م) : سقط لفظ "الضرب".
[5] في (م) : سقط لفظ "زائدا".
[6] في (أ) : "أكثر".

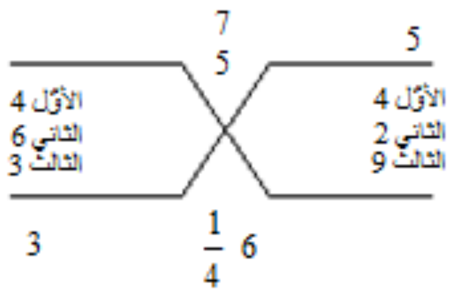
والوجه الثالث : تضرب ما بين الكفّتين في العدد المفروض لك[1]، فإن كان الخطأن زائدين أو ناقصين، قسّمت الضرب على ما بينهما، وإن كان أحدُهما زائدًا والآخر ناقصًا قسّمت الضرب[2] على مجمُوعهما، يحصل المطلوب. فافهمه.

[وإنّما كانت الكفّات أيضًا تبنى لاستخراج المجهول لأنّه قد يخرج بها ما ليس فيه تناسب.

فمن ذلك مسألة : ثلاثة رجال تبايعوا دابّة. فقال الأوّل للثاني: اعطني نصف ما معك، إلى ما معي يكون معي[3] ثمنُ الدّابة. وقال الثاني |54 ظ| للثالث: اعطني ثلث ما معك، إلى ما معي يكون معي ثمنُ الدّابة. وقال الثالث للأوّل: اعطني رُبع ما معك، إلى ما معي يكون معي ثمنُ الدّابة.

فنتّخذ كفّة للثلاثة الرجال، ونفرض فيها للأوّل ما شئنا، فكأنّه أربعة. وللثاني ما شئنا أيضًا، فكأنّه إثنان. فيكون ثمنُ الدّابة بحسب ذلك خمسة. فنجعله على القبّة، وهو الذي نقابل به. ويكون للثالث بحسب ذلك أيضًا تسعة. فإذا زدنا رُبع ما مع الأوّل، اجتمع عشرة. فقد أخطأت الكفّة الأولى في أعدادها الثلاثة بخمسة زائدة. ثمّ نتّخذ كفّة أخرى، نفرض فيها للأوّل أربعة، التي فرضناها له أوّلا، ونجعل للثاني ما شئنا. وظاهر أنّه لا يكون ثمانية فأكثر، (لأنّه يؤدّي إلى أن لا يكون بيد الثالث شيء)[4]. فاعلمه. فكأنّه ستّة. فيكون ثمنُ الدّابة الذي نقابل به سبعة. نجعله على القبّة أيضًا. ويجب من ذلك أن يكون للثالث ثلاثة. وإذا زيد عليه رُبع ما مع الأوّل كان أربعة. فقد أخطأت الكفّة الثانية بثلاثة ناقصة، و صورة ذلك هذه[5] :

[1] في (ط) وفي (ت) : سقط لفظ "لك".
[2] في (ب) : سقط لفظ "الضرب".
[3] في (ط) وفي (ت) : سقط لفظ "معي" هنا وفي كامل هذه الجملة.
[4] في **رفع الحجاب** : الجملة "لأنّه يؤدّي ... شيء" غائبة.
[5] في (أ) : غابت الصورة.



فنضرب خطأ كلّ كفّة في ما بيد كلّ واحد من الكفّة الأخرى، ونقسم مجموع الضربين[1]، على ما ذُكر، يخرج ما بيد كلّ واحد منهم وثمن الدّابة.][2]

فيكون للأوّل[3] ما بيده، وهي الأربعة، وللثاني[4] أربعة ونصف، وللثالث[5] خمسة ورُبع، وثمن الدّابة ستّة ورُبع.

ولو شئنا الخروج عن[6] الكسر، فنضرب المسألة كلّها في أقلّ عدد [55 و] ينقسم على أيمّتها. وقد علمناه ممّا تقدّم، وهو أربعة. فيكون للأوّل، بحسب ذلك، ستّة عشر، وللثاني ثمانية عشر، وللثالث أحد وعشرون، وثمن الدّابة خمسة وعشرون. فافهمه[7].

[[8]] وإن شئنا، فنُعيد[9] ما فرضناه للأوّل في الكفّة الأولى[10] ونترك[11] ما فرضناه للثاني على حاله. فإنّ هذا شرط أن يكون لواحد منهم عدد واحد[12] في الكفّتين جميعًا.

---

[1] في (أ) : " على مجموع الضربين". وفي (ط) وفي (ت) : " مجموع الخطأين "، وكلاهما مخطئ.
[2] الجملة : "وإنما كانت الكفات أيضا ... ثمن الدابّة." منقولة من **رفع الحجاب**، صفحة 299: 1-14.
[3] في (ت) : "الأوّل".
[4] في (ت) : "الثاني".
[5] في (ت) : "الثالث".
[6] في (م) : "على" وفي (ت) : "من".
[7] في (ب) : "فاعلمه".
[8] كلّ الفقرات التالية إلى نهاية الباب منقولة من **رفع الحجاب**، صفحة 299: 25 إلى 300: 20. ولم يدخل الهواري إلا بعض التغييرات في التخاطب مع القارئ وأضاف صور الكفات.
[9] في **رفع الحجاب** : "فغيّر" وفي (ب) : "نغيّر".
[10] في **رفع الحجاب** وفي (ط) وفي (ت) : سقط لفظ "الأولى".
[11] في (أ) : "ننزل".
[12] في (ط) وفي (ت) : سقط لفظ "واحد".

ولو فرض لنا ثمن الدّابة، لجعلناه على القبّة وجعلنا بعضه للأوّل ومثلي[1] باقيه للثاني. ونسقط ربع[2] ما جعلناه للأوّل من ثمن الدّابة، يبقى ما للثالث. (ثمّ نأخذ ما مع الثاني وثلث ما مع الثالث) [3]، ونقابل به الثمن المفروض. وكذلك نفعل في الكفّة الأخرى : نجعل للأوّل (من ثمن الدّابة)[4] ما شئنا، ولا بد أن يكون غير العدد الأوّل. ونكمّل العمل كما[5] تقدّم.

ومنه مسألة أخرى[6]، أربعون طائرا، ما بين أوزّ[7] و دجاج و زرازير بأربعين درهمًا. الزّرازير ثمانية بدرهم، والدّجاج واحدة بدرهمين، والأوزّ واحدة بثلاثة دراهم. كم أخذ من كلّ صنف من الطير؟

فهذا نوع ليس جميع[8] المثل تصحّ فيه. فإنّ له شرطين :
أحدهما أن يكون العدد صحيحًا لا كسر فيه.
والثاني أن يكون ثمن الواحد الأقلّ إذا ضرب في عدد الطير خرج أقلّ من جملة الثمن، وثمن الواحد الأكثر إذا ضرب كذلك خرج أكثر من الثمن.
وظاهر في هذه المسألة أنّ عدد الزّرازير ينبغي[9] أن[10] يكون ثمانية، أو ستة عشر، أو أربعة وعشرين، أو إثنين وثلاثين، لا غير ذلك.
فإن كان ثمانية، يبقى من الطير إثنان وثلاثون ومن الدراهم[11] تسعة وثلاثون.
وإذا اختبرنا ذلك [55 ظ] بالشرط الثاني، كان ضرب أشخاص الطير الباقية في (أقل ثمن الواحد)[12] منها أكثر من عدد الثمن، فلا يصحّ ذلك.
وإن جعلنا الزرازير ستّة عشر[13]، واختبرنا (الباقي بالطير)[14] فالباقي من الثمن كذلك، فلا يصحّ أيضًا.

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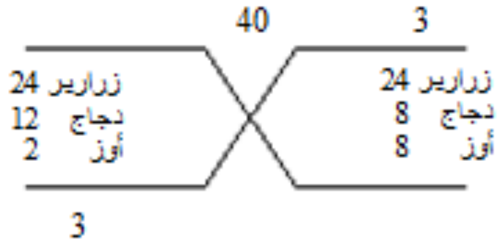
[1] في (ط) وفي (ت) : "ثمني" ، وهذا غير صحيح.
[2] في (ط) : سقط لفظ "ربع".
[3] في **رفع الحجاب** : سقطت الجملة " ثمّ نأخذ ... مع الثالث ".
[4] في (ط) وفي (ت) : سقطت العبارة "من ثمن الدّابة".
[5] في (ت) : "على ما".
[6] في (ت) : عبارة "وهي" زائدة.
[7] في (ب) : سقط لفظ "أوز".
[8] في (أ) : "جمع".
[9] في (ب) : "يتعين".
[10] في (أ) : سقط لفظ "أن".
[11] في (أ) وفي (ب) : "الثمن".
[12] في (أ) وفي (ت) : " ثمن الواحد الأقلّ". وفي (ط) : "في ثمن الواحد".
[13] في (أ) : سقط لفظ "عشر".
[14] في (ط) وفي (ت) : سقطت عبارة "الباقي بالطير".

وإن (جعلنا الزّرازير)[1] أربعة وعشرين (واختبرنا)[2] الباقي كذلك، صحّ فيه الشرطان. فنضع الزّرازير أربعة وعشرين ونضع الدّجاج ما شئنا، فكأنّنه ثمانية. فيكون الأوزّ ثمانية، باقي العدد. فنخطأ في الثمن بثلاثة دراهم زائدة[3].
ثمّ نتّخذ كفّة أخرى، نجعل الزّرازير فيها أربعة وعشرين، كما كانت في الأولى – وهذا شرط في صحة[4] العمل أن يكون عددًا مُكررًا في الكفّتين – ونجعل الدّجاج ما شئنا غير الأول، فكأنّه أربعة عشر. فيكون عدد الأوزّ إثنين. فنخطأ بثلاثة دراهم ناقصة، وهذه صورتها[5] :



فنعمل على ما تقدّم، يخرج المطلوب امّا عدد كلّ صنف من الطير وامّا ثمن كلّ صنف، أيّهما أردنا استخراجه أوّلا.
[فيكون ثمنُ الزّرازير ثلاثة وعددها أربعة وعشرين، والأوزّ (خمسة، وثمنُها)[6] خمسة عشر، والدّجاج أحد عشر وثمنُها إثنان وعشرون][7].
ولو جعلنا الزّرازير إثنين وثلاثين[8]، لم يصحّ ذلك، لانخرام الشرط في الباقي.
فليس لهذه المسألة إلّا جواب واحد. فقس على هاتين المسئلتين ما أشبههما.

ومثل هذه المسائل لا يخرج بالوجه الثاني (من عمل الكفّات)[9]، لأنّه خاص بالتّناسب، كما قدّمنا.

وما كان من مسائل الضرب مما لا تناسب فيه، فلا يخرج بالكفّات. فاعلمه.
وقد كمل القسم الأوّل، بحمد الله وحسن عونه[10].

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1 وفي (م) وفي (ط) وفي (ت) : "جعلنا"، وفي (ب) : "جعلناه".
2 في (ط) وفي (ت) : سقطت عبارة "واختبرنا ".
3 في (ب) : سقط لفظ "زائدة".
4 في (أ) وفي (ط) وفي (ت) : سقط لفظ "صحة".
5 في (أ) : الصورة ناقصة. وفي (ط) : الصورة خاطئة.
6 في (م) : سقطت العبارة "خمسة، وثمنها".
7 في **رفع الحجاب** : الجملة غائبة.
8 في (أ) : سقط لفظ "ثلاثين" ، وهذا خطأ.
9 في (م) وفي (ط) : سقط العبارة "من عمل الكفات".
10 في (ب) : "بحمد الله وعونه ومنّه وكرمه".

القسم الثاني :

في الجبر والمقابلة

ويتعلق به من الأعمال خمسة أبواب.

مخطوط المدينة المنورة : 56 و – 63 و
مخطوط استانبول : 97 ظ – 103 ظ
مخطوط أكسفرد (ب) : 154 ظ – 159 و
مخطوط تهران (ط) : 51 و – 56 ظ
مخطوط تونس (ت) : 28 و – 32 و

الباب الأوّل : في معنى الجبر والمقابلة وبيان[1] ضروبه

الجبر هو الإصلاح، كما ذكرنا في الجزء الأوّل من الكتاب،
والمُقابلة طرح كلّ نوع من نظيره حتى لا يكون في الجهتين نوعان من جنس واحد،
والمُعادلة[2] هو[3] أن تُجبر[4] النّاقص إلى الزّائد وتُطرح[5] الزّائد من الزائد والنّاقص من النّاقص من الأشياء المُتجانسة.

(وسيأتي مُتمثّلا مُتبيّنافي باب الجمع والطرح إن شاء الله تعالى)[6].

ومدار الجبر على ثلاثة أنواع : العدد والأشياء والأموال.

فالأشياء هي الجذور،
لأنّ كلّ مجهول من الأعداد هي[7] شيء وجذر (لمُربّعه وإن عُلم.)[8] :

والمال ما يجتمع من ضرب (الجذر في مثله)[9].
(ويُسمّى بذاك المال تمييزا)[10] عن غيره.

وهذه الأنواع[11] الثلاثة يعدل بعضها بعضًا بالإفراد والتّركيب. فيكون من ذلك ستّة ضروب[12]: ثلاثة مُفردة وثلاثة مُركّبة.

فأوّل[13] المُفردات، على ما جرى عليه الاصطلاح، أموال تعدل جذورا.

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[1] في (م): "بيانه".
[2] في (م): "المقاباة"، وهو خطأ.
[3] في (أ) وفي (ط) وفي (ت) : "هي".
[4] في (س) و (ت) : " يُجبر ".
[5] في (س) و (ت) : " يُطرح ".
[6] وفي (ط) وفي (ت) : " ويأتي مُمثّلا مُبيّنا إن شاء الله تعالى في باب الطرح ". وفي (أ) سقط لفظ "والطرج".
[7] في (ب) : "هو".
[8] في (أ) : "وجذر مربّعه مال. واعلم أن". وفي (ب) : " وجذر لمُربّعه. وهو مال. واعلم أن".
[9] في (ط) : "المال ألشيء في مثله". وفي (ت) : " المال في الشيء ", ، والحالتان خاطتان.
[10] في (أ) : "وسمي المال ممبزا". وفي (ط) : "وسمي بذلك تمييزا".
[11] في (ب) : زيادة لفظ "الأنواع".
[12] في (ط) وفي (ت) : " ضروب ستّة ".
[13] في (أ) :"فأما"، والمعنى مختل.

مثاله : ثلاثة أموال تعدل سبعة[1] أشياء.

والثاني : أموال تعدل عددًا[2].

مثاله : خمسة أموال تعدل عشرين.

والثالث : جذور تعدل عددًا[3].

مثاله : ثلاثة أجذار تعدل إثني عشر.

والثلاثة[4] المُركّبة،

أوّلها، وهو الضرب الرّابع، ينفرد فيه العدد.

مثاله : مال وعشرة أجذار[5] تعدل أربعة وعشرين.

والخامس ينفرد فيه الجذر.

مثاله : مال وأربعة تعدل خمسة أجذار[6].

والسّادس ينفرد فيه المال.

مثاله : مال[7] يعدل أربعة أجذار[8] وخمسة.

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[1] في (ط) وفي (ت) : "ستّة ".
[2] في (أ) وفي (ت) وفي (م) وفي (ط) : " أعدادًا ".
[3] في (أ) : " أعدادًا ".
[4] في (أ) : " الثانية ".
[5] في (ب) : " أجذاره ".
[6] في (ب) : " أجذاره ".
[7] في (م) : سقط لفظ "مال"، وهو خطأ.
[8] في (ط) وفي (ب) : " أجذاره ".

الباب الثاني في العمل بالضروب الستة

أمّا الثلاثة المفردة[1]، فإنّك تقسم على الأموال[2] ما عادلها[3] وعلى الجذور في عدمها، ويخرج لك في[4] القسمة من الضرب الأوّل والثالث الجذر[5]، ومن الثاني المال[6].
وإذا [56 ظ] عُلم الجذر عُلم المال بضرب[7] الجذر في مثله، وإذا عُلم المال عُلم منه الجذر.

مثال منها[8] : لو قيل ثلاثة أموال تعدل خمسة عشر شيئا.
ومعنى هذه المسألة : أيّ مال إذا أخذنا جذره خمسة عشر[9] مرّة[10] كان الجميع مُساويًا لثلاثة أمثال المال؟
وهي من الضّرب الأوّل. والعمل فيها كما[11] ذُكر: أن نقسم الخمسة عشر، عدد الأشياء، على الثلاثة، عدد الأموال، يخرج خمسة، وهي جذر المال المجهول[12]. وذلك خمسة وعشرون.

ومثال منه آخر : لو قيل مالان يعدلان ثمانية عشر.
ومعنى هذه المسألة أيضًا : أيّ مال إذا حملنا عليه مثله كان مُساويًا لثمانية عشر؟
وهي من الضّرب الثاني. والعمل فيها أن نقسم الثمانية عشر على الإثنين، عدد الأموال، كما ذُكر[13]، يخرج تسعة، وهو المال المجهول. وجذره ثلاثة.

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1 في (ب) : " المنفردة ".
2 في (أ) : " الأوّل ".
3 في (أ) وفي (ب) : "مُعادلها".
4 في (أ) وفي (ب) : " من ".
5 في (أ) وفي (ب) وفي (س) : " الجذور ".
6 في (أ) : سقط لفظ "المال".
7 في (أ) : "فيُضرب".
8 في (ط) وفي (ت) : "منه".
9 في (ب) : سقط لفظ عشر". وهذا خطأ.
10 في (أ) : سقط لفظ "مرة".
11 في (ت) : "ما".
12 في (م) : سقط لفظ "المجهول".
13 في (ط) وفي (ت) : "تقدّم".

ومثال منه آخر : لو قيل خمسة أشياء تعدل عشرين.
ومعنى هذه المسألة أيضًا : أيّ مال إذا أخذنا خمسة أجذاره كان جميعها مُساويًا للعشرين؟
وهي من الضّرب الثالث. والعمل فيها أن نقسم العشرين على الخمسة، عدد الأشياء، لعدم الأموال[1]. يخرج أربعة، وهي جذر المال المجهول. وذلك ستة عشر.

والعمل في الضرب الرّابع أن تُنصّف عدد الأجذار[2] وتُربّع التّنصيف[3] وتحمله على العدد وتأخذ جذر المجتمع وتسقط منه التّنصيف، يبقى الجذر.

مثال ذلك[4] : لو قيل مال وشيئان يعدل خمسة عشر.
ومعناه : أيّ مال إذا حملنا عليه جذريه، كان مُساويًا لخمسة عشر؟
فنأخذ نصف الشيئين بواحد، (ونربّعه بواحد)[5]، (ونحمل عليه)[6] العدد بستّة عشر[7]. نأخذ جذرها بأربعة، نسقط منها الواحد، التّنصيف، تبقى ثلاثة. وهي الشيء [57 و] المجهول، والمال تسعة.

وإن شئنا الخروج إلى المال أوّلا[8] قبل الجذر، فنحمل على العدد نصف مُربّع عدة[9] الأجذار. ونحفظ المُجتمع. ثمّ نطرح مُربّع العدد من مُربّع المحفوظ ونأخذ جذر الباقي ونطرحه من المحفوظ. فما بقي فهو المال.
(فكأنا أردنا عمله)[10] في المثال : فنحمل[11] نصف مُربّع عدد الشيئين، وذلك إثنان[12]، على الخمسة عشر (بسبعة عشر)[13]. نحفظها. ثمّ نُربّع العدد بخمسة وعشرين ومائتين. نطرحها من مُربّع المحفوظ، وذلك تسعة وثمانون ومائتان.

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[1] في (أ) وفي (ب) : "المال".
[2] في (ب) وفي (س) : "أجذاره".
[3] في (س) وفي (ط) وفي (ت) : "النّصف".
[4] في (ط) وفي (ت) : "مثاله".
[5] في (أ) : "وتربيعه واحد".
[6] في (أ) وفي (ب) : "ونحمله على".
[7] في (أ) : سقط لفظ "عشر" . وهذا خطأ.
[8] في (م) : "أو"، لا يصح هنا.
[9] في (أ) وفي (ط) : "عدد". وفي (م) سقط اللفظ.
[10] في (ت) : "وإذا أردنا في حمله ".
[11] في (م) : سقط لفظ "فنحمل".
[12] في (أ) : "شيئان" ، وهذا خطأ.
[13] في (أ) : سقطت العبارة "بسبعة عشر".

الباقي أربعة وستّون. نأخذ جذرها بثمانية، نطرحها من السّبعة عشر. الباقي تسعة، وهو المال المطلوب[1].

والخامس : تطرح العدد من مُربّع نصف عدد الأجذار وتأخذ جذر الباقي، فإن حملته على التّنصيف كان جذر المال الأكبر، وإن نقصته[2] كان جذر المال الأصغر.

مثال من ذلك : لو قيل مال وثمانية تعدل ستّة أشياء.
ومعناه : أيّ مال إذا حملنا عليه ثمانية، كان الجميع[3] مُساويًا لستّة أجذاره؟
فنأخذ نصف الأشياء ونُربّعه بتسعة، ونسقط منه العدد، الباقي واحدا[4]. نأخذ[5] جذره بواحد، فإن حملناه[6] على التّنصيف كان أربعة، وهو جذر المال الأكبر. فالمال (الأكبر[7] ستّة عشر. وإن نقصناه منه بقي إثنان، وهو جذر المال)[8] الأصغر. فالمال أربعة.

(واعلم أنّه)[9] متى خرج مُربّع النّصف مثل العدد فالنصف هو الجذر والمال هو العدد.

مثال ذلك : لو قيل مال وتسعة تعدل ستّة أشياء.
فنُربّع نصف الأشياء بتسعة[10]، وذلك مثل العدد. فالعدد هو المال، والتّنصيف هو الجذر.
ولو اطردنا[11] العمل، لطرحنا العدد من المُرّبع، يبقى لا شيء. نأخذ جذره بلا شيء، نحمل لا شيء[12] على التّنصيف أو ننقصه منه، |57 ظ| يبقى التّنصيف هو الجذر ومُربّعه هو[13] العدد. فاعلمه.

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[1] في (م) و في (ط) وفي (ت) : سقط لفظ "المطلوب".
[2] في (م) : "نصفته"، لا يصح هنا.
[3] في (ط) وفي (ت) : "المجتع".
[4] في (م) : "واحد واحد"، لا يصح هنا.
[5] في (أ) : "خذ".
[6] في (أ) : "حملنا".
[7] في (م) وفي (ب) وفي (ط) : سقط لفظ "الاكبر".
[8] في (ت) : سقطت الجملة "الأكبر ... وهو جذر المال.".
[9] في (س) : سقطت الجملة "واعلم أنه".
[10] في (أ) : "ونصف رُبع الأشياء" ، وهو غير صحيح هنا.
[11] في (أ) : "اختصرنا"، وفي (ب) : "أردنا"، وفي (ت) : "طرحنا" ، وهو غير صحيح هنا.
[12] في (ت) : "جذره".
[13] في (ت) : سقط.

وإن شئنا[1] الخروج إلى المال أوّلا قبل الجذر، فنطرح العدد من نصف مُربّع عدد الأجذار ونحفظ الباقي، ثمّ نطرح مُربّع العدد من مُربّع المحفوظ. فإن حملنا جذر الباقي على المحفوظ كان المال الأكبر، وإن نقصناه منه كان الباقي المال الأصغر.
وإنّما يكون هذا في ما يكون فيه العدد أقلّ من مُربّع نصف عدد الأجذار.

فلو أردنا عمله في المثال المتقدّم : فنطرح العدد من نصف مُربّع عدد الأجذار، وذلك ثمانية عشر. الباقي عشرة. نحفظها. ثمّ نطرح مُربّع العدد، وذلك أربعة وستّون، من مُربّع المحفوظ. الباقي ستّة وثلاثون. نأخذ جذرها بستّة. فإن حملناها على المحفوظ كان ستّة عشر، المال الأكبر، وإن نقصناها من المحفوظ، الباقي أربعة[2]، المال الأصغر. فاعلمه.

السّادس مثل الرّابع في العمل إلّا أنّك[3] تحمل التّنصيف آخرا على جذر المُجتمع يكون الجذر.

مثال ذلك : لو قيل مال يعدل جذريه وثلاثة.
ومعناه أيّ مال يُساوي جذريه وثلاثة؟ فنُربّع نصف الشيئين ونحمله على العدد بأربعة. نأخذ جذرها بإثنين، نحمل عليها التّنصيف بثلاثة. وهو الشيء المجهول، والمال تسعة.

وإن شئنا الخروج إلى المال أوّلا قبل الجذر، فنحمل ضعف العدد على مُربّع الأجذار. ونحفظ نصف المجتمع، ثمّ نسقط مُربّع العدد من مُربّع المحفوظ. فما بقي حمل جذره[4] على المحفوظ، فما كان فهو المال.

فإن[5] أردنا عمله في المثال : فنحمل ضعف العدد بستّة على مُربّع الأجذار بعشرة. نحفظ نصفها بخمسة، ثم نسقط مُربّع العدد من مُربّع المحفوظ. (الباقي ستّة عشر، | 58 و | نحمل جذرها بأربعة على المحفوظ)[6]. فما كان فهو المال المجهول، وذلك تسعة.

---

[1] في (ت) : " شئت".
[2] في (ت) : "أربعة أمثال"، وهذا خطأ.
[3] في (ط) وفي (ت) : "أنّا".
[4] في (أ) : "احمل جذره"، وفي (ب) : "حمل جذر"، وهو خطأ.
[5] في (أ) : "فإذا".
[6] في (أ) : سقطت الجماة "الباقي ... المحفوظ".

وكلّ ما أتاك في[1] الضّروب الثلاثة المُركّبة أكثر من مال واحد، فحطّه إلى مال واحد وحطّ بذلك الإسم جميع المُعادلة.

وكلّ ما أتاك فيها أقل من مال واحد، فاجبره إلى مال واحد واجبر بذلك الإسم جميع المعادلة.
ووجه العمل في الجبر والحطّ كما تقدّم.

وإن شئت، فاقسم ألقاب المسألة على ما فيها من عدد الأموال، فما خرج فهو راجع إلى[2] المسألة، فقابل بعضه ببعض.

وذكرُه للمُركّبة وحدها لأنّ[3] البسيطة لا يحتاج فيها إلى جبر وإن كان فيها أقلّ من مال واحد، ولا إلى حطّ وإن كان فيها أكثر من مال واحد.

مثال[4] ذلك : لو قيل مالان وستّة أشياء تعدل ستّة وثلاثين.
فنحطّ المالين، على ما تقدّم، إلى مال واحد، وذلك بضربهما في نصف، ونحطّ به الأشياء والعدد. فترجع المسألة إلى مال وثلاثة أشياء تعدل ثمانية عشر. وهو الضرب الرّابع.

وكذلك العمل[5] في الخامس والسّادس.

ولو عملنا بالوجه الثاني، فنقسم الإثنين، عدد الأموال، على نفسها، يخرج واحد. ونقسم عليها أيضًا عدد الأشياء، تخرج ثلاثة. ونقسم عليها أيضًا العدد. فترجع المسألة إلى مال وثلاثة أشياء تعدل ثمانية عشر، كما تقدّم.

---

[1] في (أ) : " من ".
[2] في (م) : لفظ "إلى" زائد.
[3] في (أ) : "في"، وفي (ب) : "دون البسيطة لان".
[4] في (أ) وفي (ب) : "مثال في".
[5] في (ط) وفي (ت) : سقط لفظ "العمل".

مثال منه آخر : لو قيل نصف مال وشيئان تعدل ستّة.

فنجبر النّصف إلى مال واحد، وذلك بضربه (في إثنين ونضربها أيضًا في)[1] الشيئين والعدد. فترجع المسألة إلى مال وأربعة أشياء تعدل إثني عشر. وهو الضرب الرّابع أيضًا.

و كذلك العمل في الخامس و السّادس.

ولو عملنا بالوجه الثاني، لقسمنا النّصف على مثله، يخرج واحد. ونقسم عليه أيضًا عدد الأشياء، يخرج أربعة. [58 ظ] ونقسم عليه العدد أيضًا، يخرج إثنى عشر. فترجع المسألة إلى مال واحد وأربعة أشياء تعدل إثني عشر، كما تقدّم. فافهمه[2].

---

[1] في (ط) وفي (ت) : سقطت الجملة " إثنين ونضربها أيضا في ".

[2] في (ب) : سقطت العبارة "فافهمه".

الباب الثالث في الجمع و الطرح

جمع الأجناس المختلفة بواو العطف.

مثل مال[1] وستّة أشياء وعشرة دراهم.

والمُستثنى المُختلف بلا طرح[2].

[مثاله : لو قيل اجمع مالا إلّا شيئًا إلى عشرة دراهم. فالمجتمع[3] مال وعشرة دراهم إلّا شيئًا. فيبقى المُستثنى كما كان لم يطرح[4] من شيء.

والمُتّفق بطرح الأقلّ من الأكثر، (يعني المسألة)[5].

مثاله : لو قيل اجمع مالين إلّا مالا إلى عشرة دراهم. فنطرح المال المُستثنى من المالين، يبقى المُجتمع: مال وعشرة دراهم.

ومثال منه آخر[6] : لو قيل اجمع مالا إلّا شيئين إلى[7] عشرة أشياء. فنطرح الشيئين من العشرة الأشياء[8]، يبقى المُجتمع: مال وثمانية[9] أشياء][10].

ومثال منه آخر[11]: لو قيل اجمع مالا إلّا خمسة أشياء إلى[12] عشرة أشياء. فنطرح خمسة الأشياء من العشرة الأشياء، يبقى المجتمع[13]: مال و[14] خمسة أشياء.

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[1] في (ب) : سقط لفظ "مال".
[2] في (ت) : " بالطرخ"، وهو خطأ.
[3] في (أ) : " فالجواب"، وفي (ط) وفي (ت) : " فالجمع ".
[4] في (ط) وفي (ت) : " ينطرح ".
[5] في (س) : سقطت العبارة " يعني المسألة "، وفي (أ) وفي (ب) وفي (م) : " يبقى المستثنى ".
[6] في (م) : سقط لفظ "آخر".
[7] في (أ) : " الا " ، وهذا خطأ.
[8] في **رفع الحجاب** وفي (ب): سقط لفظ "الأشياء". وهذا خطأ.
[9] في (أ) : " ثلاثة " ، وهذا خطأ.
[10] الامثاة الثلاثة منقولة من **رفع الحجاب**، صفحة 313: 3-9.
[11] في **رفع الحجاب**، صفحة 314 يختلف المثال : " اجمع مالا إلّا عشرة أشياء إلى خمسة أشياء. فتطرح خمسة الأشياء من عشرة الأشياء، يبقى المجتمع: مال الا خمسة الأشياء ".
[12] في (أ) : " الا " ، وهذا خطأ.
[13] في (ط) : سقط لفظ "المجتمع".
[14] في (أ) : " الا " ، وهذا خطأ.

وطرح الأجناس المُختلفة بحرف الاستثناء.

مثاله : لو قيل اطرح شيئا من مال، فالباقي مال إلّا شيئا.

ومثال آخر : لو قيل اطرح عشرة دراهم من مالين وثلاثة أشياء، فالباقي: مالان[1] وثلاثة أشياء إلّا عشرة دراهم.

والمستثنى، أمّا أن يكون من الجانبين، أو من[2] أحدهما. وقد يكون نوعًا واحدًا أو نوعين مختلفين.
والعمل في ذلك[3] أن تزيد مُستثنى كلّ جهة على الجهتين معًا، وحينئذ تطرح. وهكذا العمل في المُتعادلين إن كان فيهما استثناء[4].

مثال من ذلك : لو قيل اطرح اثنين وشيئا من مال [59 و] إلّا ثلاثة أشياء.
فنجد[5] الأشياء من جهة المطروح منه ناقصة، وهي ثلاثة. فنزيدها[6]
على الجهتين معًا، وذلك معنى المُعادلة في الجبر، لأنّا لمّا زدنا في المال ما خص[7] منه، وهي الثلاثة الأشياء، فقد جبرناه (إلى الزيادة)[8] بعد النّقص، فصار[9] أكثر ممّا طلب أن يطرح منه. فنعادل بأن نجبر المطروح بقدر ما زدنا في المطروح منه، (بثلاثة الأشياء)[10]، لأنّ طرح عددين كطرحهما وقد زدنا على كلّ واحد منهما أو نقّصنا من كلّ واحد منهما عددا ما بعينه، فتصير المسألة كأنّه قيل: اطرح إثنين وأربعة أشياء من مال. فنعمل كما تقدّم. الباقي بعد هذا[11] هو المطلوب، وذلك: مال إلّا أربعة أشياء وإلّا إثنين. فاعلمه.

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[1] وفي (ط) وفي (ت) : "مال" ، وهذا خطأ.
[2] وفي (ط) وفي (ت) : سقط لفظ "من".
[3] في (ب) : "في ذلك كلّه".
[4] في (م) : "مستثنى".
[5] وفي (ط) وفي (ت) : "فخُذ".
[6] في (ت) : "فتزدها".
[7] في (ب) : "نقص".
[8] في (أ) : "بالزيادة". وفي (ط) وفي (ت) سقطت العبارة "إلى الزيادة".
[9] في (أ) : عبارة "فصار" ناقصة.
[10] في (ط) : "وذلك بالثلاثة الأشياء". وفي (ت) : "وذلك ثلاثة بالثلاثة الأشياء".
[11] في (أ) : لفظ "هذا" ناقص.

ومثال منه آخر : لو قيل اطرح إثنين وخمسين درهما إلّا خمسة أشياء من كعبين وثلاثين درهما.
فنجد الأشياء من (جهة المطروح)[1] ناقصة وهي خمسة. فنزيدها على الجهتين معًا، كما تقدّم. فتصير المسألة كأنّه قيل: اطرح إثنين وخمسين درهما من كعبين وخمسة أشياء وثلاثين درهما. فنجد الدّراهم من[2] الجهتين. فنزيل قدر أقلّهما من الجهتين معًا. فترجع المسألة كأنه قيل اطرح إثنين وعشرين درهما من كعبين وخمسة أشياء.

وذلك هو المقابلة في المعادلة، لأنّا لمّا أزلنا[3] الثلاثين من المطروح منه، صار أقلّ ممّا (أثبته المطروح)[4] منه، (فنعادل بأن نزيل من المطروح[5] أيضًا بقدر ما أزلنا من المطروح منه)[6]. (وعمله أيضًا كما)[7] تقدّم. فنعمل في الباقي كما تقدّم. فما بقي بعده، فهو المطلوب، وذلك كعبان وخمسة أشياء إلّا[8] إثنين وعشرين درهما.
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مثال منه آخر : لو قيل اطرح إثني عشر درهما إلّا أربعة أشياء من ثلاثة أموال إلّا شيئين.
فنجد في المطروح منه شيئين ناقصين وفي المطروح أربعة أشياء ناقصة أيضًا. فنجمعها، لأنّها من جنس واحد بستّة أشياء، نزيدها على الجهتين معًا، أو نجبر كلّ واحد من المطروح والمطروح منه[9] بما استثنى منه، ونزيد مثل ذلك على الآخر، كما تقدّم. فتصير المسألة كأنّه قيل اطرح إثني عشر درهما وشيئين من ثلاثة أموال وأربعة أشياء. فنقابل ونطرح، فما بقي بعد هو المطلوب، وذلك ثلاثة أموال وشيئين[10] إلّا أثني عشر درهما.

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[1] في (أ) : "خمسة".
[2] في (ب) : "في".
[3] في (أ) : "أنزلنا".
[4] في (ط) وفي (ت) : "ابتدى الطرح".
[5] في (ط) وفي (ت) : "الطرح".
[6] في (أ) : سقطت الجملة : "فنعادل ... المطروح منه".
[7] في (م) وفي (ط) : "وعليه أيضا مما". وفي (ب) : "علّته".
[8] في (ب) : سقط لفظ "الا" ، وهو خطأ.
[9] في (أ) : "من المطروح منه "، وفي الجملة نقص.
[10] في (م) : "شيء"، وهو خطأ.

مثال منه آخر : لو قيل اطرح كعبًا[1] إلّا مالين من ثلاثين[2] درهما إلّا أربعة أشياء.

فنجبر ونطرح، فما بقي فهو المطلوب. وذلك ثلاثون درهما ومالان إلّا كعبًا وإلّا أربعة أشياء.

[1] في (ب) : "كعبان"، وهو خطأ.
[2] في (أ) : "ثلاثة"، وهو خطأ.

فصل نذكر فيه أمثلة من المتعادلين

فمن ذلك لو قيل : مال إلّا ثلاثة أشياء تعدل إثنين وشيئا.
فنجد الأشياء من جهة المال ناقصة، وهي ثلاثة. (فنجبر به[1] المال)[2] ونزيدها على المُعادلين[3] أيضًا، كما تقدّم – لأنّ المُتساويين إذا زيد عليهما مُتساويان أو نقص منهما مُتساويان، كان الحاصل منهما بعد ذلك[4] متساويا – فتصير المسألة إلى مال يعدل إثنين وأربعة أشياء. وهو الضّرب السّادس.

ومثال منه آخر : لو قيل مال إلّا ثلاثة أشياء تعدل أربعة وعشرين درهما إلّا خمسة أشياء.
فنجبر ونقابل كما تقدّم. فتصير المسألة إلى مال وشيئين تعدل أربعة وعشرين درهما. وهو الضّرب الرّابع.

وإن شئنا، فنطرح أوّلا الثلاثة الأشياء، النّاقصة من جهة المال، من الخمسة الأشياء، النّاقصة | 60 و | من جهة العدد، يبقى شيئان ناقصان من جهة العدد. ونكمل العمل، يخرج الضّرب الرّابع أيضًا.

ومثال منه آخر : لو قيل مال[5] إلّا عشرة دراهم تعدل (مالا إلّا)[6] شيئان ونصفًا.
فنجبر ونقابل. فتصير المسألة إلى عشرة دراهم تعدل [شيئين ونصف شيء، وهو الضّرب الثالث.

ومثال منه آخر : لو قيل مال وعشرة دراهم يعدل] [7] واحدا وخمسين درهمًا إلّا أربعة أشياء.
فنجبر، وتصير المسألة إلى (مال وأربعة أشياء تعدل واحدا وأربعين درهمًا)[8].
وهو الضّرب الرّابع.

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[1] في (ب) وفي (ط) وفي (ت) : "بها ".
[2] في (أ) : "فتجبر ها بالمال".
[3] في (ب) : المعادل". وفي (ط) وفي (ت) : المتعادلين".
[4] في (ط) وفي (ت) : سقط لفظ "ذلك".
[5] في (ب) : "مالان".
[6] في (م) : سقط لفظ " الا ". وفي (ب) : "مالين الا".
[7] في (ط) وفي (ت) : سقطت الجمل " شيئين ونصف شيء، وهو الضّرب الثالث. ومثال منه آخر : لو قيل مال وعشرة دراهم يعدل ".
[8] في (م) وفي (أ) : " أربعة أشياء تعدل واحدا وخمسين درهمًا ". والكل لا يصح.

ومثال منه آخر : لو قيل مال وخمسة أشياء تعدل عشرة دراهم ومالين إلّا شيئا.
فنجبر ونقابل. فتصير المسألة إلى : ستّة أشياء تعدل مالا وعشرة دراهم.
(وهو الضّرب الخامس.

ومثال منه آخر : لو قيل مالان[1] وعشرة[2] درهما)[3] إلّا شيئان يعدل ثلاثة أموال[4] وسبعة دراهم إلّا سبعة أشياء.
فنجبر ونقابل. فتصير المسألة إلى : مال[5] يعدل ثلاثة دراهم وخمسة أشياء. وهو الضّرب السّادس. فاعلمه.

وكذلك العمل في ما يرد عليك[6] من مثل هذا، والله المستعان.

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[1] في (ب) : "مال ".
[2] في (ط) وفي (ت) : "عشرون"، وهو لا يصحّ قي هذا المقام.
[3] في (م) : سقطت الجمل "وهو الضّرب الخامس. ومثال منه آخر : لو قيل مالان وعشرة درهما".
[4] في (ت) : "أشياء "، وهو لا يصحّ قي هذا المقام.
[5] في (م) وفي (ط) : "مالان" وهذا لا يصح. أما في (ب) : "مالان" مقبولة.
[6] في (م) وفي (أ) : سقط لفظ "عليك".

الباب الرابع في الضرب و معرفة الاس و الاسم

أمّا الأسّ، فاعلم أنّ[1] أسّ الأشياء واحد، وأسّ الأموال إثنان، وأسّ الكعوب[2] ثلاثة.
وأمّا الاسم، فاسم الواحد أشياء[3]، واسم الإثنين أموال، واسم الثلاثة كعوب، وما بعد ذلك ثلاثة لكلّ كعب وإثنان للمال.
(والكعبُ هو مُسطّح الشيء في المال. سُمّي به لأنّه كعب وإن لم تعلم له كمّية)[4].

فلو[5] قيل ما أسّ مال[6] مال؟ فنقول أربعة. (ولو قيل ما أسّ مال كعب؟ فنقول خمسة)[7]. فلو قيل ما أسّ مال مال مال؟ فنقول ستّة. ولو قيل[8] ما أسّ مال كعب مال كعب؟ فنقول عشرة.
والعمل في ذلك أن نأخذ لكلّ مال إثنين أبدًا، لأنّه أسّه، كما ذُكر، ولكلّ كعب ثلاثة أبدًا، لأنّه أسّه، كما ذُكر أيضًا. [60 ظ] فما كان، فهو أسّ[9] المسؤول عنه.

ولو قيل: ما أسّ مال كعب مال (مال؟ لقلنا)[10] تسعة. ولو قيل[11]: (ما أسّ كعب مال كعب كعب مال مال؟)[12]، لقلنا خمسة عشر[13].
وكذلك العمل فيما أشبه ذلك.

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[1] في (م) : سقط لفظ "أن".
[2] في (أ) : "الكعب".
[3] في (ت) : "شيء".
[4] في (س) : الجملة "الكعب ... كمية" غائبة، ويعتبرها ناسخ (ت) من شرح الهواري.
[5] في (ب) : " فإن ".
[6] في (أ) : سقط لفظ "مال".
[7] في (ت) : الجملة " ولو قيل ما أسّ مال كعب؟ فنقول خمسة " غائبة.
[8] في (ب) : سقط لفظ "قيل".
[9] في (ت) : سقط لفظ "أسّ".
[10] في (م) : سقطت العبارات "مال؟ لقلنا".
[11] في (ب) : سقط لفظ "قيل".
[12] في (ب) : " مال كعب مال كعب مال كعب ".
[13] في (ط) وفي (ت) : " ولو قيل ما أسّ كعب مال كعب كعب مال، لقلنا ثلاثة عشر ".

وعكسه، لو قيل ما اسم أربعة؟ فنقول مال مال. ولو قيل: ما اسم سبعة؟ فنقول كعب مال مال. ولو قيل: ما اسم ستّة؟ فنقول مال مال مال، أو كعب كعب.
والعمل في هذا أيضًا أن نفصل الأسّ[1] (ثنائيا أو ثلاثيا أو بمجموعهما)[2]، (الأوّل للمال والثاني للكعب والثالث للكعب والمال)[3].
فلو قيل مثلا، ما اسم ثمانية؟ فنقول مال مال مال مال. ولو شئنا، لقلنا[4] كعب مال كعب، أو كعب كعب مال، أو مال كعب كعب. كلّ ذلك جائز.

ولو قيل ما اسم تسعة؟ فنقول كعب كعب كعب. ولو شئنا، لقلنا[5] كعب مال مال مال. ولو شئنا أيضًا[6] قدّمناه أو أخّرناه. كلّ ذلك جائز. فاعلمه.

فاذا ضربت هذه الأنواع، فاجمع أسّ المضروب وأسّ[7] المضروب فيه، يكون مجموع الأسّين[8] أسًا للخارج. وإذا ضربت عددا في أحد هذه الأنواع، فالخارج ذلك النوع بعينه.

مثال من ذلك : لو قيل اضرب خمسة أشياء في سبعة[9] أشياء.
فنضرب عدد المضروب، وذلك خمسة، في عدد المضروب فيه، وذلك سبعة، بخمسة وثلاثين. ثمّ نجمع أسّي المضروبين باثنين. فهذه الإثنان[10] أسّ الخمسة والثلاثين الخارجة من الضرب، فتكون أموالا. فالخارج من الضرب خمسة وثلاثون مالا[11]، وهو المطلوب.

ومثال منه آخر : لو قيل اضرب عشرة أشياء في ستّة أموال.

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[1] في (أ) : " الاسم "، وهو خطأ.
[2] في (ب) : " ثناء ثناء، أو ثلاثا، أو بمجموعهما ".
[3] في (ط) وفي (ت) : " الأوّل للشيء والثاني للمال والثالث للكعب "، وفي (أ) : " الأوّل للمال والثاني للكعب والثالث للمال والكعب".
[4] في (أ) : سقط لفظ " لقلنا ".
[5] في (م) وفي (ب) : لفظ " لقلنا ".
[6] في (ت) : سقط لفظ " أيضًا ".
[7] في (س) : سقط لفظ " المضروب وأسّ ".
[8] في (م) : "الاسمين".
[9] في (ت) : " تسعة "، وهو خطأ.
[10] في (أ) : " الأموال "، وهو خطأ.
[11] في (أ) : سقط لفظ "مالا"، وهو خطأ.

فنضرب عدد المضروب في عدد المضروب فيه، يكون الخارج ستّين. ثمّ نجمع أسّي المضروبين بثلاثة، نجعلها أسّا للخارج، وهي الستّون، كما ذُكر. فما كان، فهو المطلوب. وذلك ستّون كعبًا.

ومثال منه [61 و] آخر : لو قيل اضرب شيئا في كعب. فالخارج من ضرب عدد المضروب في عدد المضروب فيه واحد، ومجموع أسّيهما أربعة. فهي أسّ للواحد الخارج من الضرب. فيكون مال مال.

ومثال منه آخر : لو قيل اضرب ستّة في أربعة أموال.
فنضرب المضروب في عدد المضروب فيه بأربعة وعشرين، وذلك ما يجتمع في الضرب من أمثال المال. فهي إذًا أربعة وعشرون مالا.

ومثال منه آخر : لو قيل اضرب سبعة في ثلاثة أموال كعب.
فنضرب المضروب في عدد المضروب فيه بأحد وعشرين، و ذلك أيضًا ما يجتمع في الضرب من أمثال مال كعب. فهي إذا إحدى وعشرون مال كعب. فاعلمه.

ومتى عادلت بين أموال الأموال والكعوب والأموال، أو الكعوب والأموال والأشياء، وشبه ذلك، ولم يكن معك عدد، فاطرح أقلّ الأسوس من أسّ كلّ واحد منهما، فما بقي تعادل بعضه ببعض على نحو ما كانت المُعادلة.

مثال من ذلك : لو قيل ثلاثة أموال مال[1] تعدل أربعة كعوب وعشرة أموال.
فنجد أسّ الأموال أقلّ أسّ المّعادلة، وهو إثنان. فإذا اسقطناه من أسّ[2] أموال الأموال، الذي هو أربعة، يبقى[3] إثنان ، اسمها أموال. ونسقطها أيضًا من أسّ الكعوب[4]، الذي هو ثلاثة، يبقى واحد، اسمه شيء. وترجع الأموال عددا[5]. فتصير المسألة إلى: ثلاثة أموال تعدل أربعة أشياء وعشرة دراهم، وذلك هو الضّرب السّادس.

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[1] في (أ) : " أموال ".
[2] في (ب) وفي (ت) : سقط لفظ "أسّ".
[3] في (ب) : "فبقي".
[4] في (ب) : "الكعب".
[5] في (أ) و (س) : " إلى عدد".

ومثال منه آخر : لو قيل ثلاثة كعوب تعدل عشرة أموال وعشرين شيئا. فنجد أسّ الأشياء أقلّ من أسّ المُعادلة ونعمل كما تقدّم. فترجع المسألة إلى: ثلاثة أموال تعدل عشرة أشياء وعشرين درهما. وهو الضّرب السّادس أيضًا.

ومثال منه آخر : لو قيل كعب وعشرة أموال [61 ظ] تعدل تسعة وثلاثين شيئا. فنعمل كما تقدّم. فترجع المسألة إلى: مال وعشرة أشياء تعدل تسعة وثلاثين درهما. وهو الضّرب الرّابع.

وكذلك العمل فيما أشبه ذلك[1] من الأمثلة.
وما لا يُمكن ردّه إلى الضّروب السّتّة، فلا يعرّج عليه إذ لا يفيد[2] شيئا.

وضرب الزّائدين أو النّاقصين أحدهما في الآخر زائد وضرب الزائد في النّاقص ناقص.

مثال من ذلك : لو قيل اضرب خمسة أشياء في ثلاثة عشر إلّا أربعة أشياء. فنضرب الخمسة أشياء في الثلاثة عشر الزائدة، ونسقط من الخارج ضرب الخمسة الأشياء أيضًا في الأربعة الأشياء الناقصة. والباقي بعد هُو المطلوب، وذلك خمسة وستّون شيئا إلّا عشرين مالا.

ومثال منه آخر : لو قيل اضرب ثمانية إلّا شيئين في سبعة إلّا أربعة أموال. فنضرب الثمانية الزائدة في السبعة الزائدة، يكون زائدًا. ونحمل عليه ضرب الشيئين النّاقصين في الأربعة الأموال النّاقصة، لأنّه زائد. ونسقط من المجتمع ضرب الثمانية الزّائدة في الأربعة الأموال النّاقصة، لأنّه ناقص، وضرب الشيئين النّاقصين في السبعة الزائدة، لأنّه ناقص. والباقي بعد، هو المطلوب. وذلك ثمانية[3] كعوب وستّة وخمسون إلّا أربعة عشر شيئا وإلّا أثنين وثلاثين مالا. فاعلمه.

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[1] في (م) : "هذا".
[2] في (م) : "يقبل".
[3] في (أ) و في (ت) : " ستّة "، وهو خطأ. والملاحظ أنّ ناسخ (ب) كتب أوّلا "ستّة"، ثمّ شطبها وعوّضها بالعدد الصحيح، وهو "ثمانية".

إذا قسمت نوعا من هذه الأنواع على نوع أدنى منه فأسقط من أسّ المقسوم أسّ المقسوم عليه. فما بقي، فهو أسّ[1] النّوع الخارج من القسمة.

ومثال ذلك : لو قيل اقسم عشرة أموال على شيئين.
فنقسم عدد الأموال على عدد الأشياء ونُسمّي الخمسة الخارجة باسم فضل ما بين أسّ الأشياء والأموال، وذلك شيء. فالخارج من القسمة، أيضا، خمسة أشياء. وهو المطلوب.

مثال منه آخر : لو قيل اقسم خمسة [62 و] عشر كعبًا على ثلاثة أشياء.
فنقسم عدد الكعوب على عدد الأشياء ونُسمّي الخمسة الخارجة باسم فضل ما بين أسّ[2] الأشياء وأسّ الكعوب، وذلك مال. فالخمسة الخارجة، إذًا، خمسة أموال. وهو المطلوب[3]. فاعلمه.

ومتى قسمت نوعا منها على مثله، فالخارج عدد.

مثاله : لو قيل اقسم إثني عشر مالا على ثلاثة أموال.
فنقسم عدد المقسوم على عدد المقسوم عليه، يخرج أربعة وليس بين أسّيهما[4] فضل. فنُسمّي باسمه الأربعة الخارجة، فهي عدد، لا محالة.

ومتى قسمت أحد هذه الأنواع على عدد، فالخارج ذلك النّوع بعينه.

مثال منه : لو قيل اقسم إثني عشر شيئا على أربعة دراهم.
فنقسم عدد المقسوم على عدد المقسوم عليه، يخرج ثلاثة. والعدد المقسوم عليه ليس له أسّ، بحيث ان يُسقط من أسّ المقسوم، يبقى أسّ المقسوم أسّا للخارج، يُسمّى باسمه. فالخارج يكون ثلاثة أشياء، وهو المطلوب.

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[1] في (أ) : سقط لفظ " أسّ ".
[2] في (ط) وفي (ت) : سقط لفظ " أسّ ".
[3] في (أ) : سقطت الجماة " وهو المطلوب ".
[4] في (أ) : " سمها الفضل "، وهذا غامض.

فان كان في المقسوم[1] استثناء، فاقسم كلّ واحد من المستثنى والمستثنى منه على المقسوم عليه ويستثنى خارج المستثنى من خارج المستثنى منه، فما كان فهو خارج القسمة.

مثال منه : لو قيل اقسم إثني عشر كعبًا إلّا ثلاثة أموال على شيئين.
فنقسم الإثني عشر كعبًا، وهي المستثنى منه، على الشيئين، المقسوم عليها، ويُستثنى من ذلك قسمة الثلاثة الأموال، وهي المستثنى، على الشيئين أيضًا. فما بقي، فهو المطلوب. وذلك ستّة أموال إلّا شيئا ونصف شيء.

ومثال منه آخر : لو قيل اقسم عشرة أموال إلّا ثلاثة أشياء على درهمين.
فنعمل كما تقدّم. فيكون الخارج خمسة أموال إلّا شيئا ونصف شيء، وهو المطلوب.

ولا يقسم الأدنى من[2] النّوعين على الأعلى إلّا بعد زوال الاشتراك بأن تطرح من أسّ كلّ واحد منهما أسّ أقلهما.
[62 ظ]

مثال ذلك : لو قيل اقسم ستّة أموال على ثلاثة كعوب.
فنقسم العدد على العدد. (وما خرج يكون مقسوما)[3] على فضل ما بين المرتبتين، وذلك شيء[4]. فيكون الخارج إثنين مقسومة على شيء.

ولا يقسم على المُستثنى منه. فافهم وتدبر. وبالله التوفيق.

مثاله : لو قيل اقسم عشرة أموال على ثلاثة إلا شيئا.
فنقول الخارج عشرة أموال مقسومة على ثلاثة إلا شيئا. الجواب كالسؤال. فافهمه[5].

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[1] في (أ) : " المطلوب ".
[2] في (أ) : " في ".
[3] في (ب) في (ط) وفي (ت) : " فما خرج فهو مقسوم ".
[4] في (م) : سقط لفظ "شيء".
[5] في (م) : "فاعلم تصب، إن شاء الله تعالى".

فـــــــــصل

ولنختم هنا[1] هذا التّأليف بثلاثة مسائل من ملح الحساب[2]، إذ لا يزال[3] الحُسّاب يختمون بها[4] مُصنّفاتهم.

أحدها : أن نأمره أن يُسقط عدده[5] من عشرة، ثمّ يسقط مُربّع الباقي من مُربّع عدده.
إن كان <مُربّع> الباقي أقلّ، يُخبرنا بالباقي، فنقسمه نحن[6] على عشرة. وما خرج نزيد عليه نصف بقيته إلى العشرة[7]، يكون المُضمر.
وإن كان مُربّع الباقي أكثر، فيُسقط منه مُربّع المُضمر ويُخبرُنا بالباقي، فنقسمه على العشرة ونسقط الخارج من عشرة[8]. ونصف الباقي هو العدد المُضمر[9].
وإن شئنا أن نأمره بإسقاط عدده المُضمر من غير العشرة، ونتّبع العمل، يحصل المطلوب.

المسألة الثانية: نأمره أن يقسم العشرة[10] بقسمين، يُضمرها. ثمّ نأمره أن يقسم مُربّع أحدهما على مُسطّحهما، ويُخبرُنا بالحاصل[11]. فإذا علمناه، فهو نسبة أحد القسمين إلى الآخر، فنقسم العشرة على تلك النّسبة.
وكذلك نفعل[12] في أيّ عدد شئنا غير العشرة (قسمتاه مضموراة)[13]. فيخرج القسمان المُضمران.

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1 في (ب) وفي (ت) : سقط لفظ " هنا ".
2 في (ب) وفي (ت) : سقطت العبارات "من ملح الحساب ".
3 في (أ) : " لم يزل ".
4 في (ب) وفي (ط) وفي (ت) : " بمثلها ".
5 في (ب) وفي (ط) وفي (ت) : "عدّة ".
6 في (ب) : اللفظ " نحن " زائد.
7 في (ت) : " نزيد على نصفه بقيته إلى العشرة. "، وفي (ب) الجملة مبتورة لا تفهم : "نزيد على الباقي مربع من العشرة ".
8 في (ب) : "الباقي من العشرة".
9 في (ت) : " ونُسقط الخارج من الباقي من العشرة. الباقي هو العدد المُضمر ".
10 في (ب) : "الباقي من العشرة".
11 في (أ) : "الخارج".
12 في (ط) : "العمل".
13 في (أ) وفي (ت) : " أيضًا "، وسقط اللفظ من (ب) ومن (ط).

المسألة الثالثة: عددٌ مُضمر وقسمان[1] مُضمران، كم هو؟ وكم كلّ واحد من قسميه؟
نأمره أن يضرب أحد القسمين في الآخر وأن يُربّع كلّ واحد منهما، ثمّ يسقط مُربّع[2] الأصغر من مُربّع [63 و] المُسطّح ويخبرنا بالباقي (ويسقط المُسطّح من مُربّع الأكبر ويخبرنا بالباقي)[3]. فنأخذ جذر (فضل ما)[4] بين المُخبر عنهما[5] يكون ما بين القسمين. فنقسم عليه مجموع[6] ما أخبرنا به، يكون (العدد المُضمر وهو)[7] مجمُوع القسمين. فإن زدنا عليه جذر[8] فضل ما بينهما كان ضعف أكبرهما، وإن نقّصناه من مجموعهما بقي ضعف أصغرهما. فاعلمه.

وهذه المسائل الثلاثة أيضًا ممّا أملى[9] عليّ شيخُنا الفقيه[10] **ابو العبّاس**، رضي الله عنه.

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[1] في (أ) : سقط لفظ " قسمان".
[2] في (م) : سقط لفظ "مربع".
[3] في (أ) : سقطت الجملة " ويسقط ... بالباقي.".
[4] في (م) : سقطت العبارة "فضل ما".
[5] في (م) : "بهما".
[6] في (ب) وفي (ت) : سقط لفظ "مجموع".
[7] في (أ) : "يكون العدد. والمضمور هو".
[8] في (م) : سقط لفظ "جذر".
[9] في (أ) : " أملاه " .
[10] في (ت) : سقط لفظ " الفقيه ".

<الخاتمة>

قال العبدُ (المُسمّى المُعترف)[1] المُقترف **عبد العزيز بن علي بن داود**[2] **الهواري**[3] **المصراتي**، (عفا عنه الله)[4]:

قد أتينا (بحمد الله وحسن عونه)[5] على ما شرطنا الإتيان به على قدر الطاقة، غير مُبرئ نفسي من الخطإ (ولا)[6] ما لا يعترى الأفكار من الزلل.

والله (سبحانه المسؤول في العصمة)[7] وفي انسدال[8] ما عُود من النّعمة. (وهو حسبنا وعليه في كلّ حال مُتّكلين)[9].

(وكان الفراغ منه يوم السبت ثامن عشر ذي القعدة عام أربعة وسبعمائة)[10].

(والله عز وجل ينفع المؤلف، والمؤلف بسببه بمنه ويمنه. والصلاة التامة على سيدنا ومولانا محمّد نبيه وعبده وعلى أهله وصحبه وأشياعه.) [11]

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[1] في (ط) وفي (ت) : "المُعترف المُسمّى".
[2] في (ط) : "داوود".
[3] في (ب) وفي (ت) : "الهوازي".
[4] في (أ) : "رحمه الله". وفي (ب) : "عفا الله له". وفي (ط) : "عفا الله عنه".
[5] في النسخ الاخرى : سقطت العبارة " بحمد الله وحسن عونه".
[6] وفي (ب) وفي (ت) : سقطت العبارة " و لا".
[7] في (أ) : "تعالى أسأله العصمة". وفي (ت) : "سبحانه المأمول والمسؤول في العصمة".
[8] في (أ) وفي (ت) : "ابذال".
[9] في (ط) وفي (ت) : "وهو حسبنا ونعم الوكيل." والجملة غائبة في باقي النسخ.
[10] في كل النسخ الاخرى : سقطت الجملة "وكان الفراغ منه يوم السبت ثامن عشر ذي القعدة عام أربعة وسبعمائة ".
[11] في (أ) : "والحمد لله ربّ العالمين والصّلاة والسّلام على أشرف المُرسلين محمّد وآله، الطيّبين الطاهرين وصحبه الأكرمين."
في (ب) : "والحمد لله وحده."
في (ط) وفي (ت) : "وصلى الله على سيّدنا محمد وآله وصحبه وسلّم تسليمًا كثيرًا إلى يوم الدّين." لكن غابت "إلى يوم الدين".

أخر النسخ

أخر نسخة المدينة المنورة (نسخة م)
كمل يوم الثلثاء ثامن عشر، شهر ربيع الأول، عام ستة وأربعين وسبعمائة. وكتب لنفسه بخط يده الفانية راجيا عفو ربه، **عبد الرحمن بن أبي بكر بن أحمد بن علي بن أحمد بن علي بن سبع بن مالك النفزي** عفا الله عنه.

آخر نسخة السليمانية باسطنبول (نسخة أ)
علّقه بنفسه عن نسخة سقيمة، للضرورة، **عمر بن عثمان بن عمر الحُسيني** بمدينة القسطنطينية المحروسة، غفر الله لمن ترحّم على كاتبه وعلى جميع المُسلمين بتاريخ عشرين رمضان المُكرّم سنة ثمانين وثمانمائة هجرة (880هـ).

آخر نسخة أكسفورد (نسخة ب)
تمّ الكتاب بحمد الله وعونه.

آخر نسخة تهران (نسخة ط)
وصلى الله على سيّدنا محمد وآله وصحبه وسلّم تسليمًا كثيرًا.

آخر نسخة تونس (نسخة ت)
وقد نُجز بعون الله القدير ليلة الجمعة رابع ليلة خلت من جمادى الآخرة من شهور سنة إثنتين وثمانين وألف، على يد العُبيد الفقير المُعترف بالعجز والتقصير تحويل مُصنّف العلم الشريف بدمشق الشام **يسين من مصطفي الإمام** بتكية المرحوم السلطان سليمان خان عليه الرحمةُ والرضوانُ والرضى الحنفي المآثر برضى عنهما وعن المسلمين آمين (1082 هـ).

الملحقات والفهارس[1]

الملحق الأول: مسائل **اللباب** المنقولة من **رفع الحجاب** لابن البنا

الملحق الثاني: الأمثلة في كتاب **اللباب** للهواري المصراتي

الجزء الأول :
القسم الأول في الاعداد الصحيحة
القسم الثاني في الكسور
القسم الثالث في الجذور
الجزء الثاني :
القسم الأول في النسبة والكفات
القسم الثاني في الجبر والمقابلة

معجم المصطلحات الواردة في كتاب **اللباب في شرح أعمال الحساب**

فهرس الأعلام في الكتاب

المراجع

[1] في الجداول التالية، الأرقام تشير إلى صفحات كل من:
تحقيق محمد سويسي (1969) لكتاب ابن البنا : **تلخيص أعمال الحساب**
وتحقيق محمد ابلاغ (1994) لكتاب ابن البنا : **رفع الحجاب عن وجوه أعمال الحساب**
وتحقيقنا هذا لكتاب الهواري المصراتي: **اللباب** في **شرح تلخيص أعمال الحساب لابن البنا**
ولورقات مخطوط شكر زاده: **أمثلة من التلخيص لابن البنا والحاوي لابن الهائم** (أسعد أفندي رقم 2/3150 بالمكتبة السليمانية باسطنبول)